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SOME INEQUALITIES FOR THE KHATRI-RAO PRODUCT OF MATRICES*

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Abstract. Several inequalities for the Khatri-Rao product of complex positive definite Hermitian matrices are established, and these results generalize some known inequalities for the Hadamard and Khatri-Rao products of matrices.

Key words. Matrix inequalities, Hadamard product, Khatri-Rao product, Tracy-Singh product, Spectral decomposition, Complex positive definite Hermitian matrix.

AMS subject classifications. 15A45, 15A69

1. Introduction. Consider complex matrices $A = (a_{ij})$ and $C = (c_{ij})$ of order $m \times n$ and $B = (b_{ij})$ of order $p \times q$. Let A and B be partitioned as $A = (A_{ij})$ and $B = (B_{ij})$, where A_{ij} is an $m_i \times n_j$ matrix and B_{kl} is a $p_k \times q_l$ matrix ($\sum m_i = m$, $\sum n_j = n$, $\sum p_k = p$, $\sum q_l = q$). Let $A \otimes B$, $A \circ C$, $A \odot B$ and $A * B$ be the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, respectively. The definitions of the mentioned four matrix products are given by Liu in [1]. Additionally, Liu [1, p. 269] also shows that the Khatri-Rao product can be viewed as a generalized Hadamard product and the Kronecker product is a special case of the Khatri-Rao or Tracy-Singh products. The purpose of this present paper is to establish several inequalities for the Khatri-Rao product of complex positive definite matrices, and thereby generalize some inequalities involving the Hadamard and Khatri-Rao products of matrices in [1, Eq. (13) and Theorem 8], [6, Eq. (3), Lemmas 2.1 and 2.2, Theorems 3.1 and 3.2], and [3, Eqs. (2) and (9)].

Let $S(m)$ be the set of all complex Hermitian matrices of order m, and $S^+(m)$ the set of all complex positive definite Hermitian matrices of order m. For M and N in $S(m)$, we write $M \geq N$ in the Löwner ordering sense, i.e., $M - N$ is positive semidefinite. For a matrix $A \in S^+(m)$, we denote by $\lambda_1(A)$ and $\lambda_m(A)$ the largest and smallest eigenvalue of A, respectively. Let B^* be the conjugate transpose matrix of the complex matrix B. We denote the $n \times n$ identity matrix by I_n, also we write I when the order of the matrix is clear.

2. Some Lemmas. In this section, we give some preliminaries.

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LEMMA 2.1. *There exists an* $mp \times \sum m_i p_i$ *real matrix* Z *such that* $Z^T Z = I$ *and*

$$A * B = Z^T (A \odot B) Z \tag{2.1}$$

for any $A \in S(m)$ *and* $B \in S(p)$ *partitioned as follows:*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \cdots & \cdots & \cdots \\ A_{t1} & \cdots & A_{tt} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & \cdots & B_{1t} \\ \cdots & \cdots & \cdots \\ B_{t1} & \cdots & B_{tt} \end{bmatrix},$$

where $A_{ii} \in S(m_i)$ *and* $B_{ii} \in S(p_i)$ *for* $i = 1, 2, \cdots, t$.

Proof. Let

$$Z_i = \begin{bmatrix} O_{i1} & \cdots & O_{i\ i-1} & I_{m_i p_i} & O_{i\ i+1} & \cdots & O_{it} \end{bmatrix}^T, \qquad i = 1, 2, \cdots, t,$$

where O_{ik} is the $m_i p_k \times m_i p_i$ zero matrix for any $k \neq i$. Then $Z_i^T Z_i = I$ and

$$Z_i^T (A_{ij} \odot B) Z_j = Z_i^T (A_{ij} \odot B_{kl})_{kl} Z_j = A_{ij} \otimes B_{ij}, \qquad i, j = 1, 2, \cdots, t.$$

Letting $Z = \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_t \end{bmatrix}$, the lemma follows by a direct computation. □

If $t = 2$ in Lemma 2.1, then Eq. (2.1) becomes Eq. (13) of [1].

COROLLARY 2.2. *There exists a real matrix* Z *such that* $Z^T Z = I$ *and*

$$M_1 * \cdots * M_k = Z^T (M_1 \odot \cdots \odot M_k) Z \tag{2.2}$$

for any $M_i \in S(m(i))$ $(1 \le i \le k,\ k \ge 2)$ *partitioned as*

$$M_i = \begin{bmatrix} N_{11}^{(i)} & \cdots & N_{1t}^{(i)} \\ \cdots & \cdots & \cdots \\ N_{t1}^{(i)} & \cdots & N_{tt}^{(i)} \end{bmatrix}, \tag{2.3}$$

where $N_{jj}^{(i)} \in S(m(i)_j)$ *for any* $1 \le i \le k$ *and* $1 \le j \le t$.

Proof. We proceed by induction on k. If $k = 2$, the corollary is true by Lemma 2.1. Suppose the corollary is true when $k = s$, i.e., there exists a real matrix P such that $P^T P = I$ and $M_1 * \cdots * M_s = P^T (M_1 \odot \cdots \odot M_s) P$, we will prove that it is true when $k = s + 1$. In fact,

$$\begin{aligned}
& M_1 * \cdots * M_{s+1} = \\
= \; & (M_1 * \cdots * M_s) * M_{s+1} \\
= \; & P^T (M_1 \odot \cdots \odot M_s) P * M_{s+1} \\
= \; & Q^T \left[P^T (M_1 \odot \cdots \odot M_s) P \odot M_{s+1} \right] Q \qquad (Q^T Q = I) \\
= \; & Q^T \left[P^T (M_1 \odot \cdots \odot M_s) P \odot \left(I_{m(s+1)} M_{s+1} I_{m(s+1)} \right) \right] Q \\
= \; & Q^T \left(P^T \odot I_{m(s+1)} \right) \left[(M_1 \odot \cdots \odot M_s) \odot M_{s+1} \right] \left(P \odot I_{m(s+1)} \right) Q.
\end{aligned}$$

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Letting $Z = \left(P \odot I_{m(s+1)}\right) Q$, the corollary follows. □

If the Khatri-Rao and Tracy-Singh products are replaced by the the Hadamard and Kronecker products in Corollary 2.2, respectively, then (2.2) becomes Lemma 2.2 in [6].

Lemma 2.3. *Let A and B be compatibly partitioned matrices, then $(A \odot B)^* = A^* \odot B^*$.*

Proof.

$$\begin{aligned}(A \odot B)^* = & \left((A_{ij} \odot B)_{ij}\right)^* = \left(((A_{ij} \otimes B_{kl})_{kl})_{ij}\right)^* = \left(((A_{ij} \otimes B_{kl})_{kl})^*\right)_{ji} \\ = & \left(((A_{ij} \otimes B_{kl})^*)_{lk}\right)_{ji} = \left((A_{ij}^* \otimes B_{kl}^*)_{lk}\right)_{ji} = \left(A_{ij}^* \odot B^*\right)_{ji} \\ = & A^* \odot B^*. \quad \square\end{aligned}$$

Definition 2.4. *Let the spectral decomposition of A ($\in S^+(m)$) be*

$$A = U_A^* D_A U_A = U_A^* \mathrm{diag}(d_1, \ \cdots, \ d_m) U_A,$$

where $d_i > 0$ for all i. For any $c \in \mathbf{R}$, we define the power of matrix A as follows

$$A^c = U_A^* D_A^c U_A = U_A^* \mathrm{diag}(d_1^c, \ \cdots, \ d_m^c) U_A.$$

Lemma 2.5. *Let $A \in S^+(m)$, $B \in S^+(p)$ and $c \in \mathbf{R}$, then*
i) $A \odot B \in S^+(mp)$, $\lambda_1(A \odot B) = \lambda_1(A)\lambda_1(B)$, and $\lambda_{mp}(A \odot B) = \lambda_m(A)\lambda_p(B)$;
ii) $(A \odot B)^c = A^c \odot B^c$.

Proof. Let $A = U_A^* D_A U_A$ and $B = U_B^* D_B U_B$ be the spectral decompositions of A and B, respectively. From Lemma 2.3 and [1, Theorem 1(a)], we derive

$$(U_A \odot U_B)^*(U_A \odot U_B) = (U_A^* \odot U_B^*)(U_A \odot U_B) = (U_A^* U_A) \odot (U_B^* U_B) = I_{mp} \tag{2.4}$$

$$\begin{aligned} A \odot B = & (U_A^* D_A U_A) \odot (U_B^* D_B U_B) = (U_A^* \odot U_B^*)(D_A \odot D_B)(U_A \odot U_B) \\ = & (U_A \odot U_B)^*(D_A \odot D_B)(U_A \odot U_B). \end{aligned} \tag{2.5}$$

The lemma follows from (2.4), (2.5), and the definitions of $A \odot B$ and $(A \odot B)^c$. □

If the Tracy-Singh product is placed by the Kronecker product in Lemma 2.5, then ii) of Lemma 2.5 becomes Lemma 2.1 in [6].

Corollary 2.6. *Let $M_i \in S^+(m(i))$ for $i = 1, 2 \cdots, k$, $n = \prod_{i=1}^{k} m(i)$ and $c \in \mathbf{R}$, then*
i) $M_1 \odot \cdots \odot M_k \in S^+(n)$, $\quad \lambda_1(M_1 \odot \cdots \odot M_k) = \prod_{i=1}^{k} \lambda_1(M_i)$ and
$\lambda_n(M_1 \odot \cdots \odot M_k) = \prod_{i=1}^{k} \lambda_{m(i)}(M_i)$;
ii) $(M_1 \odot \cdots \odot M_k)^c = M_1^c \odot \cdots \odot M_k^c$.

Proof. Using Lemma 2.5, the corollary follows by induction. □

If the Tracy-Singh product is replaced by the Kronecker product in Corollary 2.6, then ii) of Corollary 2.6 becomes Eq. (3) in [6].

Lemma 2.7. [4], [5] *Let $H \in S^+(n)$ and V be a complex matrix of order $n \times m$ such that $V^*V = I_m$, then*

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i) $(V^*H^rV)^{1/r} \le (V^*H^sV)^{1/s}$, *where r and s are two real numbers such that $s > r$, and either $s \notin (-1,1)$ and $r \notin (-1,1)$ or $s \ge 1 \ge r \ge \frac{1}{2}$ or $r \le -1 \le s \le -\frac{1}{2}$;*
ii) $(V^*H^sV)^{1/s} \le \overline{\Delta}(s,r)(V^*H^rV)^{1/r}$,
where r and s are two real numbers such that $s > r$ and either $s \notin (-1,1)$ or $r \notin (-1,1)$, $\overline{\Delta}(s,r) = \left\{\frac{r(\delta^s-\delta^r)}{(s-r)(\delta^r-1)}\right\}^{1/s}\left\{\frac{s(\delta^r-\delta^s)}{(r-s)(\delta^s-1)}\right\}^{-1/r}$, $W = \lambda_1(H)$, $w = \lambda_n(H)$ and $\delta = \frac{W}{w}$.
iii) $(V^*H^sV)^{1/s} - (V^*H^rV)^{1/r} \le \Delta(s,r)I$, *where* $\Delta(s,r) = \max_{\theta\in[0,\ 1]} \left\{[\theta W^s + (1-\theta)w^s]^{1/s} - [\theta W^r + (1-\theta)w^r]^{1/r}\right\}$, *and r, s, W, w and δ are as in* ii).

3. Main results. In this section, we establish some inequalities for the Khatri-Rao product of matrices.

THEOREM 3.1. *Let $M_i \in S^+(m(i))$ $(1 \le i \le k)$ be partitioned as in* (2.3) *and $n = \prod_{i=1}^{k} m(i)$, then*
(i) $(M_1^s * \cdots * M_k^s)^{1/s} \ge (M_1^r * \cdots * M_k^r)^{1/r}$, *where r and s are as in* i) *of Lemma 2.7;*
(ii) $(M_1^s * \cdots * M_k^s)^{1/s} \le \overline{\Delta}(s,r)(M_1^r * \cdots * M_k^r)^{1/r}$, *where $W = \prod_{i=1}^{k}\lambda_1(M_i)$ and $w = \prod_{i=1}^{k}\lambda_{m(i)}(M_i)$, and r, s, δ and $\overline{\Delta}(s,r)$ are as in* ii) *of Lemma 2.7;*
(iii) $(M_1^s * \cdots * M_k^s)^{1/s} - (M_1^r * \cdots * M_k^r)^{1/r} \le \Delta(s,r)I$, *where $W = \prod_{i=1}^{k}\lambda_1(M_i)$ and $w = \prod_{i=1}^{k}\lambda_{m(i)}(M_i)$, and r, s, δ and $\Delta(s,r)$ is as in* iii) *of Lemma 2.7.*

Proof. Let $H = M_1 \odot \cdots \odot M_k$, then $H \in S^+(n)$, $\lambda_1(H) = \prod_{i=1}^{k}\lambda_1(M_i)$ and $\lambda_n(H) = \prod_{i=1}^{k}\lambda_{m(i)}(M_i)$ from i) of Corollary 2.6. Therefore, using ii) of Corollary 2.6, Corollary 2.2, and Lemma 2.7,

$$\begin{aligned}
(M_1^r * \cdots * M_k^r)^{1/r} &= \left(Z^T(M_1^r \odot \cdots \odot M_k^r)Z\right)^{1/r}\\
&= \left(Z^T(M_1 \odot \cdots \odot M_k)^r Z\right)^{1/r}\\
&\le \left(Z^T(M_1 \odot \cdots \odot M_k)^s Z\right)^{1/s}\\
&= \left(Z^T(M_1^s \odot \cdots \odot M_k^s)Z\right)^{1/s}\\
&= (M_1^s * \cdots * M_k^s)^{1/s},
\end{aligned}$$

$$\begin{aligned}
(M_1^s * \cdots * M_k^s)^{1/s} &= \left(Z^T(M_1^s \odot \cdots \odot M_k^s)Z\right)^{1/s}\\
&= \left(Z^T(M_1 \odot \cdots \odot M_k)^s Z\right)^{1/s}\\
&\le \overline{\Delta}(s,r)\left(Z^T(M_1 \odot \cdots \odot M_k)^r Z\right)^{1/r}\\
&= \overline{\Delta}(s,r)\left(Z^T(M_1^r \odot \cdots \odot M_k^r)Z\right)^{1/r}\\
&= \overline{\Delta}(s,r)\,(M_1^r * \cdots * M_k^r)^{1/r},
\end{aligned}$$

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$$\begin{aligned} & (M_1^s * \cdots * M_k^s)^{1/s} - (M_1^r * \cdots * M_k^r)^{1/r} = \\ = \quad & \left(Z^T(M_1 \odot \cdots \odot M_k)^s Z\right)^{1/s} - \left(Z^T(M_1 \odot \cdots \odot M_k)^r Z\right)^{1/r} \\ \leq \quad & \Delta(s,r)I. \quad \square \end{aligned}$$

If the Khatri-Rao and Tracy-Singh products are replaced by the Hadamard and Kronecker products in Theorem 3.1, respectively, then (i) becomes Theorem 3.1 in [6], and (ii) and (iii) become Theorem 3.2 in [6].

THEOREM 3.2. *Let $M_i \in S^+(m(i))$ $(1 \leq i \leq k)$ be partitioned as in* (2.3)*, then*

$$(M_1 * \cdots * M_k)^{-1} \leq M_1^{-1} * \cdots * M_k^{-1}, \tag{3.1}$$

$$M_1^{-1} * \cdots * M_k^{-1} \leq \frac{(W+w)^2}{4Ww}(M_1 * \cdots * M_k)^{-1}, \tag{3.2}$$

$$M_1 * \cdots * M_k - (M_1^{-1} * \cdots * M_k^{-1})^{-1} \leq (\sqrt{W} - \sqrt{w})^2 I, \tag{3.3}$$

$$(M_1 * \cdots * M_k)^2 \leq M_1^2 * \cdots * M_k^2, \tag{3.4}$$

$$M_1^2 * \cdots * M_k^2 \leq \frac{(W+w)^2}{4Ww}(M_1 * \cdots * M_k)^2, \tag{3.5}$$

$$(M_1 * \cdots * M_k)^2 - M_1^2 * \cdots * M_k^2 \leq \frac{1}{4}(W-w)^2 I, \tag{3.6}$$

$$M_1 * \cdots * M_k \leq (M_1^2 * \cdots * M_k^2)^{1/2}, \tag{3.7}$$

$$(M_1^2 * \cdots * M_k^2)^{1/2} \leq \frac{W+w}{2\sqrt{Ww}}(M_1 * \cdots * M_k), \tag{3.8}$$

$$(M_1^2 * \cdots * M_k^2)^{1/2} - M_1 * \cdots * M_k \leq \frac{(W-w)^2}{4(W+w)} I, \tag{3.9}$$

where W and w are as in Theorem 3.1.

Proof. Noting that $G \geq H > O$ if and only if $H^{-1} \geq G^{-1} > O$ [2], we obtain (3.1), (3.2) and (3.3) by choosing $r = -1$ and $s = 1$ in Theorem 3.1. Similarly, (3.7), (3.8) and (3.9) can be obtained by choosing $r = 1$ and $s = 2$ in Theorem 1. Thereby, using that $G \geq H > 0$ implies $G^2 \geq H^2 > 0$, we derive that (3.4) and (3.5) hold.

Liu and Neudecker [3] show that

$$V^*A^2V - (V^*AV)^2 \leq \frac{1}{4}\left(\lambda_1(A) - \lambda_m(A)\right)^2 I \tag{3.10}$$

for $A \in S^+(m)$ and $V^*V = I$. Replacing A by $M_1 \odot \cdots \odot M_k$ and V by Z in (3.10), we obtain (3.6). $\square$

If we replace the Khatri-Rao product by the Hadamard product in (3.1), (3.2), (3.3), (3.4), (3.7), (3.8) and (3.9), then we obtain some inequalities in [6]. If choosing $t = 2$ and considering the real positive definite matrices in Theorem 3.2, then Theorem 3.2 becomes Theorem 8 in [1]. If choosing $t = 2$ and replacing the Khatri-Rao product by the Hadamard product in (3.6) and (3.8), respectively, then we obtain Eqs. (2) and (9) of [3].

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