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FROM CONVERGENCE IN MEASURE TO CONVERGENCE OF MATRIX-SEQUENCES THROUGH CONCAVE FUNCTIONS AND SINGULAR VALUES*

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Abstract. Sequences of matrices with increasing size naturally arise in several areas of science, such as, for example, the numerical discretization of differential and integral equations. An approximation theory for sequences of this kind has recently been developed, with the aim of providing tools for computing their asymptotic singular value and eigenvalue distributions. The cornerstone of this theory is the notion of approximating classes of sequences (a.c.s.), which is also fundamental to the theory of generalized locally Toeplitz (GLT) sequences, and hence to the spectral analysis of PDE discretization matrices. Drawing inspiration from measure theory, here it is introduced a class of functions which are proved to be complete pseudometrics inducing the a.c.s. convergence. It is also shown that each of these pseudometrics gives rise to a natural isometry between the spaces of GLT sequences and measurable functions. Furthermore, it is highlighted that the a.c.s. convergence is an asymptotic matrix version of the convergence in measure, thus suggesting a way to obtain matrix theory results from measure theory results.

Key words. Singular value and eigenvalue asymptotics, Convergence in measure, Matrix-sequences, PDE discretizations, Generalized locally Toeplitz sequences, Concave functions.

AMS subject classifications. 15A18, 28A20, 15A60, 15B05, 26A51.

1. Introduction and main results. Let $\mathbb{C}^{n\times n}$ be the space of complex $n\times n$ matrices. Throughout this paper, a matrix-sequence is a sequence of the form $\{A_n\}_n$ with $A_n\in\mathbb{C}^{n\times n}$. Matrix-sequences naturally arise in several contexts. For example, when discretizing a linear differential or integral equation by a linear numerical method (such as the finite difference method, the finite element method, the modern isogeometric analysis, etc.), the actual computation of the numerical solution reduces to solving a linear system $A_n\mathbf{u}_n=\mathbf{g}_n$. The size n of this system diverges to infinity as the mesh discretization parameter tends to 0, and we are then in the presence of a matrix-sequence $\{A_n\}_n$. It is often observed in practice that $\{A_n\}_n$ belongs to the class of the so-called generalized locally Toeplitz (GLT) sequences, and in particular it enjoys an asymptotic singular value and eigenvalue distribution as $n\to\infty$; we refer the reader to [4] for a nice introduction to this subject and to [2, 10, 11, 13, 14, 20, 21, 23] for more advanced studies. Another noteworthy example concerns the finite sections of an infinite Toeplitz matrix. An infinite Toeplitz matrix is a matrix of the form

$$[a_{i-j}]_{i,j=1}^{\infty}=\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & \ddots & \ddots & \ddots \\ \vdots & a_2 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{1.1}$$

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The nth section of (1.1) is the $n \times n$ matrix defined by $A_n = [a_{i-j}]_{i,j=1}^n$. In the case where the entries a_k are the Fourier coefficients of a function $a \in L^1([-\pi, \pi])$, the matrix A_n is denoted by $T_n(a)$ and is referred to as the nth Toeplitz matrix generated by a. The asymptotic singular value and eigenvalue distribution of the matrix-sequence $\{T_n(a)\}_n$ has been deeply investigated in recent times, starting from Szegő's first limit theorem [5, 16, 27] and the Avram–Parter theorem [1, 5, 17], up to the works by Tyrtyshnikov–Zamarashkin [25, 26, 28] and Tilli [22, 24].

In the last 20 years, an asymptotic approximation theory for matrix-sequences has been developed. The aim was to obtain sufficiently powerful tools — which we formally state in Theorems 1.3 and 1.4 — for computing the asymptotic singular value and eigenvalue distribution of a "difficult" matrix-sequence $\{A_n\}_n$ from the asymptotic singular value and eigenvalue distributions of "simpler" matrix-sequences $\{B_{n,m}\}_n$ that "converge" to $\{A_n\}_n$ in a suitable way as $m \to \infty$. The cornerstone of all this approximation theory is the concept of approximating classes of sequences (a.c.s.), which we report in Definition 1.1 and which is due to Serra-Capizzano [18], though the underlying idea was already present in previous works of the same author [19] and in Tilli's pioneering paper on locally Toeplitz (LT) sequences [23]. After Definition 1.1, we provide the precise notions of asymptotic singular value and eigenvalue distribution for a matrix-sequence. It is worth stressing that the a.c.s., along with Theorems 1.3 and 1.4, form the basis of the theory of LT sequences [10, 23] and GLT sequences [2, 4, 11, 13, 14, 20, 21]. For some of their concrete applications, we refer the reader to [4, Sections 3–4], [10, Section 5.3], [11, Section 6.2.2], [13, Chapter 10], [14, Chapter 8], [15, Section 3], and also [8, 9, 12].

In what follows, we use the abbreviation "a.c.s." for both the singular "approximating class of sequences" and the plural "approximating classes of sequences"; it will be clear from the context whether "a.c.s." is singular or plural. We denote by μ_k the Lebesgue measure in $\mathbb{R}^k$ and by $C_c(\mathbb{R})$ (resp., $C_c(\mathbb{C})$) the space of continuous complex-valued functions with bounded support defined on $\mathbb{R}$ (resp., $\mathbb{C}$). The composite function $f \circ g$ is denoted by $f(g)$. If $A \in \mathbb{C}^{n\times n}$, the singular values and the eigenvalues of A are denoted by $\sigma_1(A), \ldots, \sigma_n(A)$ and $\lambda_1(A), \ldots, \lambda_n(A)$, respectively. It is always understood that the singular values are arranged in non-increasing order: $\sigma_{\max}(A) = \sigma_1(A) \geq \cdots \geq \sigma_n(A) = \sigma_{\min}(A)$. The symbol $\|\cdot\|$ will denote both the 2-norm of vectors and the associated operator norm for matrices. We recall that $\|A\| = \sigma_{\max}(A)$ for every matrix A.

Definition 1.1 (Approximating class of sequences). Let $\{A_n\}_n$ be a matrix-sequence and $\{\{B_{n,m}\}_n\}_m$ a sequence of matrix-sequences. We say that $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ if the following condition is met: for every m there exists n_m such that, for $n \geq n_m$,

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad \operatorname{rank}(R_{n,m}) \leq c(m)n, \quad \|N_{n,m}\| \leq \omega(m),$$

where the quantities n_m, $c(m)$, $\omega(m)$ depend only on m, and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

Roughly speaking, $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$ if, for all sufficiently large m, the sequence $\{B_{n,m}\}_n$ approximates (asymptotically) the sequence $\{A_n\}_n$ in the sense that A_n is eventually equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size n) plus a small-norm matrix.

Definition 1.2 (Asymptotic singular value and eigenvalue distribution of a matrix-sequence). Let $\{A_n\}_n$ be a matrix-sequence and let $f : D \to \mathbb{C}$ be a measurable function defined on a set $D \subset \mathbb{R}^k$ with $0 < \mu_k(D) < \infty$.

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- We say that $\{A_n\}_n$ has an asymptotic singular value distribution described by f, and we write $\{A_n\}_n \sim_\sigma f$, if, for every $F \in C_c(\mathbb{R})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(|f|)\mathrm{d}\mu_k.$$

- We say that $\{A_n\}_n$ has an asymptotic eigenvalue distribution described by f, and we write $\{A_n\}_n \sim_\lambda f$, if, for every $F \in C_c(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(f)\mathrm{d}\mu_k.$$

We are now in the position of stating the fundamental Theorems 1.3 and 1.4, which originally appeared in [15, 18]. For their proofs, see [13, Section 5.3].

THEOREM 1.3. *Let $\{A_n\}_n$, $\{B_{n,m}\}_n$ be matrix-sequences and let f, $f_m : D \to \mathbb{C}$ be measurable functions defined on a set $D \subset \mathbb{R}^k$ with $0 < \mu_k(D) < \infty$. Suppose that:*

(i) $\{B_{n,m}\}_n \sim_\sigma f_m$, *for every* m;
(ii) $\{\{B_{n,m}\}_n\}_m$ *is an a.c.s. for* $\{A_n\}_n$;
(iii) $f_m \to f$ *in measure.*

Then $\{A_n\}_n \sim_\sigma f$.

THEOREM 1.4. *Let $\{A_n\}_n$, $\{B_{n,m}\}_n$ be matrix-sequences formed by Hermitian matrices and let f, f_m : $D \to \mathbb{C}$ be measurable functions defined on a set $D \subset \mathbb{R}^k$ with $0 < \mu_k(D) < \infty$. Suppose that:*

(i) $\{B_{n,m}\}_n \sim_\lambda f_m$, *for every* m;
(ii) $\{\{B_{n,m}\}_n\}_m$ *is an a.c.s. for* $\{A_n\}_n$;
(iii) $f_m \to f$ *in measure.*

Then $\{A_n\}_n \sim_\lambda f$.

In very recent times, it was discovered that the notion of a.c.s. is a notion of convergence in the space of matrix-sequences

$$\mathscr{E} = \{\{A_n\}_n : \ A_n \in \mathbb{C}^{n\times n}\}. \tag{1.2}$$

To be more precise, for $A \in \mathbb{C}^{n\times n}$ let

$$p(A) = \inf\left\{\frac{\operatorname{rank}(R)}{n} + \|N\| : \ R, N \in \mathbb{C}^{n\times n}, \ R + N = A\right\}, \tag{1.3}$$

and for $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, set

$$p_{\text{a.c.s.}}(\{A_n\}_n) = \limsup_{n\to\infty} p(A_n), \tag{1.4}$$

$$d_{\text{a.c.s.}}(\{A_n\}_n, \{B_n\}_n) = p_{\text{a.c.s.}}(\{A_n - B_n\}_n). \tag{1.5}$$

It was proved in [2, 7] that $d_{\text{a.c.s.}}$ is a distance on $\mathscr{E}$ which turns $\mathscr{E}$ into a complete topological (pseudometric) space $(\mathscr{E}, \tau_{\text{a.c.s.}})$ where the statement "$\{\{B_{n,m}\}_n\}_m$ converges to $\{A_n\}_n$" is equivalent to "$\{\{B_{n,m}\}_n\}_m$ is

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an a.c.s. for $\{A_n\}_n$". In view of this, we will use the convergence notation $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ to indicate that $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$. Note that Theorem 1.3 admits a simple topological interpretation: for any measurable set $D \subset \mathbb{R}^k$ with $0 < \mu_k(D) < \infty$, consider the space of measurable functions

$$\mathfrak{M}_D = \{f : D \to \mathbb{C} : \ f \text{ is measurable}\}, \tag{1.6}$$

and define τ_{measure} to be the (pseudometric) topology of convergence in measure over $\mathfrak{M}_D$; then, Theorem 1.3 is equivalent to saying that the set of pairs $(\{A_n\}_n, f) \in \mathscr{E} \times \mathfrak{M}_D$ such that $\{A_n\}_n \sim_\sigma f$ is closed in $\mathscr{E} \times \mathfrak{M}_D$ equipped with the product (pseudometric) topology $\tau_{\text{a.c.s.}} \times \tau_{\text{measure}}$. An analogous interpretation can be given for Theorem 1.4.

Besides proving the existence and completeness of a pseudometric topology $\tau_{\text{a.c.s.}}$ associated with the notion of a.c.s., papers [2, 7] highlighted a strong connection between $\tau_{\text{a.c.s.}}$ and τ_{measure}, a connection which had somehow already been suggested in [19]. It is worth spending a few words to illustrate this connection in some detail as it will be the starting point of the analysis carried out in this paper. In what follows, we use a notation borrowed from probability theory to indicate sets: for a generic function $\gamma : E \to \mathbb{C}$, the set $\{\tau \in E : \ \gamma(\tau) \neq 0\}$ is denoted by $\{\gamma \neq 0\}$, the set $\{\tau \in E : \ |\gamma(\tau)| > \varepsilon\}$ is denoted by $\{|\gamma| > \varepsilon\}$, etc. For $f, g \in \mathfrak{M}_D$ let

$$p_{\text{measure}}(f) = \inf\left\{\frac{\mu_k\{f_R \neq 0\}}{\mu_k(D)} + \|f_N\|_{L^\infty(D)} : \ f_R, f_N \in \mathfrak{M}_D, \ f_R + f_N = f\right\}, \tag{1.7}$$

$$d_{\text{measure}}(f, g) = p_{\text{measure}}(f - g), \tag{1.8}$$

where $\|f_N\|_{L^\infty(D)} = \operatorname{ess\,sup}_D |f_N|$. It was proved in [7] that d_{measure} is a pseudometric on $\mathfrak{M}_D$ inducing τ_{measure}. Denote by ν_n the counting measure on $D_n = \{1, \dots, n\}$ and by $\boldsymbol{\sigma}(A) = (\sigma_1(A), \dots, \sigma_n(A))$ the vector of the singular values of $A \in \mathbb{C}^{n\times n}$, considered as a function on D_n defined by the rule $i \mapsto \sigma_i(A)$. Then, Eq. (1.3) can be rewritten as

$$p(A) = \inf\left\{\frac{\nu_n\{\boldsymbol{\sigma}(R) \neq 0\}}{\nu_n(D_n)} + \|\boldsymbol{\sigma}(N)\|_{L^\infty(D_n)} : \ R, N \in \mathbb{C}^{n\times n}, \ R + N = A\right\}, \tag{1.9}$$

and we therefore notice an evident connection via singular values between (1.4)–(1.5) and (1.7)–(1.8). It was also proved in [2] that $p_{\text{a.c.s.}}(\{A_n\}_n) = p_{\text{measure}}(f)$ whenever $\{A_n\}_n \sim_\sigma f$.

Other pseudometrics on $\mathfrak{M}_D$ inducing the topology τ_{measure} can be obtained from the following construction. Let $\varphi : [0, \infty) \to [0, \infty)$ be a concave bounded continuous function such that $\varphi(0) = 0$ and $\varphi > 0$ on $(0, \infty)$. It can be proved that any such function φ is non-decreasing and subadditive, i.e., $\varphi(x + y) \le \varphi(x) + \varphi(y)$ for all $x, y \in [0, \infty)$, and if for $f, g \in \mathfrak{M}_D$ we define

$$p^\varphi_{\text{measure}}(f) = \frac{1}{\mu_k(D)} \int_D \varphi(|f|) \mathrm{d}\mu_k, \tag{1.10}$$

$$d^\varphi_{\text{measure}}(f, g) = p^\varphi_{\text{measure}}(f - g), \tag{1.11}$$

then $d^\varphi_{\text{measure}}$ is a complete pseudometric on $\mathfrak{M}_D$ inducing τ_{measure}. In view of the connection between (1.4)–(1.5) and (1.7)–(1.8) via Eq. (1.9), the analogs of (1.10)–(1.11) are defined for $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$ as follows:

$$p^\varphi_{\text{a.c.s.}}(\{A_n\}_n) = \limsup_{n\to\infty} p^\varphi(A_n), \tag{1.12}$$

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$$d^{\varphi}_{\mathrm{a.c.s.}}(\{A_n\}_n,\{B_n\}_n)=p^{\varphi}_{\mathrm{a.c.s.}}(\{A_n-B_n\}_n), \tag{1.13}$$

where, for $A\in\mathbb{C}^{n\times n}$,

$$p^{\varphi}(A)=\frac{1}{\nu_n(D_n)}\int_{D_n}\varphi(\boldsymbol{\sigma}(A))\mathrm{d}\nu_n=\frac{1}{n}\sum_{i=1}^{n}\varphi(\sigma_i(A)). \tag{1.14}$$

In the present paper we provide a positive answer to a question raised in [7, Section 4] and we show that $d^{\varphi}_{\mathrm{a.c.s.}}$ is a complete pseudometric on $\mathscr{E}$ which induces the a.c.s. topology $\tau_{\mathrm{a.c.s.}}$. Besides the theoretical interest, this result can be used to solve a problem that is often encountered in practical applications, namely the problem of establishing whether a given sequence of matrix-sequences $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for another matrix-sequence $\{A_n\}_n$: it suffices to choose a suitable function φ and establish if either $d^{\varphi}_{\mathrm{a.c.s.}}(\{B_{n,m}\}_n,\{A_n\}_n)\to 0$ or $d^{\varphi}_{\mathrm{a.c.s.}}(\{B_{n,m}\}_n,\{A_n\}_n)\not\to 0$ as $m\to\infty$. We also show that $p^{\varphi}_{\mathrm{a.c.s.}}(\{A_n\}_n)=p^{\varphi}_{\mathrm{measure}}(f)$ whenever $\{A_n\}_n\sim_\sigma f$. As a consequence, we will see that, if we identify two matrix-sequences $\{A_n\}_n,\{B_n\}_n\in\mathscr{E}$ whenever $\{A_n-B_n\}_n\sim_\sigma 0$ and two measurable functions $\kappa,\xi\in\mathfrak{M}_{[0,1]\times[-\pi,\pi]}$ whenever $\kappa-\xi=0$ almost everywhere (a.e.), then there exists a natural isometry between the (metric) spaces $(\mathscr{G},d^{\varphi}_{\mathrm{a.c.s.}})$ and $(\mathfrak{M}_{[0,1]\times[-\pi,\pi]},d^{\varphi}_{\mathrm{measure}})$, where $\mathscr{G}$ is the subset of $\mathscr{E}$ consisting of GLT sequences. Here are our main results.

THEOREM 1.5. *Let $\varphi:[0,\infty)\to[0,\infty)$ be a concave bounded continuous function such that $\varphi(0)=0$ and $\varphi>0$ on $(0,\infty)$, and let $\mathscr{E}$ be the space of matrix-sequences (1.2). Then, the function $d^{\varphi}_{\mathrm{a.c.s.}}$ defined in (1.13) is a complete pseudometric on $\mathscr{E}$ inducing the a.c.s. topology $\tau_{\mathrm{a.c.s.}}$.*

THEOREM 1.6. *Let $\varphi:[0,\infty)\to[0,\infty)$ be a bounded continuous function such that $\varphi(0)=0$, let $\mathscr{E}$ be the space of matrix-sequences (1.2), and let $\mathfrak{M}_D$ be the space of measurable functions (1.6), with $D\subset\mathbb{R}^k$ being a measurable set such that $0<\mu_k(D)<\infty$. Then, for every $(\{A_n\}_n,f)\in\mathscr{E}\times\mathfrak{M}_D$,*

$$\{A_n\}_n\sim_\sigma f \quad\Longrightarrow\quad p^{\varphi}_{\mathrm{a.c.s.}}(\{A_n\}_n)=p^{\varphi}_{\mathrm{measure}}(f),$$

where $p^{\varphi}_{\mathrm{measure}}$ and $p^{\varphi}_{\mathrm{a.c.s.}}$ are defined, respectively, in (1.10) and (1.12).

THEOREM 1.7. *Let $\varphi:[0,\infty)\to[0,\infty)$ be a concave bounded continuous function such that $\varphi(0)=0$ and $\varphi>0$ on $(0,\infty)$, and let $\{A_n\}_n$ and $\{B_n\}_n$ be GLT sequences with symbols κ and ξ, respectively. Then,*

$$d^{\varphi}_{\mathrm{a.c.s.}}(\{A_n\}_n,\{B_n\}_n)=d^{\varphi}_{\mathrm{measure}}(\kappa,\xi),$$

where $d^{\varphi}_{\mathrm{measure}}$ and $d^{\varphi}_{\mathrm{a.c.s.}}$ are defined, respectively, in (1.11) and (1.13).

Before concluding this introduction, it is important to emphasize that the proofs of Theorems 1.5–1.7 reported in Sections 2–4 will highlight further deep connections — in addition to those appearing in [2, 7] — between the a.c.s. convergence of matrix-sequences and the convergence in measure of functions. Actually, the insights one will gain from a careful consideration of Sections 2–4 may be regarded as a further main contribution of this paper. We refer in particular to Theorem 2.4, Remark 2.5, the proof of Theorem 2.6, and Section 3. It is not to be excluded that the deep connections between the a.c.s. convergence and the convergence in measure highlighted in this paper and in [2, 7] may lead to a “bridge”, in the precise mathematical sense established in [6], between measure theory and the asymptotic linear algebra theory underlying the notion of a.c.s.

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2. Proof of Theorem 1.5. We begin by reporting the statement of the Rotfel'd theorem [3, Theorem IV.2.14].

THEOREM 2.1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave function such that $\varphi(0) = 0$. Then, for all $A, B \in \mathbb{C}^{n\times n}$,*

$$\sum_{i=1}^{n} \varphi(\sigma_i(A+B)) \leq \sum_{i=1}^{n} \varphi(\sigma_i(A)) + \sum_{i=1}^{n} \varphi(\sigma_i(B)).$$

To prove that $d^{\varphi}_{\text{a.c.s.}}$ is a complete pseudometric on $\mathscr{E}$, we resort to a more general result. Theorem 2.1 from [2] shows that the pseudometric $d_{\text{a.c.s.}}$ is complete, and the proof only uses the definition (1.5) and the properties of the function p in (1.3) which ensure that $d_{\text{a.c.s.}}$ is a pseudometric. This observation leads to the following general theorem, which is interesting also in itself and may be considered as a further main result of this paper in addition to Theorems 1.5–1.7.

THEOREM 2.2. *Let $q : \bigcup_n \mathbb{C}^{n\times n} \to \mathbb{R}$ be a function such that:*

1. *$0 \leq q(A) \leq L$, for every $A \in \mathbb{C}^{n\times n}$ and every n, where L is a constant independent of both A and n;*
2. *$q(O_n) = 0$, for every n, where O_n is the $n \times n$ zero matrix;*
3. *$q(A) = q(-A)$, for every $A \in \mathbb{C}^{n\times n}$ and every n;*
4. *$q(A+B) \leq q(A) + q(B)$, for every $A, B \in \mathbb{C}^{n\times n}$ and every n.*

For $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, let

$$p_{[q]}(\{A_n\}_n) = \limsup_{n\to\infty} q(A_n), \tag{2.15}$$

$$d_{[q]}(\{A_n\}_n, \{B_n\}_n) = p_{[q]}(\{A_n - B_n\}_n). \tag{2.16}$$

Then, $d_{[q]}$ is a complete pseudometric on $\mathscr{E}$.

Proof. The conditions 1–4 on q imply that, for all $\{A_n\}_n, \{B_n\}_n, \{C_n\}_n \in \mathscr{E}$,

1. $0 \leq d_{[q]}(\{A_n\}_n, \{B_n\}) \leq L$,
2. $d_{[q]}(\{A_n\}_n, \{A_n\}_n) = 0$,
3. $d_{[q]}(\{A_n\}_n, \{B_n\}_n) = d_{[q]}(\{B_n\}_n, \{A_n\}_n)$,
4. $d_{[q]}(\{A_n\}_n, \{C_n\}_n) \leq d_{[q]}(\{A_n\}_n, \{B_n\}_n) + d_{[q]}(\{B_n\}_n, \{C_n\}_n)$.

Consequently, the function $d_{[q]}$ is a pseudometric on $\mathscr{E}$.

Let $\{\{B_{n,m}\}_n\}_m$ be a Cauchy sequence for the pseudometric $d_{[q]}$. We recall that the convergence of the sequence is equivalent to the convergence of any subsequence, with the same limit. Extract from $\{\{B_{n,m}\}_n\}_m$ a subsequence, which we call again $\{\{B_{n,m}\}_n\}_m$, such that, for every pair of indices s, t,

$$d_{[q]}(\{B_{n,s}\}_n, \{B_{n,t}\}_n) \leq 2^{-\min\{s,t\}}$$

or, equivalently,

$$\limsup_{n\to\infty} q(B_{n,s} - B_{n,t}) \leq 2^{-\min\{s,t\}}.$$

If we consider any two consecutive indices m and $m+1$, we have

$$\limsup_{n\to\infty} q(B_{n,m} - B_{n,m+1}) \leq 2^{-m}.$$

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Given $\varepsilon > 0$, the argument of the limsup is eventually less then $2^{-m} + \varepsilon$. We choose $\varepsilon = 2^{-m}$ and we find a strictly increasing sequence of indices N_m such that

$$q(B_{n,m} - B_{n,m+1}) \leq 2^{-m+1}, \quad \forall\, n \geq N_m.$$

We can now define the matrix-sequence $\{A_n\}_n$ that will turn out to be the limit of $\{\{B_{n,m}\}_n\}_m$. Let

$$A_n = B_{n,m} \quad \text{whenever} \quad N_{m+1} > n \geq N_m.$$

For any m and $n \geq N_m$, the q-distance between A_n and $B_{n,m}$ can be estimated as follows: let $M \geq m$ be such that $N_{M+1} > n \geq N_M$, then

$$q(B_{n,m} - A_n) = q(B_{n,m} - B_{n,M}) \leq \sum_{k=m}^{M-1} q(B_{n,k} - B_{n,k+1}).$$

Considering that $n \geq N_M > N_k$ for all indices k in the summation, we obtain

$$\sum_{k=m}^{M-1} q(B_{n,k} - B_{n,k+1}) \leq \sum_{k=m}^{M-1} 2^{-k+1} \leq 2^{-m+1} \sum_{k=0}^{M-m-1} 2^{-k} \leq 2^{-m+1} \sum_{k=0}^{\infty} 2^{-k} = 2^{-m+2}.$$

We conclude that, for any m and $n \geq N_m$,

$$q(B_{n,m} - A_n) \leq 2^{-m+2}.$$

Thus, for any m,

$$d_{[q]}(\{B_{n,m}\}_n, \{A_n\}_n) = \limsup_{n\to\infty} q(B_{n,m} - A_n) \leq 2^{-m+2}$$

and $d_{[q]}(\{B_{n,m}\}_n, \{A_n\}_n) \to 0$ as $m \to \infty$. ∎

As a corollary of Theorems 2.1 and 2.2, we obtain that $d^{\varphi}_{\text{a.c.s.}}$ is a complete pseudometric on $\mathscr{E}$.

Corollary 2.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave bounded function such that $\varphi(0) = 0$. Then, $d^{\varphi}_{\text{a.c.s.}}$ is a complete pseudometric on $\mathscr{E}$.*

Proof. We want to apply Theorem 2.2 with

$$q(A) = p^{\varphi}(A) = \frac{1}{n} \sum_{i=1}^{n} \varphi(\sigma_i(A)),$$

so let us check the four conditions. Since φ is nonnegative and bounded, we have $0 \leq p^{\varphi}(A) \leq L$ for some constant L. Since $\varphi(0) = 0$ and the singular values of the zero matrix O_n are all zeros, we have $p^{\varphi}(O_n) = 0$. Since the singular values of $-A$ are the same as the singular values of A, we have $p^{\varphi}(-A) = p^{\varphi}(A)$. Finally, the inequality $p^{\varphi}(A+B) \leq p^{\varphi}(A) + p^{\varphi}(B)$ follows from Theorem 2.1. We conclude that $q = p^{\varphi}$ satisfies the four conditions of Theorem 2.2 and the thesis follows. ∎

To conclude the proof of Theorem 1.5 we still have to show that $d^{\varphi}_{\text{a.c.s.}}$ induces on $\mathscr{E}$ the a.c.s. topology $\tau_{\text{a.c.s.}}$. To this end, we will make use of the following characterization theorem for a.c.s., which may certainly be regarded as another main result of this paper, especially if we consider it in the light of Remark 2.5. Recall that ν_n denotes the counting measure on $D_n = \{1, \ldots, n\}$, so $\nu_n(E)$ is just the cardinality of E for all $E \subseteq D_n$.

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THEOREM 2.4. *Let $\{A_n\}_n$ be a matrix-sequence and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of matrix-sequences. The following conditions are equivalent:*

1. $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$;
2. $\lim_{m\to\infty} \limsup_{n\to\infty} \dfrac{\nu_n\{i \in D_n :\ \sigma_i(A_n - B_{n,m}) > \varepsilon\}}{n} = 0,$ *for all* $\varepsilon > 0$.

Proof. ($1 \implies 2$) Suppose that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$. By definition, for every m there exists n_m such that, for $n \geq n_m$,

$$A_n - B_{n,m} = R_{n,m} + N_{n,m}, \quad \operatorname{rank}(R_{n,m}) \leq c(m)n, \quad \|N_{n,m}\| \leq \omega(m)$$

with

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

Let $V \subseteq_{\text{sp.}} \mathbb{C}^n$ mean that V is a subspace of $\mathbb{C}^n$. By the minimax principle for singular values [3, Problem III.6.1],

$$\begin{aligned}
\sigma_i(A_n - B_{n,m}) &= \max_{\substack{V\subseteq_{\text{sp.}}\mathbb{C}^n\\ \dim V=i}} \min_{\substack{\mathbf{x}\in V\\ \|\mathbf{x}\|=1}} \|(A_n - B_{n,m})\mathbf{x}\| \\
&= \max_{\substack{V\subseteq_{\text{sp.}}\mathbb{C}^n\\ \dim V=i}} \min_{\substack{\mathbf{x}\in V\\ \|\mathbf{x}\|=1}} \|(R_{n,m} + N_{n,m})\mathbf{x}\| \\
&\leq \|N_{n,m}\| + \max_{\substack{V\subseteq_{\text{sp.}}\mathbb{C}^n\\ \dim V=i}} \min_{\substack{\mathbf{x}\in V\\ \|\mathbf{x}\|=1}} \|R_{n,m}\mathbf{x}\| \leq \omega(m) + \sigma_i(R_{n,m}).
\end{aligned}$$

If $n \geq n_m$ then $\operatorname{rank}(R_{n,m}) \leq c(m)n$, so for every $i > c(m)n$ we have $\sigma_i(R_{n,m}) = 0$ and $\sigma_i(A_n - B_{n,m}) \leq \omega(m)$. We conclude that, for every m,

$$\limsup_{n\to\infty} \frac{\nu_n\{i \in D_n :\ \sigma_i(A_n - B_{n,m}) > \omega(m)\}}{n} \leq c(m).$$

Considering that $c(m)$ and $\omega(m)$ tend to 0 as $m \to \infty$, for each fixed $\varepsilon > 0$ and $\delta > 0$, we can find $M = M(\varepsilon, \delta)$ such that, for $m \geq M$,

$$\omega(m) < \varepsilon \quad \text{and} \quad c(m) < \delta.$$

Hence, for $m \geq M$,

$$\limsup_{n\to\infty} \frac{\nu_n\{i \in D_n :\ \sigma_i(A_n - B_{n,m}) > \varepsilon\}}{n} \leq \limsup_{n\to\infty} \frac{\nu_n\{i \in D_n :\ \sigma_i(A_n - B_{n,m}) > \omega(m)\}}{n} \leq c(m) < \delta,$$

and the thesis is proved.

($2 \implies 1$) Suppose now that, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} f_m(\varepsilon) = 0,$$

where

$$f_m(\varepsilon) = \limsup_{n\to\infty} \frac{\nu_n\{i \in D_n :\ \sigma_i(A_n - B_{n,m}) > \varepsilon\}}{n}.$$

Then, we can find a sequence of positive numbers ε_m such that

$$\lim_{m\to\infty} \varepsilon_m = \lim_{m\to\infty} f_m(\varepsilon_m) = 0;$$

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see, e.g., [13, Exercise 3.3]. By definition of limsup, for every m there exists n_m such that, for $n \geq n_m$,

$$\frac{\nu_n\{i \in D_n : \ \sigma_i(A_n - B_{n,m}) > \varepsilon_m\}}{n} \leq f_m(\varepsilon_m) + \frac{1}{m}.$$

Let

$$A_n - B_{n,m} = U_{n,m}\Sigma_{n,m}V_{n,m}^*$$

be a singular value decomposition of $A_n - B_{n,m}$. Let $\widehat{\Sigma}_{n,m}$ (resp., $\widetilde{\Sigma}_{n,m}$) be the matrix obtained from $\Sigma_{n,m}$ by setting to zeros all the singular values of $A_n - B_{n,m}$ that are less than or equal to (resp., greater than) ε_m. If we define

$$R_{n,m} = U_{n,m}\widehat{\Sigma}_{n,m}V_{n,m}^*, \quad N_{n,m} = U_{n,m}\widetilde{\Sigma}_{n,m}V_{n,m}^*,$$

then

$$A_n = B_{n,m} + N_{n,m} + R_{n,m}, \quad \mathrm{rank}(R_{n,m}) \leq \left(f_m(\varepsilon_m) + \frac{1}{m}\right)n, \quad \|N_{n,m}\| \leq \varepsilon_m,$$

and so $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$. ∎

Remark 2.5 (Connections between convergence in measure and a.c.s. convergence). Let $D \subset \mathbb{R}^k$ be a measurable set with $0 < \mu_k(D) < \infty$, and let $f, f_m \in \mathfrak{M}_D$. It is well-known that the following two conditions are equivalent:

1′. $f_m \to f$ in measure;
2′. $\displaystyle\lim_{m\to\infty} \frac{\mu_k\{|f - f_m| > \varepsilon\}}{\mu_k(D)} = 0$, for all $\varepsilon > 0$.

This is actually the definition of convergence in measure, up to the inessential normalization constant $\mu_k(D)$. Considering that condition 2 of Theorem 2.4 can be rewritten as

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{\nu_n\{\boldsymbol{\sigma}(A_n - B_{n,m}) > \varepsilon\}}{\nu_n(D_n)} = 0, \quad \text{for all } \varepsilon > 0,$$

it is clear that the equivalence ($1 \iff 2$) in the world of matrix-sequences $\mathscr{E}$ is the analog of the equivalence ($1' \iff 2'$) in the world of measurable functions $\mathfrak{M}_D$. Note also that the function

$$p^{(\varepsilon)}(A) = \frac{\nu_n\{\boldsymbol{\sigma}(A) > \varepsilon\}}{\nu_n(D_n)}, \quad A \in \mathbb{C}^{n\times n},$$

is obtained from the function

$$p^{(\varepsilon)}_{\text{measure}}(f) = \frac{\mu_k\{|f| > \varepsilon\}}{\mu_k(D)}, \quad f \in \mathfrak{M}_D,$$

by the same rule which allows one to obtain (1.9) from (1.7) and (1.14) from (1.10).

The next theorem concludes the proof of Theorem 1.5.

Theorem 2.6. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave bounded continuous function such that $\varphi(0) = 0$ and $\varphi > 0$ on $(0,\infty)$. Then, $d^{\varphi}_{\text{a.c.s.}}$ induces on $\mathscr{E}$ the a.c.s. topology $\tau_{\text{a.c.s.}}$.*

Proof. Considering that $\tau_{\text{a.c.s.}}$ is the pseudometric topology induced by the distance $d_{\text{a.c.s.}}$ in (1.5), in order to show that $d^{\varphi}_{\text{a.c.s.}}$ induces $\tau_{\text{a.c.s.}}$ it is enough to show that $d_{\text{a.c.s.}}$ and $d^{\varphi}_{\text{a.c.s.}}$ have the same convergent sequences, i.e., that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ if and only if $d^{\varphi}_{\text{a.c.s.}}(\{A_n\}_n, \{B_{n,m}\}_n) \to 0$ as $m \to \infty$. In view of Theorem 2.4, this is the same as proving the equivalence between the following conditions:

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i. $\lim_{m\to\infty}\limsup_{n\to\infty}\dfrac{\nu_n\{\boldsymbol{\sigma}(A_n-B_{n,m})>\varepsilon\}}{n}=0$, for all $\varepsilon>0$;

ii. $\lim_{m\to\infty}\limsup_{n\to\infty}\dfrac{1}{n}\displaystyle\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n=0$.

Note that we are using the integral expression of $d^{\varphi}_{\text{a.c.s.}}$ because this will highlight further connections between the a.c.s. convergence and the convergence in measure. Indeed, a careful comparison between the proof we are going to see and a possible proof, appearing, e.g., in [13, p. 257], that the distance $d^{\varphi}_{\text{measure}}$ in (1.11) induces on $\mathfrak{M}_D$ the topology of convergence in measure reveals that the two proofs are essentially the same(!)

(i $\Longrightarrow$ ii) Considering that φ is non-decreasing and bounded, for every m,n and $\varepsilon>0$ we have

$$\begin{aligned}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n &= \frac{1}{n}\int_{\{\boldsymbol{\sigma}(A_n-B_{n,m})>\varepsilon\}}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n\\ &\quad+\frac{1}{n}\int_{\{\boldsymbol{\sigma}(A_n-B_{n,m})\le\varepsilon\}}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n\\ &\le \|\varphi\|_\infty\frac{\nu_n\{\boldsymbol{\sigma}(A_n-B_{n,m})>\varepsilon\}}{n}+\varphi(\varepsilon).\end{aligned}$$

Thus, if condition i is satisfied, then for every $\varepsilon>0$ we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n\le\varphi(\varepsilon),$$

i.e.,

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n=0,$$

because φ is continuous and $\varphi(0)=0$. We conclude that condition ii is satisfied.

(ii $\Longrightarrow$ i) Considering that φ is non-decreasing, for every m,n and $\varepsilon>0$ we have

$$\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n\ge\frac{1}{n}\int_{\{\boldsymbol{\sigma}(A_n-B_{n,m})>\varepsilon\}}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n\ge\varphi(\varepsilon)\,\frac{\nu_n\{\boldsymbol{\sigma}(A_n-B_{n,m})>\varepsilon\}}{n}.$$

Thus, if condition ii is satisfied, then for every $\varepsilon>0$ we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\frac{\nu_n\{\boldsymbol{\sigma}(A_n-B_{n,m})>\varepsilon\}}{n}\le\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{\varphi(\varepsilon)}\,\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n-B_{n,m}))\mathrm{d}\nu_n=0,$$

and condition i is satisfied; note that we could divide by $\varphi(\varepsilon)$ because $\varphi>0$ on $(0,\infty)$. ∎

Remark 2.7 (The case of zero-distributed sequences). A zero-distributed sequence is a matrix-sequence $\{Z_n\}_n$ such that $\{Z_n\}_n\sim_\sigma 0$, where 0 here denotes the identically zero function (defined on some measurable subset D of some $\mathbb{R}^k$ with $0<\mu_k(D)<\infty$). In other words, $\{Z_n\}_n$ is zero-distributed if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}F(\sigma_i(Z_n))=F(0)$$

for all $F\in C_c(\mathbb{R})$. We know from [7, Theorem 4] that the "standard" pseudometric $d_{\text{a.c.s.}}$ inducing $\tau_{\text{a.c.s.}}$ satisfies

$$d_{\text{a.c.s.}}(\{A_n\}_n,\{B_n\}_n)=0\quad\iff\quad\{A_n-B_n\}_n\sim_\sigma 0.$$

Since, by Theorem 1.5, $d^{\varphi}_{\text{a.c.s.}}$ is another pseudometric inducing $\tau_{\text{a.c.s.}}$, we have

$$d^{\varphi}_{\text{a.c.s.}}(\{A_n\}_n,\{B_n\}_n)=0\quad\iff\quad d_{\text{a.c.s.}}(\{A_n\}_n,\{B_n\}_n)=0\quad\iff\quad\{A_n-B_n\}_n\sim_\sigma 0.$$

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Remark 2.8 (New tools for testing the a.c.s. convergence). A problem that is often encountered in practice, especially in the applications of the theory of GLT sequences, is the problem of establishing whether a given sequence of matrix-sequences $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for another matrix-sequence $\{A_n\}_n$; see, for example, [13, Chapter 10]. Such a problem arises whenever it is necessary to derive the asymptotic singular value or eigenvalue distribution of a "difficult" matrix-sequence $\{A_n\}_n$ from the asymptotic singular value or eigenvalue distribution of "simpler" matrix-sequences $\{B_{n,m}\}_n$ by means of the fundamental Theorems 1.3 and 1.4. Theorem 1.5 offers new tools for solving this problem. Indeed, it suffices to choose a simple function φ with the properties expressed in Theorem 1.5 and to check if either $d^{\varphi}_{\text{a.c.s.}}(\{B_{n,m}\}_n, \{A_n\}_n) \to 0$ (in which case $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}$) or $d^{\varphi}_{\text{a.c.s.}}(\{B_{n,m}\}_n, \{A_n\}_n) \not\to 0$ (in which case $\{\{B_{n,m}\}_n\}_m$ is not an a.c.s. for $\{A_n\}$). For instance, we may choose one of the two very simple functions

$$\varphi_1(x) = \min(x, 1),$$

$$\varphi_2(x) = \frac{x}{x+1},$$

and test the a.c.s. convergence by means of one of the two corresponding distances

$$d^{\varphi_1}_{\text{a.c.s.}}(\{A_n\}_n, \{B_{n,m}\}_n) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \min(\sigma_i(A_n - B_{n,m}), 1),$$

$$d^{\varphi_2}_{\text{a.c.s.}}(\{A_n\}_n, \{B_{n,m}\}_n) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_i(A_n - B_{n,m})}{\sigma_i(A_n - B_{n,m}) + 1}.$$

3. Proof of Theorem 1.6. Since $\{A_n\}_n \sim_\sigma f$, by definition we have

$$\lim_{n\to\infty} \frac{1}{n} \int_{D_n} F(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n = \frac{1}{\mu_k(D)} \int_D F(|f|)\mathrm{d}\mu_k \tag{3.17}$$

for every $F \in C_c(\mathbb{R})$. The sequence $\{A_n\}_n$, just like any other matrix-sequence possessing a singular value distribution, is sparsely unbounded (s.u.), which means that for every $\varepsilon > 0$ there exist $M, N > 0$ such that, for $n \geq N$,

$$\frac{\nu_n\{\boldsymbol{\sigma}(A_n) > M\}}{n} \leq \varepsilon;$$

see [13, Section 5.4]. On the other hand, the function f is measurable, so for any $\varepsilon > 0$ there exists $M > 0$ such that

$$\frac{\mu_k\{|f| > M\}}{\mu_k(D)} \leq \varepsilon.$$

Fix a real-valued function $F \in C_c(\mathbb{R})$ such that $\chi_{[0,M]}\varphi \leq F \leq \varphi$ over $[0, \infty)$, with $\chi_{[0,M]}$ being the characteristic (indicator) function of the set $[0, M]$. Note that such a function F exists because φ is continuous. Considering that φ is also bounded, we obtain four inequalities. On the one hand,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int_{D_n} F(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n &\geq \limsup_{n\to\infty} \frac{1}{n} \int_{\{\boldsymbol{\sigma}(A_n)\leq M\}} \varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n \\
&= \limsup_{n\to\infty} \frac{1}{n} \left[\int_{D_n} \varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n - \int_{\{\boldsymbol{\sigma}(A_n)>M\}} \varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n \right] \\
&\geq \limsup_{n\to\infty} \frac{1}{n} \int_{D_n} \varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n - \varepsilon\|\varphi\|_\infty,
\end{aligned}$$

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$$\lim_{n\to\infty}\frac{1}{n}\int_{D_n}F(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n \le \liminf_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n.$$

On the other hand,

$$\begin{aligned}\frac{1}{\mu_k(D)}\int_D F(|f|)\mathrm{d}\mu_k &\ge \frac{1}{\mu_k(D)}\int_{\{|f|\le M\}}\varphi(|f|)\mathrm{d}\mu_k\\ &= \frac{1}{\mu_k(D)}\left[\int_D \varphi(|f|)\mathrm{d}\mu_k - \int_{\{|f|>M\}}\varphi(|f|)\mathrm{d}\mu_k\right]\\ &\ge \frac{1}{\mu_k(D)}\int_D \varphi(|f|)\mathrm{d}\mu_k - \varepsilon\|\varphi\|_\infty,\end{aligned}$$

$$\frac{1}{\mu_k(D)}\int_D F(|f|)\mathrm{d}\mu_k \le \frac{1}{\mu_k(D)}\int_D \varphi(|f|)\mathrm{d}\mu_k.$$

Recalling (3.17), we arrive at the following inequalities:

$$\liminf_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n \ge \frac{1}{\mu_k(D)}\int_D \varphi(|f|)\mathrm{d}\mu_k - \varepsilon\|\varphi\|_\infty,$$

$$\limsup_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n - \varepsilon\|\varphi\|_\infty \le \frac{1}{\mu_k(D)}\int_D \varphi(|f|)\mathrm{d}\mu_k.$$

In the limit where $\varepsilon \to 0$, we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n \ge \frac{1}{\mu_k(D)}\int_D \varphi(|f|)\mathrm{d}\mu_k \ge \limsup_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n,$$

which implies

$$p^{\varphi}_{\text{a.c.s.}}(\{A_n\}_n) = \lim_{n\to\infty}\frac{1}{n}\int_{D_n}\varphi(\boldsymbol{\sigma}(A_n))\mathrm{d}\nu_n = \frac{1}{\mu_k(D)}\int_D \varphi(|f|)\mathrm{d}\mu_k = p^{\varphi}_{\text{measure}}(f).$$

This completes the proof of Theorem 1.6.

4. Proof of Theorem 1.7: Isometry between GLT sequences and measurable functions. We first recall from [2, 13] everything that we need to know about GLT sequences for the purposes of this section. We refer the reader to [4] for a more extended introduction to the theory of GLT sequences and to [2, 10, 11, 13, 14, 20, 21, 23] for advanced studies.

A GLT sequence $\{A_n\}_n$ is a special matrix-sequence equipped with a function $\kappa \in \mathfrak{M}_{[0,1]\times[-\pi,\pi]}$, the so-called symbol. We use the notation $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ to indicate that $\{A_n\}_n$ is a GLT sequence with symbol κ. The symbol of a GLT sequence is unique in the sense that if $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $\{A_n\}_n \sim_{\mathrm{GLT}} \xi$ then $\kappa = \xi$ a.e. in $[0,1]\times[-\pi,\pi]$. Conversely, if $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $\kappa = \xi$ a.e. in $[0,1]\times[-\pi,\pi]$ then $\{A_n\}_n \sim_{\mathrm{GLT}} \xi$. For any measurable function $\kappa \in \mathfrak{M}_{[0,1]\times[-\pi,\pi]}$ there exists a matrix-sequence $\{A_n\}_n$ such that $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ then $\{A_n\}_n \sim_\sigma \kappa$. Any linear combination of GLT sequences is again a GLT sequence with symbol given by the same linear combination of the symbols. Thus, if $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $\{B_n\}_n \sim_{\mathrm{GLT}} \xi$, then $\{\alpha A_n + \beta B_n\}_n \sim_{\mathrm{GLT}} \alpha\kappa + \beta\xi$ for all $\alpha, \beta \in \mathbb{C}$. This shows in particular that the set of GLT sequences $\mathscr{G}$ is a subspace of $\mathscr{E}$ with respect to the natural (componentwise) operations of addition and scalar multiplication of matrix-sequences.

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Proof of Theorem 1.7. Suppose that $\{A_n\}_n \sim_{\rm GLT} \kappa$ and $\{B_n\}_n \sim_{\rm GLT} \xi$. Then $\{A_n - B_n\}_n \sim_{\rm GLT} \kappa - \xi$ and, consequently, $\{A_n - B_n\}_n \sim_\sigma \kappa - \xi$. By applying Theorem 1.6, we obtain

$$d^{\varphi}_{\rm a.c.s.}(\{A_n\}_n, \{B_n\}_n) = p^{\varphi}_{\rm a.c.s.}(\{A_n - B_n\}_n) = p^{\varphi}_{\rm measure}(\kappa - \xi) = d^{\varphi}_{\rm measure}(\kappa, \xi),$$

which concludes the proof. $\square$

Theorem 1.7 yields a natural isometry between the spaces $(\mathscr{G}, d^{\varphi}_{\rm a.c.s.})$ and $(\mathfrak{M}_{[0,1]\times[-\pi,\pi]}, d^{\varphi}_{\rm measure})$, which is completely analogous to the isometry identified in [2] between $(\mathscr{G}, d_{\rm a.c.s.})$ and $(\mathfrak{M}_{[0,1]\times[-\pi,\pi]}, d_{\rm measure})$. To be more precise, suppose we identify two matrix-sequences $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$ if and only if $\{A_n - B_n\}_n \sim_\sigma 0$, i.e., by Remark 2.7, if and only if $d^{\varphi}_{\rm a.c.s.}(\{A_n\}_n, \{B_n\}_n) = 0$. Then, we are introducing in $\mathscr{G}$ an equivalence relation, with respect to which $(\mathscr{G}, d^{\varphi}_{\rm a.c.s.})$ becomes a metric space. Similarly, suppose we identify two functions $\kappa, \xi \in \mathfrak{M}_{[0,1]\times[-\pi,\pi]}$ if and only if $\kappa - \xi = 0$ a.e., i.e., if and only if $d^{\varphi}_{\rm measure}(\kappa, \xi) = 0$ a.e. Then, we are introducing in $\mathfrak{M}_{[0,1]\times[-\pi,\pi]}$ an equivalence relation, with respect to which $(\mathfrak{M}_{[0,1]\times[-\pi,\pi]}, d^{\varphi}_{\rm measure})$ becomes a metric space. Consider the map which associates to (the equivalence class of) a GLT sequence $\{A_n\}_n$ (the equivalence class of) its symbol κ. This map is well-defined, as long as we keep in mind the equivalence relations introduced in $\mathscr{G}$ and $\mathfrak{M}_{[0,1]\times[-\pi,\pi]}$, and it is a sujective isometry by Theorem 1.7 and the fact that any $\kappa \in \mathfrak{M}_{[0,1]\times[-\pi,\pi]}$ is the symbol of some GLT sequence $\{A_n\}_n$.

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