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LIGHTS OUT! ON CARTESIAN PRODUCTS*

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Abstract. The game *LIGHTS OUT!* is played on a 5×5 square grid of buttons; each button may be on or off. Pressing a button changes the on/off state of the light of the button pressed and of all its vertical and horizontal neighbors. Given an initial configuration of buttons that are on, the object of the game is to turn all the lights out. The game can be generalized to arbitrary graphs. In this paper, Cartesian Product graphs (that is, graphs of the form $G\Box H$, where G and H are arbitrary finite, simple graphs) are investigated. In particular, conditions for which $G\Box H$ is universally solvable (every initial configuration of lights can be turned out by a finite sequence of button presses), using both closed neighborhood switching and open neighborhood switching, are provided.

Key words. Matrix, Determinant, Graph, Lights Out, Fibonacci polynomials.

AMS subject classifications. 05C50, 15A15, 15A03, 15B33.

1. Introduction. The popular electronic game *LIGHTS OUT!* was released by Tiger Electronics in 1995. The game is played on a 5×5 square grid of buttons, where each button is either on or off (lit or unlit). When you start the game, it generates a random puzzle or configuration of lit and unlit buttons. The object of the game is simple - turn the lights out. When you press a button, not only does it change the state of that light (from on to off or vice versa), but it also changes the state of the adjacent lights (those that are directly above, below, or next to the pressed button).

We represent the state of each light by an element of $GF(2)$, the field of integers modulo 2; 1 means on, 0 means off. Throughout this paper, *all calculations are done modulo* 2.

The traditional game is played on a square grid but can be generalized to arbitrary graphs. Recall that the *Cartesian Product* of G with H, denoted $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $v \sim v'$ in H, or (2) $v = v'$ and $u \sim u'$ in G. In this paper, we consider the game applied to graphs of the form $G\Box H$, where G and H are arbitrary finite, simple graphs. In particular, we address the question of whether or not $G\Box H$ is *universally solvable*, i. e. whether or not every initial configuration on $G\Box H$ is *solvable* (all of the lights can be turned off by a finite sequence of button presses).

Let $G = (V, E)$ be a simple graph of order n. For each $v \in V$, the *open neighborhood* $N(v)$ of v is the set of vertices adjacent to v, $N(v) = \{u \in V : (u, v) \in E\}$. The *closed neighborhood* $N[v]$ of v is the open neighborhood along with v itself, $N[v] = N(v) \cup \{v\}$. In the traditional game, pressing a vertex v changes the state of v as well as that of the vertices adjacent to v. This is called *closed neighborhood switching.* In a variation of the game, pressing a vertex v does not change the state of v, only that of the vertices adjacent to v. This is called *open neighborhood switching.*

We say G is *closed universally solvable* if every initial configuration is solvable using closed neighbor-

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hood switching. We say G is *open universally solvable* if every initial configuration is solvable using open neighborhood switching. Throughout the paper, we denote by $A(G)$ the adjacency matrix of G. Then $A(G)$ is the open neighborhood matrix of G, and $A(G)+I_n$, where I_n is the $n \times n$ identity matrix, is the closed neighborhood matrix of G.

THEOREM 1.1. [6, 9, 10] *A graph is universally solvable if and only if its adjacency matrix is invertible over $GF(2)$.*

Consequently, G is closed universally solvable if and only if $A(G)+I_n$ is invertible, and G is open universally solvable if and only if $A(G)$ is invertible. Since a matrix A is invertible if and only if $\det(A) \neq 0$, we approach the problem of determining whether or not a graph G is open (closed) universally solvable by studying the determinant of the adjacency matrix $A(G)$ (the closed neighborhood matrix $A(G)+I_n$).

For an extensive survey of the work that has been done on the game, see [4]. Sutner [9, 10] was the first to study the game, and he did so in the context of cellular automata. He showed that for any graph, it is possible to turn all the lights off if initially all the lights are on. Amin et al. [1, 2, 3] concentrated on universally solvable graphs. Goldwasser et al. [5, 6] used Fibonacci polynomials to determine for which pairs (m,n) the grid graph $G_{m,n}$ is closed universally solvable and for which pairs it is open universally solvable. We note that $G_{m,n}$ is the Cartesian Product $P_m \Box P_n$, where P_i is a path with i vertices. In this paper, we generalize the game to graphs of the form $G \Box H$, where G and H are arbitrary finite, simple graphs.

2. Cartesian Products. Let G and H be finite, simple graphs, with $|G| = m$, $|H| = p$, and let $n = mp$. Let $B = [b_{ij}] = A(G)$, the adjacency matrix of G, and let $C = A(H)$, the adjacency matrix of H. We make the following observation.

OBSERVATION 2.1. *The adjacency matrix of $G \Box H$ is*

$$A(G\Box H) = \begin{bmatrix} C & b_{12}I_p & \dots & b_{1m}I_p \\ b_{21}I_p & C & & \vdots \\ \vdots & & \ddots & b_{(m-1)m}I_p \\ b_{m1}I_p & \dots & b_{m(m-1)}I_p & C \end{bmatrix},$$

where $b_{ij}I_p$ is either the $p \times p$ identity matrix or the $p \times p$ matrix of zeros as $b_{ij} \in GF(2)$ for all $1 \le i, j \le m$. Note that $b_{ii} = 0$ for all $1 \le i \le m$ as B is the adjacency matrix of G. The rows and columns could be permuted to get an adjacency matrix with p $m \times m$ blocks with B on the main diagonal.

The following result on the determinant of a block matrix will play a key role in our computations.

THEOREM 2.2. [8] *Let R be a commutative subring of $M_n(F)$, where F is a field (or a commutative ring), and let $\mathbf{M} \in M_m(R)$. Then*

$$det_F\mathbf{M} = det_F(det_R\mathbf{M}).$$

For example, if $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are $n \times n$ matrices over F that mutually commute, then Theorem 2.2 says

$$det_F\mathbf{M} = det_F(\mathbf{AD} - \mathbf{BC}).$$

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By Observation 2.1, $A(G\square H)$ is a block matrix consisting of the blocks $C = A(H)$, the $p\times p$ identity matrix I_p, and the $p\times p$ matrix of zeros O_p. These three blocks are pairwise commutative, so the next result follows from Theorem 2.2. We let $p_A(x)$ denote the characteristic polynomial of the $n\times n$ matrix A.

THEOREM 2.3. *Let G and H be finite, simple graphs, with $|G| = m$, $|H| = p$, and $n = mp$. Let $B = A(G)$ and $C = A(H)$. Then*

$$\det[A(G\square H)] = \det[p_B(C)]$$

and

$$det[A(G\square H) + I_n] = \det[p_B(C + I_p)].$$

We will also need the following result.

THEOREM 2.4. *Let $f(x)$ be a polynomial and let A be an $n\times n$ matrix. Then $f(A)$ is singular if and only if* $\gcd(f(x), p_A(x)) \neq 1$.

Proof. Suppose $q(x) = \gcd(f(x), p_A(x)) = 1$. We can express $q(x)$ as a linear combination of two polynomials $p_1(x)$ and $p_2(x)$. So $1 = q(x) = p_1(x)f(x) + p_2(x)p_A(x)$. Then $I_n = q(A) = p_1(A)f(A)$ as $p_A(A) = 0$. Hence, $f(A)$ is nonsingular.

Conversely, suppose $\gcd(f(x), p_A(x)) \neq 1$. Then $r(x) = \gcd(f(x), m_A(x)) \neq 1$, where $m_A(x)$ is the minimal polynomial of A. Since $r(x) \neq 1$ divides $m_A(x)$, $r(A)$ is singular. To see this, observe that $m_A(x) = r(x)s(x)$ for some $s(x)$. So $m_A(A) = r(A)s(A) = 0$, with $s(A) \neq 0$ as $m_A(x)$ is the minimal polynomial of A. Since $r(A)s(A)\vec{x} = \vec{0}$ for every $\vec{x}$, there exists a vector $\vec{x}$ such that $\vec{y} = s(A)\vec{x} \neq \vec{0}$. Hence, $r(A)\vec{y} = \vec{0}$ and $r(A)$ is singular. Since $f(x) = r(x)t(x)$ for some $t(x)$, it follows that $f(A)$ is singular. ∎

We now provide conditions for which $G\square H$ is closed universally solvable and open universally solvable, and illustrate with an example.

THEOREM 2.5. *Let G and H be finite, simple graphs, with $|G| = m$, $|H| = p$, and $n = mp$. Let $B = A(G)$ and $C = A(H)$. Then $G\square H$ is closed universally solvable if and only if* $\gcd(p_B(x + 1), p_C(x)) = 1$.

Proof. Suppose $q(x) = \gcd(p_B(x + 1), p_C(x)) \neq 1$. Since $q(x) \neq 1$ divides $p_C(x)$, $q(C)$ is singular. Suppose $p_B(x+1) = s(x)q(x)$ for some $s(x)$. Then $p_B(C+I_p) = s(C)q(C)$ is singular since $q(C)$ is singular. By Theorem 2.3, $\det[A(G\square H) + I_n] = \det[p_B(C + I_p)] = 0$. Hence, $A(G\square H) + I_n$ is singular, and $G\square H$ is not closed universally solvable.

Conversely, suppose $A(G\square H)+I_n$ is singular. Then $p_B(C+I_p)$ is singular. By Theorem 2.4, $\gcd(p_B(x+1), p_C(x)) \neq 1$. ∎

THEOREM 2.6. *Let G and H be finite, simple graphs, with $|G| = m$, $|H| = p$, and $n = mp$. Let $B = A(G)$ and $C = A(H)$. Then $G\square H$ is open universally solvable if and only if* $\gcd(p_B(x), p_C(x)) = 1$.

Proof. Suppose $r(x) = \gcd(p_B(x), p_C(x)) \neq 1$. Since $r(x) \neq 1$ divides $p_C(x)$, $r(C)$ is singular. Suppose $p_B(x) = t(x)q(x)$ for some $t(x)$. Then $p_B(C) = t(C)q(C)$ is singular since $q(C)$ is singular. By Theorem 2.3, $\det[A(G\square H)] = \det[p_B(C)]$. Hence, $A(G\square H)$ is singular, and $G\square H$ is not open universally solvable.

Conversely, suppose $A(G\square H)$ is singular. Then $p_B(C)$ is singular. So by Theorem 2.4, $\gcd(p_B(x), p_C(x)) \neq 1$. ∎

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EXAMPLE 2.7. Let $G = K_4$, the complete graph on four vertices, and let $H = C_4$, the cycle graph of order 4. Then

$$B = A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$C = A(C_4) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

and

$$A(K_4 \square C_4) = \begin{bmatrix} C & I_4 & I_4 & I_4 \\ I_4 & C & I_4 & I_4 \\ I_4 & I_4 & C & I_4 \\ I_4 & I_4 & I_4 & C \end{bmatrix}.$$

Observe that $p_B(x) = (x+1)^4$ and $p_C(x) = x^4$. By Theorem 2.3, $\det[A(K_4 \square C_4)] = \det[p_B(C)] = \det[(C + I_4)^4] = [\det(C + I_4)]^4 = 1$, and $\det[A(K_4 \square C_4) + I_4] = \det[p_B(C + I_4)] = \det[C^4] = [\det(C)]^4 = 0$. Hence, $K_4 \square C_4$ is open universally solvable but not closed universally solvable.

Alternatively, $\gcd(p_B(x), p_C(x)) = \gcd((x+1)^4, x^4) = 1$, so $K_4 \square C_4$ is open universally solvable by Theorem 2.6. In addition, $\gcd(p_B(x+1), p_C(x)) = \gcd(x^4, x^4) = x^4$, so $K_4 \square C_4$ is not closed universally solvable by Theorem 2.5.

If $f(x)$ is any polynomial over $GF(2)$, we define the *conjugate* of $f(x)$ by $f(x+1)$. Interestingly, if $p_B(x) = p_B(x+1)$ (so $p_B(x)$ is *self-conjugate*) for some graph G with $B = A(G)$, then $G \square H$ will be both closed universally solvable and open universally solvable if $\gcd(p_B(x), p_C(x)) = 1$, where $C = A(H)$. Goldwasser et al. [7] showed that the only self-conjugate Fibonacci polynomials are $f_0(x) = 0$, $f_1(x) = 1$, and $f_5(x) = x^4 + x^2 + 1 = (x^2 + x + 1)^2$. Let $B_n = A(P_n)$, the adjacency matrix of the path on n vertices. Goldwasser et al. [6] observed that

$$p_{B_n}(x) = x p_{B_{n-1}}(x) + p_{B_{n-2}}(x),$$

where $p_{B_1}(x) = x$ and $p_{B_0}(x) = 1$. The sequence $\{p_{B_i}(x)\}$ satisfies the Fibonacci recurrence with initial conditions shifted by one, so $p_{B_i} = f_{i+1}$ $(i = 0, 1, 2, \dots)$. This relationship allows us to find an explicit formula for the characteristic polynomial of $A(P_n)$. In particular, if $G = P_4$, then $p_{B_4}(x) = f_5(x) = p_{B_4}(x+1)$. So if $\gcd(p_{B_4}(x), p_C(x)) = 1$ for some graph H with $C = A(H)$, then $P_4 \square H$ will be both closed universally solvable and open universally solvable. This is the case for $P_4 \square K_n$ (the characteristic polynomial for $A(K_n)$ is given in the proof of Proposition 3.1).

As an immediate consequence of Theorem 2.6, $G \square G$ is not open universally solvable for any graph G.

COROLLARY 2.8. *Let G be a finite, simple graph, and let $B = A(G)$. Then $G \square G$ is not open universally solvable.*

Proof. Since $\gcd(p_B(x), p_B(x)) = p_B(x) \neq 1$, the result follows immediately from Theorem 2.6. ∎

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While $G\square G$ is not open universally solvable for any graph G, $G\square G$ may or may not be closed universally solvable. For example, $P_3\square P_3$ is closed universally solvable and $K_3\square K_3$ is not closed universally solvable. In fact, $G\square G$ is closed universally solvable if and only if the characteristic polynomial of G is not divisible by both $f(x)$ and the conjugate of $f(x)$ for any polynomial $f(x)$ of degree at least one, and the result follows immediately from Theorem 2.5.

Corollary 2.9. *Let G be a finite, simple graph, and let $B = A(G)$. Then $G\square G$ is closed universally solvable if and only if $p_B(x)$ is not divisible by both $f(x)$ and $g(x) = f(x+1)$ for any polynomial $f(x)$ of degree at least one.*

3. Cartesian Products of common graph families. We now investigate Cartesian Products involving some common graph families and decide whether or not they are closed universally solvable and open universally solvable. The results are summarized in Tables 1, 2, and 3.

Proposition 3.1. *If n and m are even, then $K_n\square K_m$ is closed universally solvable but not open universally solvable. If m is odd and n is either even or odd, then $K_n\square K_m$ is not closed universally solvable and not open universally solvable.*

Proof. Let $G = K_n$, $H = K_m$, $B = A(K_n)$, and $C = A(K_m)$. If n is even, $p_B(x) = \big(p_{A(K_2)}(x)\big)^{n/2} = \big((x+1)^2\big)^{n/2} = (x+1)^n$. Moreover, $p_B(x+1) = \big((x+1)+1\big)^n = x^n$. Hence, if both n and m are even, then $x+1$ divides both $p_B(x)$ and $p_C(x)$. In addition, $\gcd(p_B(x+1), p_C(x)) = \gcd(x^n, (x+1)^m) = 1$. Therefore, $K_n\square K_m$ is closed universally solvable but not open universally solvable if n and m are even by Theorems 2.5 and 2.6, respectively (see the entries in red text in Table 1).

If m is odd, $p_C(x) = x\big(p_{A(K_2)}(x)\big)^{\frac{m-1}{2}} = x\big((x+1)^2\big)^{\frac{m-1}{2}} = x(x+1)^{m-1}$. Moreover, $p_C(x+1) = (x+1)\big((x+1)+1\big)^{m-1} = (x+1)(x^{m-1})$. Hence, if n is even and m is odd, then $x+1$ divides both $p_B(x)$ and $p_C(x)$. In addition, x divides both $p_B(x+1)$ and $p_C(x)$. If n and m are both odd, then $x(x+1)$ divides both $p_B(x)$ and $p_C(x)$. In addition, x divides both $p_B(x+1)$ and $p_C(x)$. Therefore, $K_n\square K_m$ is not closed universally solvable and not open universally solvable if m is odd and n is either even or odd by Theorems 2.5 and 2.6, respectively. ∎

Let $B_n = A(P_n)$, the adjacency matrix of the path on n vertices, and let $A_n = A(C_n)$, the adjacency matrix of the cycle of order n. Then by expansion on the first row we get

$$\begin{aligned} p_{A_n}(x) &= \det \begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & x & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & x & 1 \\ 1 & 0 & \dots & 0 & 0 & 1 & x \end{bmatrix} \\ &= xp_{B_{n-1}}(x) \\ &= xf_n(x), \end{aligned}$$

where $f_n(x)$ is the nth Fibonacci polynomial. We use a result on the divisibility of polynomials by Goldwasser et al. [6] in the proof of the next proposition.

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Proposition 3.2. *If n and m are any positive integers, then $C_n\square K_m$ is not closed universally solvable. If n is any positive integer and m is odd, then $C_n\square K_m$ is not open universally solvable. If m is even, then $C_n\square K_m$ is open universally solvable if and only if n is not a multiple of 3.*

Proof. Let $A_n = A(C_n)$ and let $C = A(K_m)$. As seen in the proof of Proposition 3.1, $x+1$ is a factor of $p_C(x)$ for both m even and m odd. Since $p_{A_n}(x) = xf_n(x)$, $p_{A_n}(x+1) = (x+1)f_n(x+1)$ and $\gcd(p_{A_n}(x+1), p_C(x)) \neq 1$. Hence, $C_n\square K_m$ is not closed universally solvable for any positive integers n and m.

If m is odd, then $p_C(x) = x(x+1)^{m-1}$. So $\gcd(p_{A_n}(x), p_C(x)) \neq 1$, and $C_n\square K_m$ is not open universally solvable.

If m is even, then $p_C(x) = (x+1)^m$. Since $t = 3$ is the minimal integer for which $(x+1)|f_t(x)$, $(x+1)|f_r(x)$ if and only if $t|r$. Hence, $C_n\square K_m$ is open universally solvable if and only if n is not a multiple of 3 (see the entries in blue text in Table 1). ∎

Proposition 3.3. *If n is odd, then $K_n\square P_{m-1}$ is closed universally solvable and open universally solvable if and only if m is not a multiple of 2 or 3. If n is even, then $K_n\square P_{m-1}$ is closed universally solvable if and only if m is not a multiple of 2. If n is even, then $K_n\square P_{m-1}$ is open universally solvable if and only if m is not a multiple of 3.*

Proof. Let $B = A(K_n)$ and let $B_{m-1} = A(P_{m-1})$. If n is odd, then $p_B(x+1) = (x+1)x^{n-1}$ and $p_B(x) = x(x+1)^{n-1}$. Since $p_{B_{m-1}}(x) = f_m(x)$, x is a factor of $p_{B_{m-1}}(x)$ if and only if m is a multiple of 2 and $x+1$ is a factor of $p_{B_{m-1}}(x)$ if and only if m is a multiple of 3. Hence, $K_n\square P_{m-1}$ is closed universally solvable and open universally solvable if and only if m is not a multiple of 2 or 3 (see the entries in magenta text in Table 1).

If n is even, then $p_B(x+1) = x^n$. Since $p_{B_{m-1}}(x) = f_m(x)$, x is a factor of $p_{B_{m-1}}(x)$ if and only if m is a multiple of 2. Hence, $K_n\square P_{m-1}$ is closed universally solvable if and only if m is not a multiple of 2 (see the entries in brown text in Table 1).

If n is even, then $p_B(x) = (x+1)^n$. Since $p_{B_{m-1}}(x) = f_m(x)$, $x+1$ is a factor of $p_{B_{m-1}}(x)$ if and only if m is a multiple of 3. Hence, $K_n\square P_{m-1}$ is open universally solvable if and only if m is not a multiple of 3 (see the entries in violet text in Table 1). ∎

Proposition 3.4. *If n and m are any positive integers, then $C_n\square C_m$ is not open universally solvable. In addition, $C_n\square C_m$ is closed universally solvable if and only if n and m are not multiplies of 3 (provided n and m are not both 5).*

Proof. Let $A_k = A(C_k)$. Then $p_{A_k}(x) = xf_k(x)$. Hence, x is a factor of $p_{A_n}(x)$ and $p_{A_m}(x)$, and so $C_n\square C_m$ is not open universally solvable.

We have $p_{A_n}(x+1) = (x+1)f_n(x+1)$ and $p_{A_m}(x) = xf_m(x)$. Then $x+1$ is a factor of $f_m(x)$ if and only if m is a multiple of 3. Moreover, x is a factor of $f_n(x+1)$ if and only if n is a multiple of 3. Hence, $C_n\square C_m$ is closed universally solvable if and only if n and m are not multiplies of 3 (see the entries in blue text in Table 2). There is one exception to this rule. If $n = m = 5$, then $f_5(x) = f_5(x+1)$ as $f_5(x)$ is the only self-conjugate Fibonacci polynomial, and so $C_5\square C_5$ is not closed universally solvable (see the entry in orange text in Table 2). ∎

Proposition 3.5. *The graph $C_n\square P_{m-1}$ is open universally solvable if and only if m is not a multiple of 2 and m and n are not multiples of each other. In addition, $C_n\square P_{m-1}$ is closed universally solvable if and only if m is not a multiple of 3 and $P_{n-1}\square P_{m-1}$ is closed universally solvable.*

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Proof. Let $A_n = A(C_n)$ and let $B_{m-1} = A(P_{m-1})$. Then $p_{A_n}(x) = xf_n(x)$ and $p_{B_{m-1}}(x) = f_m(x)$. Observe that x is a factor of $f_m(x)$ if and only if m is even. In addition, $f_n(x)$ and $f_m(x)$ have common divisors if and only if m and n are multiples of each other. Hence, $C_n \square P_{m-1}$ is open universally solvable if and only if m is not a multiple of 2 and m and n are not multiples of each other (see the entries in red text in Table 2).

We have $p_{A_n}(x+1) = (x+1)f_n(x+1)$. Since $x+1$ is a factor of $f_m(x)$ if and only if m is a multiple of 3, $C_n \square P_{m-1}$ is not closed universally solvable if m is a multiple of 3 (see the entries in brown text in Table 2). Moreover, $C_n \square P_{m-1}$ is not closed universally solvable if $\gcd(f_n(x+1), f_m(x)) \neq 1$ (i.e., $P_{n-1} \square P_{m-1}$ is not closed universally solvable). ∎

Goldwasser et al. [5, 6] used Fibonacci polynomials to determine for which pairs (m, n) the grid graph $P_m \square P_n$ is closed universally solvable and for which pairs it is open universally solvable. For certain values of n, we can easily determine when $P_n \square P_n$ is closed universally solvable, as illustrated in the following example. We utilize several of the properties of Fibonacci polynomials over $GF(2)$.

Example 3.6. If $n = 2k$ for some positive integer k, then $p_{B_{n-1}}(x) = f_n(x) = xf_k^2(x)$. In addition, if $n = 3j$ for some positive integer j, then $p_{B_{n-1}}(x) = f_n(x) = f_3(x)f_j(xf_3(x)) = (x+1)^2 f_j(x(x+1)^2)$. Finally, if $n = 6l$ for some positive integer l, then $p_{B_{n-1}}(x) = f_n(x) = f_{2(3l)}(x) = xf_{3l}^2(x) = x\big((x+1)^2 f_l(x(x+1)^2)\big)^2$. In other words, $p_{B_{n-1}}(x) = f_n(x)$ is divisible by x and $x+1$ if n is a multiple of 6, and so $P_{n-1} \square P_{n-1}$ is not closed universally solvable by Corollary 2.9 (see the entries in green text in Table 3).

If $n = 2^k$ for some positive integer k, then $f_n(x) = p_{B_{n-1}}(x) = x^{n-1}$. Since $f_n(x)$ is a power of x and the conjugate of x is $x+1$, $f_n(x)$ is not divisible by both a polynomial and its conjugate. Hence, $P_{n-1} \square P_{n-1}$ is closed universally solvable by Corollary 2.9 (see the entries in orange text in Table 3).

Recall that $f_5(x)$ is self-conjugate; that is, $f_5(x) = f_5(x+1)$. Moreover, if $n = 5k$ for some positive integer k, then $p_{B_{n-1}}(x) = f_n(x) = f_5(x)f_k(xf_5(x))$. Since $f_5(x)$ is self-conjugate, $p_{B_{n-1}}(x+1) = f_n(x+1) = f_5(x+1)f_k((x+1)f_5(x+1)) = f_5(x)f_k((x+1)f_5(x))$. Hence, $f_5(x)$ divides both $p_{B_{n-1}}(x)$ and $p_{B_{n-1}}(x+1)$. Therefore, $P_{n-1} \square P_{n-1}$ is not closed universally solvable by Corollary 2.9 (see the entries in purple text in Table 3).

If $f_m(x)$ is divisible by a self-conjugate polynomial $g(x)$, then $f_n(x)$ is also divisible by $g(x)$, where $n = mk$ for some positive integer k. Observe that $f_{17}(x)$ is divisible by the self-conjugate polynomial $g(x) = x^4 + x + 1$. So if $n = 17k$ where k is a positive integer, then $f_n(x)$ is also divisible by $g(x)$. Hence, $g(x)$ divides both $p_{B_{n-1}}(x)$ and $p_{B_{n-1}}(x+1)$. Therefore, $P_{n-1} \square P_{n-1}$ is not closed universally solvable by Corollary 2.9.

The *n-cube* Q_n, $n \geq 1$, is defined as the repeated Cartesian product of n paths of length two. That is, $Q_1 = P_2$ and $Q_n = Q_{n-1} \square P_2$ for $n \geq 2$. The n-cube is often referred to as the *nth hypercube*. If $V(P_2) = \{0, 1\}$, then the vertex set of Q_n can be viewed as the set of n-tuples $(v_1, v_2, \ldots, v_n)$, where $v_i \in \{0, 1\}$. Moreover, two n-tuples share an edge if they differ in exactly one coordinate. The hypercubes Q_3 and Q_4 are shown in Figure 1.

Definition 3.7. Let $C_1 = [x]$, where $x \in GF(2)$. For each positive integer k, define C_{2^k} recursively by

$$C_{2^k} = \begin{bmatrix} C_{2^{k-1}} & I_{2^{k-1}} \\ I_{2^{k-1}} & C_{2^{k-1}} \end{bmatrix},$$

where $I_{2^{k-1}}$ is the $(2^{k-1}) \times (2^{k-1})$ identity matrix.

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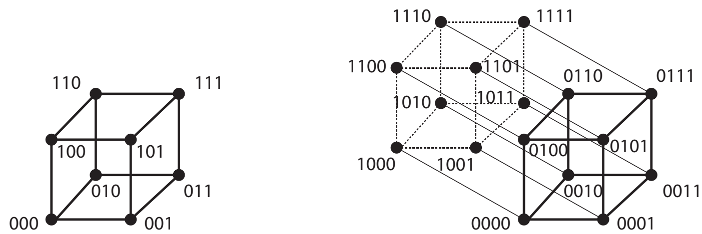

FIGURE 1. *The hypercubes Q_3 and Q_4.*

LEMMA 3.8. *For each nonnegative integer k, $\det(C_{2^k}) = (x+1)^{2^k}$ if k is odd and $\det(C_{2^k}) = x^{2^k}$ if k is even.*

Proof. The proof is by induction. Note that $\det(C_1) = \det([x]) = x$ and $\det(C_2) = \det\left(\begin{bmatrix} x & 1 \\ 1 & x \end{bmatrix}\right) = x^2+1 = (x+1)^2$. Suppose $\det(C_{2^k}) = (x+1)^{2^k}$ for some odd integer $k > 1$ and $\det(C_{2^k}) = x^{2^k}$ for some even integer $k > 0$. Then

$$\begin{aligned}
\det(C_{2^{k+2}}) &= \det\left(\left[\begin{array}{cc} C_{2^{k+1}} & I_{2^{k+1}} \\ I_{2^{k+1}} & C_{2^{k+1}} \end{array}\right]\right) \\
&= \det\Big((C_{2^{k+1}})^2 - (I_{2^{k+1}})^2\Big) && \text{(by Theorem 2.2)} \\
&= \Big[\det(C_{2^{k+1}} + I_{2^{k+1}})\Big]^2 \\
&= \left[\det\left(\left[\begin{array}{cc} C_{2^k}+I_{2^k} & I_{2^k} \\ I_{2^k} & C_{2^k}+I_{2^k} \end{array}\right]\right)\right]^2 \\
&= \Big[\det\Big((C_{2^k}+I_{2^k})^2 - (I_{2^k})^2\Big)\Big]^2 \\
&= \Big[\det(C_{2^k})^2\Big]^2 \\
&= \Big[\det(C_{2^k})\Big]^4 \\
&= \begin{cases} ((x+1)^{2^k})^4 & \text{if } k \text{ is odd} \\ (x^{2^k})^4 & \text{if } k \text{ is even} \end{cases} \\
&= \begin{cases} (x+1)^{2^{k+2}} & \text{if } k \text{ is odd} \\ x^{2^{k+2}} & \text{if } k \text{ is even} \end{cases}
\end{aligned}$$

by the induction hypothesis. ∎

Observe that for each positive integer k, the characteristic polynomial of the adjacency matrix of the k-cube Q_k is $p_{A(Q_k)}(x) = \det(C_{2^k})$, where C_{2^k} is defined as in Definition 3.7. We can determine which hypercubes are closed universally solvable and open universally solvable.

THEOREM 3.9. *For each positive integer k, the k-cube Q_k is not closed universally solvable but is open universally solvable if k is odd, and the k-cube Q_k is closed universally solvable but not open universally solvable if k is even.*

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Proof. We have $Q_k = Q_{k-1} \square P_2$ for each positive integer k. Let $B = A(Q_{k-1})$ and let $C = A(P_2)$. Then $p_C(x) = x^2 + 1 = (x+1)^2$. If k is odd (so $k-1$ is even), $p_B(x) = x^{2^{k-1}}$ by Lemma 3.8. Then $\gcd(p_B(x+1), p_C(x)) = \gcd((x+1)^{2^{k-1}}, (x+1)^2) = (x+1)^2 \neq 1$ and $\gcd(p_B(x), p_C(x)) = \gcd(x^{2^{k-1}}, (x+1)^2) = 1$. Thus, Q_k is not closed universally solvable but is open universally solvable for k odd by Theorems 2.5 and 2.6, respectively. If k is even (so $k-1$ is odd), $p_B(x) = (x+1)^{2^{k-1}}$ by Lemma 3.8. Then $\gcd(p_B(x+1), p_C(x)) = \gcd(x^{2^{k-1}}, (x+1)^2) = 1$ and $\gcd(p_B(x), p_C(x)) = \gcd((x+1)^{2^{k-1}}, (x+1)^2) = (x+1)^2 \neq 1$. Thus, Q_k is closed universally solvable but not open universally solvable for k even by Theorems 2.5 and 2.6, respectively. ☐

4. Conclusion. We conclude by stating two conjectures concerning the nullity of the adjacency matrix of a Cartesian Product.

CONJECTURE 4.1. *Let G and H be finite, simple graphs, with $|G| = m$, $|H| = p$, and $n = mp$. Let $B = A(G)$, $C = A(H)$, and $q(x) = \gcd(p_B(x+1), p_C(x))$. Then the nullity of $A(G \square H) + I_n$ is at least* $\deg q(x)$.

CONJECTURE 4.2. *Let G and H be finite, simple graphs, with $|G| = m$, $|H| = p$, and $n = mp$. Let $B = A(G)$, $C = A(H)$, and $r(x) = \gcd(p_B(x), p_C(x))$. Then the nullity of $A(G \square H)$ is at least* $\deg r(x)$.

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TABLE 1
Summary of solvable graph families.

$\square$	K_2	K_3	K_4	K_5	K_6	K_7
K_2	C, nO	nC, nO	C, nO	nC, nO	C, nO	nC, nO
K_3	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
K_4	C, nO	nC, nO	C, nO	nC, nO	C, nO	nC, nO
K_5	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
K_6	C, nO	nC, nO	C, nO	nC, nO	C, nO	nC, nO
K_7	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
C_3	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
C_4	nC, O	nC, nO	nC, O	nC, nO	nC, O	nC, nO
C_5	nC, O	nC, nO	nC, O	nC, nO	nC, O	nC, nO
C_6	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
C_7	nC, O	nC, nO	nC, O	nC, nO	nC, O	nC, nO
C_8	nC, O	nC, nO	nC, O	nC, nO	nC, O	nC, nO
C_9	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
P_2	C, nO	nC, nO	C, nO	nC, nO	C, nO	nC, nO
P_3	nC, O	nC, nO	nC, O	nC, nO	nC, O	nC, nO
P_4	C, O	C, O	C, O	C, O	C, O	C, O
P_5	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
P_6	C, O	C, O	C, O	C, O	C, O	C, O
P_7	nC, O	nC, nO	nC, O	nC, nO	nC, O	nC, nO
P_8	C, nO	nC, nO	C, nO	nC, nO	C, nO	nC, nO

C = closed universally solvable, nC = not closed universally solvable, O = open universally solvable, nO = not open universally solvable.

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TABLE 2
Summary of solvable graph families.

$\square$	C_3	C_4	C_5	C_6	C_7	C_8	C_9
C_3	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
C_4	nC, nO	C, nO	C, nO	nC, nO	C, nO	C, nO	nC, nO
C_5	nC, nO	C, nO	nC, nO	nC, nO	C, nO	C, nO	nC, nO
C_6	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
C_7	nC, nO	C, nO	C, nO	nC, nO	C, nO	C, nO	nC, nO
C_8	nC, nO	C, nO	C, nO	nC, nO	C, nO	C, nO	nC, nO
C_9	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
P_2	nC, nO	nC, O	nC, O	nC, nO	nC, O	nC, O	nC, nO
P_3	nC, nO	C, nO	C, nO	nC, nO	C, nO	C, nO	nC, nO
P_4	C, O	C, O	nC, nO	C, O	C, O	C, O	C, O
P_5	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO	nC, nO
P_6	C, O	C, O	C, O	C, O	C, nO	C, O	nC, O
P_7	nC, nO	C, nO	C, nO	nC, nO	C, nO	C, nO	nC, nO
P_8	nC, nO	nC, O	nC, O	nC, nO	nC, O	nC, O	nC, nO
P_9	nC, nO	C, nO	nC, nO	nC, nO	C, nO	C, nO	nC, nO

C = closed universally solvable, nC = not closed universally solvable, O = open universally solvable, nO = not open universally solvable.

TABLE 3
Summary of solvable graph families.

$\square$	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
P_2	C, nO	nC, O	C, O	nC, nO	C, O	nC, O	C, nO	nC, O	C, O	nC, nO
P_3	nC, O	C, nO	C, O	nC, nO	C, O	C, nO	nC, O	C, nO	C, O	nC, nO
P_4	C, O	C, O	nC, nO	C, O	C, O	C, O	C, O	nC, nO	C, O	C, O
P_5	nC, nO	nC, nO	C, O	nC, nO	C, O	nC, nO	nC, nO	nC, nO	C, O	nC, nO
P_6	C, O	C, O	C, O	C, O	C, nO	C, O	nC, O	C, O	C, O	C, O
P_7	nC, O	C, nO	C, O	nC, nO	C, O	C, nO	nC, O	C, nO	C, O	nC, nO
P_8	C, nO	nC, O	C, O	nC, nO	nC, O	nC, O	C, nO	nC, O	C, O	nC, nO
P_9	nC, O	C, nO	nC, nO	nC, nO	C, O	C, nO	nC, O	nC, nO	C, O	nC, nO
P_{10}	C, O	C, O	C, O	C, O	C, O	C, O	C, O	C, O	C, nO	C, O
P_{11}	nC, nO	nC, nO	C, O	nC, nO	C, O	nC, nO	nC, nO	nC, nO	C, O	nC, nO

C = closed universally solvable, nC = not closed universally solvable, O = open universally solvable, nO = not open universally solvable.