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UPPER BOUNDS ON THE Q-SPECTRAL RADIUS OF BOOK-FREE AND/OR $K_{S,T}$-FREE GRAPHS*

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Abstract. In this paper, two results about the signless Laplacian spectral radius $q(G)$ of a graph G of order n with maximum degree Δ are proved. Let $B_n = K_2 + \overline{K_n}$ denote a book, i.e., the graph B_n consists of n triangles sharing an edge. The results are the following:

(1) Let $1 < k \leq l < \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$-free graph of order n with maximum degree Δ. Then

$$q(G) \leq \frac{1}{4}\left[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}\right]$$

with equality if and only if G is a strongly regular graph with parameters $(\Delta,\, k,\, l)$.

(2) Let $s \geq t \geq 3$, and let G be a connected $K_{s,t}$-free graph of order n $(n \geq s + t)$. Then

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

Key words. Complete bipartite subgraph, Zarankiewicz problem, Signless Laplacian spectral radius.

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1. Introduction. Our graph notation follows Bollobás [1]. In particular, let $G = (V(G), E(G))$ be a simple graph. Denote by $v(G)$ the order of G and $e(G)$ the size of G, that is to say, $v(G) = |V(G)|$, and $e(G) = |E(G)|$. Set $\Gamma_G(u) = \{v | uv \in E(G)\}$, and $d_G(u) = |\Gamma_G(u)|$, or simply $\Gamma(u)$ and $d(u)$, respectively. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimal degree and maximal degree of graph G, respectively.

For a simple graph G of order n, let $D(G) = diag(d_1, d_2, \ldots, d_n)$, and $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of G with $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. The largest eigenvalue of $A(G)$ and $Q(G)$ are called spectral radius and signless Laplacian spectral radius (or Q-spectral radius) of G and denoted by $\rho(G)$ and $q(G)$, respectively.

Let X be a set of vertices of G. Then $G[X]$ is the graph induced by X, and $e(X) = e(G[X])$. Let P_k, C_k and K_k be the path, cycle, and complete graph of order k, respectively. If all vertices of G have the same degree k, then G is k-*regular*. A k-regular graph is called *strongly regular* with parameters (k, a, c) whenever each pair of adjacent vertices have $a \geq 0$ common neighbors, and each pair of non-adjacent vertices have $c \geq 1$ common neighbors.

The main results of this paper are in the spirit of the trend in the famous Zarankiewicz problem [9]:

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Problem A. How many edges can a graph of order n have if it does not contain a complete bipartite subgraph $K_{s,t}$?

In 1996, Füredi [4] gave an upper bound on the above Zarankiewicz problem. In 2010, Nikiforov [6] improved his result. That is, if G is a $K_{s,t}$ -free graph of order n, then

$$e(G) \leq \frac{1}{2}(s-t+1)^{1/t}n^{2-1/t} + \frac{1}{2}(t-1)n^{2-2/t} + \frac{1}{2}(t-2)n.$$

The spectral version of the Zarankiewicz problem is the following one:

Problem B. How large can be the spectral radius $\rho(G)$ of a graph G of order n that does not contain $K_{s,t}$?

There are some results for some value of s and t.

In 2007, the upper bound on the signless Laplacian spectral radius of $K_{2,l+1}$-free graph as the corollary of the following Lemma 1.1 was proved in [9] by Shi and Song.

Lemma 1.1. *Let $0 \leq k \leq l \leq \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$-free graph of order n with maximum degree Δ. Then*

$$\rho(G) \leq \frac{1}{2}\left[k - l + \sqrt{(k-l)^2 + 4\Delta + 4l(n-l)}\right]$$

with equality if and only if G is a stongly regular with parameters (Δ, k, l).

In 2007, Nikiforov [7] improved the above bound showing that:

Lemma 1.2. *Let $l \geq k \geq 0$. If G is a $\{B_{k+1}, K_{2,l+1}\}$-free graph of order n with maximum degree Δ. Then*

$$\rho(G) \leq \min\left\{\Delta, \frac{1}{2}\left[k - 1 + 1 + \sqrt{(k-l+1)^2 + 4l(n-1)}\right]\right\}.$$

If G is connected, equality holds if and only if one of the following conditions holds:

(1) $\Delta^2 - \Delta(k-l+1) \leq l(n-1)$ and G is Δ-regular;

(2) $\Delta^2 - \Delta(k-l+1) > l(n-1)$ and every two vertices of G have k common neighbors if they are adjacent, and l common neighbors, otherwise.

Setting $l = \Delta$ or $k = l$, Lemma 1.2 strengthens Corollaries 1 and 2 of [8].

In 2010, Nikiforov [6] also gave a bound as the following lemma.

Lemma 1.3. *Let $s \geq t \geq 2$, and let G be a $K_{s,t}$-free graph of order n. If $t = 2$, then*

$$\rho(G) \leq \frac{1}{2} + \sqrt{(s-1)(n-1) + 1/4}.$$

If $t \geq 3$, then

$$\rho(G) \leq (s-t+1)^{1/t}n^{1-1/t} + (t-1)n^{1-2/t} + t - 2$$

and

$$e(G) < \frac{1}{2}(s-t+1)^{1/t}n^{2-1/t} + \frac{1}{2}(t-1)n^{2-2/t} + \frac{1}{2}(t-2)n.$$

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A newer trend in extremal graph theory is the Zarankiewicz problem for the signless Laplacian spectral radius of graphs:

PROBLEM C How large can the signless Laplacian spectral radious of a graph of order G be, if it does not contain $K_{s,t}$ as a subgraph?

When $s = t = 2$, we notice that the $K_{2,2}$-free graph is the same as C_4-free graph. Also in 2013, de Freitus et al. [2] have proved that if G contains no C_4, then

$$q(G) < q(F_n),$$

unless $G = F_n$, where F_n is the friendship graph of order n. For n odd, F_n is a union of $\lfloor n/2 \rfloor$ triangles sharing a single common vertex, and for n even, F_n is obtained by hanging an edge to the common vertex of F_{n-1}.

In Section 2, we will prove the following results which give upper bounds on the signless Laplacian spectral radius of Book-free and/or $K_{2,l+1}$-free ($l > 1$) graphs of order n with maximum degree Δ.

THEOREM 1.4. *Let $1 < k \leq l < \Delta < n$ and G be a connected $\{B_{k+1}, K_{2,l+1}\}$-free graph of order n with maximum degree Δ. Then*

$$q(G) \leq \frac{1}{4}\left[3\Delta + k - 2l + 1 + \sqrt{(3\Delta + k - 2l + 1)^2 + 16l(\Delta + n - 1)}\right] \tag{1.1}$$

with equality if and only if G is a strongly regular graph with parameters (Δ, k, l).

Because every graph is obviously $K_{2,\Delta+1}$-free, Theorem 1.4 readily implies a sharp upper bound for book-free graph.

COROLLARY 1.5. *Let $1 < k < \Delta < n$ and G be a connected B_{k+1}-free graph of order n with maximum degree Δ. Then*

$$q(G) \leq \frac{1}{4}\left[\Delta + k + 1 + \sqrt{(\Delta + k + 1)^2 + 32\Delta(n - 1)}\right]$$

with equality if and only if G is a strongly regular graph with parameters(Δ, k, Δ).

Because a $K_{2,l}$-free graph is also B_l-free. Theorem 1.4 with $k = l$ also implies a sharp upper bound for $K_{2,l}$-free graphs.

COROLLARY 1.6. *Let $1 < l < \Delta$ and G be a connected $K_{2,l+1}$-free graph of order n with maximum degree Δ. Then*

$$q(G) \leq \frac{1}{4}\left[3\Delta - l + 1 + \sqrt{(3\Delta - l + 1)^2 + 32l(n - 1)}\right]$$

with equality if and only if G is a strongly regular graph with parameters (Δ, l, l).

Furthermore, we will discuss $s \geq t \geq 3$. Let G be a connected graph of order n. Since G contains no $K_{s,t}$ when $n < s + t$, we only discuss the case $n \geq s + t$.

THEOREM 1.7. *Let $s \geq t \geq 3$, and let G be a connected $K_{s,t}$-free graph of order n ($n \geq s + t$). Then*

$$q(G) \leq n + (s - t + 1)^{1/t} n^{1-1/t} + (t - 1)(n - 1)^{1-3/t} + t - 3.$$

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2. Some known lemmas. In this section, we state two known results that will be used in this paper.

Lemma 2.1. *Let $s \geq 2$, $t \geq 2$, $0 \leq k \leq s-2$, and let $G(A,B)$ be a bipartite graph with parts A and B. Suppose that G contains no copy of $K_{s,t}$ with a vertex class of size s in A and a vertex class of size t in B. Then $G(A,B)$ has at most*

$$(s-k-1)^{1/t}|B||A|^{1-1/t} + (t-1)|A|^{1+k/t} + k|B|$$

edges.

Lemma 2.2. ([3, 5]) *For every graph G, we have*

$$q(G) \leq \max_{u \in V(G)} \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \right\}.$$

3. Proofs.

Proof of Theorem 1.4. Let Q_i denote the ith row vector of $Q = Q(G)$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron-eigenvector of Q corresponding to $q(G)$. Then $x_i > 0$ for $1 \leq i \leq n$. Since G is $\{B_{k+1},\, K_{2,l+1}\}$-free, each pair of adjacent vertices has at most k common neighbors and each pair of non-adjacent vertices has at most l common neighbors. Thus,

$$\sum_{i=1}^{n} \sum_{v_p, v_q \in \Gamma(v_i)} x_p x_q \leq k \sum_{v_p v_q \in E(G)} x_p x_q + l \sum_{v_p v_q \notin E(G)} x_p x_q. \tag{3.2}$$

Note that $\mathbf{x}^T A(K_n)\mathbf{x} \leq \rho(K_n) = n-1$. Thus,

$$\begin{aligned} q(G) = \mathbf{x}^T Q\mathbf{x} = \mathbf{x}^T D\mathbf{x} + \mathbf{x}^T A\mathbf{x} &= \sum_{i=1}^{n} d_i x_i^2 + 2 \sum_{v_i v_p \in E(G)} x_i x_p \\ &\leq \Delta + \mathbf{x}^T A(K_n)\mathbf{x} - 2 \sum_{v_i v_p \notin E(G)} x_i x_p \\ &\leq \Delta + n - 1 - 2 \sum_{v_i v_p \notin E(G)} x_i x_p. \end{aligned}$$

Also we can obtain

$$\begin{aligned} q(G) = \mathbf{x}^T Q\mathbf{x} &= \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} 2q_{i,j} x_i x_j + \sum_{i=1}^{n} d_i x_i^2 \\ &\leq \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} q_{i,j}(x_i^2 + x_j^2) + \sum_{i=1}^{n} d_i x_i^2 \\ &= \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} q_{i,j} x_i^2 + \sum_{i=1}^{n} d_i x_i^2 \\ &= 2 \sum_{i=1}^{n} d_i x_i^2. \end{aligned}$$

So

$$\sum_{i=1}^{n} d_i x_i^2 \geq \frac{q}{2}.$$

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Then

$$
\begin{aligned}
q^2(G) = \|Q\mathbf{x}\|^2 = \sum_{i=1}^{n}(Q_i\mathbf{x})^2 &= \sum_{i=1}^{n}\left(d_ix_i + \sum_{v_iv_p\in E(G)} x_p\right)^2 \\
&= \sum_{i=1}^{n}\left[d_i^2x_i^2 + 2d_ix_i\sum_{v_iv_p\in E(G)} x_p + \left(\sum_{v_iv_p\in E(G)} x_p\right)^2\right] \\
&= \sum_{i=1}^{n} d_i^2x_i^2 + 2\sum_{i=1}^{n} d_i \sum_{v_iv_p\in E(G)} x_ix_p + \sum_{i=1}^{n} d_ix_i^2 + 2\sum_{i=1}^{n}\sum_{v_p,v_q\in\Gamma(v_i)} x_px_q \\
&\leq (\Delta+1)\sum_{i=1}^{n} d_ix_i^2 + 2\Delta\sum_{i=1}^{n}\sum_{v_iv_p\in E(G)} x_ix_p + 2k\sum_{v_pv_q\in E(G)} x_px_q + 2l\sum_{v_pv_q\notin E(G)} x_px_q \qquad (3.3)\\
&= (\Delta+1)\sum_{i=1}^{n} d_ix_i^2 + (4\Delta+2k)\sum_{v_iv_p\in E(G)} x_ix_p + 2l\sum_{v_pv_q\notin E(G)} x_px_q \\
&\leq (2\Delta+k)(\sum_{i=1}^{n} d_ix_i^2 + 2\sum_{v_iv_p\in E(G)} x_ix_p) \\
&\quad (\Delta+k-1)\sum_{i=1}^{n} d_ix_i^2 + 2l\sum_{v_pv_q\notin E(G)} x_px_q \\
&\leq (2\Delta+k)q - \frac{\Delta+k-1}{2}q + l(\Delta+n-1-q) \\
&= \frac{1}{2}(3\Delta+k-2l+1)q + l(\Delta+n-1).
\end{aligned}
$$

Solving the inequality gives the upper bound

$$
q(G) \leq \frac{1}{4}\left[3\Delta+k-2l+1+\sqrt{(3\Delta+k-2l+1)^2+16l(\Delta+n-1)}\right].
$$

If the upper bound of (1.1) is attained then all inequalities in the above argument must be equalities. In particular, from (3.2) and $x_i > 0$ for $1 \leq i \leq n$, we have that each pair of adjacent vertices in G has exactly k common neighbors and each pair of non-adjacent vertices in G has exactly l common neighbors. Moreover, by (3.3), G must be Δ-regular. Thus, G must be a strongly regular graph with parameters (Δ, k, l). ∎

Proof of Theorem 1.7. By Lemma 2.2, let w be a vertex of G such that

$$
d(w) + \frac{1}{d(w)}\sum_{i\in\Gamma(w)} d(i) = \max_{u\in V(G)}\{d(u) + \frac{1}{d(u)}\sum_{v\in\Gamma(u)} d(v)\}.
$$

Then

$$
q(G) \leq d(w) + \frac{1}{d(w)}\sum_{i\in\Gamma(w)} d(i).
$$

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Note first that if $d(w) \leq s+t-1$, then

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i\in\Gamma(w)} d(i) \leq d(w) + \Delta(G) \\ &\leq s+t-1+n-1 = s+t+n-2 \\ &\leq n + (s-t+1)^{1/t} n^{1-1/t} + (t-1)(n-1)^{1-3/t} + t - 3. \end{aligned}$$

Therefore, we shall assume that $s+t-1 \leq d(w) \leq n-1$. Let U and W be disjoint sets satisfying $|U| = d(w)$ and $|W| = n-1$, and let φ_U and φ_W be bijections

$$\varphi_U : U \to \Gamma(w), \varphi_W : W \to V(G)\backslash\{w\}.$$

Define a bipartite graph H with vertex classes U and W by joining $u \in U$ and $v \in W$ whenever $\{\varphi_U(u), \varphi_W(v)\} \in E(G)$.

Then we can get that H does not contain a copy of $K_{s-1,t}$ with $s-1$ vertices in W and t vertices in U. Indeed, the map $\psi : V(H) \to V(G)$ defined as

$$\psi(x) = \begin{cases} \varphi_U(x), & \text{if } x \in U, \\ \varphi_W(x), & \text{if } x \in W. \end{cases}$$

is a homomorphism of H into $G - w$. Suppose to the contrary that $F \subset H$ is a copy of $K_{s-1,t}$ with a set of S of $s-1$ vertices in W and a set of T of t vertices in U. Clearly S and T are the vertex classes of F. Note that $\psi(F)$ is a copy of $K_{s-1,t}$ in $G - w$, and $\psi(S) = \varphi_W(S) \subset V(G) \setminus \{w\}$ and $\psi(T) = \varphi_U(T) \subset \Gamma_G(w)$ are the vertex classes of $\psi(F)$ of size $s-1$ and size t, respectively. Now, adding w to $\psi(F)$, we see that G contains a $K_{s,t}$, a contradiction proving the clain.

Suppose that $0 \leq k \leq \min\{s,t\} - 2$. Setting $k' = k-1, s' = s-1, t' = t, A = W, B = U$, then from Lemma 2.1, we have

$$\begin{aligned} e(H) &\leq (s-k-1)^{1/t}|U||W|^{1-1/t} + (k-1)|U| + (t-1)|W|^{1+(k-1)/t} \\ &= (s-k-1)^{1/t} d(w) n^{1-1/t} + (k-1)d(w) + (t-1)(n-1)^{1+(k-1)/t}. \end{aligned}$$

On the other hand, we have that

$$e(H) = \sum_{v\in\Gamma(w)} d(v) - d(w),$$

and so,

$$\sum_{v\in\Gamma(w)} d(v) \leq ((s-k-1)^{1/t} n^{1-1/t} + k) d(w) + (t-1)(n-1)^{1+(k-1)/t}.$$

From Lemma 2.2, we have

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i\in\Gamma(w)} d(i) \\ &\leq d(w) + \frac{(t-1)(n-1)^{1+(k-1)/t}}{d(w)} + (s-k-1)^{1/t} n^{1-1/t} + k. \end{aligned}$$

Since the function

$$f(x) = x + \frac{(t-1)(n-1)^{1+(k-1)/t}}{x}$$

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is convex for $x > 0$, its maximum in any closed interval is attained at one of the endpoints of the interval. In the case $s + t - 1 \leq d(w) \leq n - 1$, then,

$$\begin{aligned} q(G) &\leq d(w) + \frac{1}{d(w)} \sum_{i \in \Gamma(w)} d(i) \\ &\leq \max\left\{ s + t - 1 + \frac{(t-1)(n-1)^{1+(k-1)/t}}{s+t-1}, n - 1 + \frac{(t-1)(n-1)^{1+(k-1)/t}}{n-1} \right\} \\ &\quad + (s-k-1)^{1/t} n^{1-1/t} + k \\ &\leq (s-k-1)^{1/t} n^{1-1/t} + k + \frac{(t-1)(n-1)^{1+(k-1)/t}}{n-1} + n - 1 \\ &= (s-k-1)^{1/t} n^{1-1/t} + k + (t-1)(n-1)^{(k-1)/t} + n - 1. \end{aligned}$$

Now, if $s \geq t \geq 3$, setting $k = t - 2$, we obtain

$$q(G) \leq n + (s-t+1)^{1/t} n^{1-1/t} + (t-1)(n-1)^{1-3/t} + t - 3. \quad \square$$

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