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GROUPS OF MATRICES THAT ACT MONOPOTENTLY*

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Abstract. In the present article, the authors continue the line of inquiry started by Cigler and Jerman, who studied the separation of eigenvalues of a matrix under an action of a matrix group. The authors consider groups $\mathcal{G}$ of matrices of the form $\left[\begin{smallmatrix} G & 0 \\ 0 & z \end{smallmatrix}\right]$, where z is a complex number, and the matrices G form an irreducible subgroup of $\mathbb{GL}_n(\mathbb{C})$. When $\mathcal{G}$ is not essentially finite, the authors prove that for each invertible A the set $\mathcal{G}A$ contains a matrix with more than one eigenvalue.

The authors also consider groups $\mathcal{G}$ of matrices of the form $\left[\begin{smallmatrix} G & x \\ 0 & 1 \end{smallmatrix}\right]$, where the matrices G comprise a bounded irreducible subgroup of $\mathbb{GL}_n(\mathbb{C})$. When $\mathcal{G}$ is not finite, the authors prove that for each invertible A the set $\mathcal{G}A$ contains a matrix with more than one eigenvalue.

Key words. Invertible matrices, Matrix groups, Distinct eigenvalues, Irreducible groups, Unitary group, Monopotent matrices.

AMS subject classifications. 15A18.

1. Introduction. A classical "inverse multiplicative eigenvalue problem" (see [7]) asks, whether for a given square matrix A there is a diagonal matrix D such that DA has the prescribed spectrum. The question can be stated over $\mathbb{R}$ or $\mathbb{C}$. The default field in our paper is $\mathbb{C}$.

In 1958, M.E. Fisher and A.T. Fuller [6] showed that when A is a real square matrix with positive principal leading minors (of all orders), there is a diagonal matrix D with a positive diagonal such that all of the (complex) eigenvalues of DA are positive and algebraically simple. A decade later C.S. Ballantine [1] extended the result to complex matrices under the hypotheses that all principal leading minors of A are non-zero and D is allowed to be an invertible complex diagonal matrix. In 1975, S. Friedland [7] improved on Ballantine's theorem by showing that one can arrange for DA to have whatever complex spectrum one desires (including the algebraic multiplicities of the eigenvalues) by using general complex diagonal matrices D.

Fast forward to 2012 and the paper of X.-L. Feng, Z. Li and T.-Z. Huang [5] in which the authors pose the following related question: Is every complex invertible matrix diagonally equivalent to a matrix with distinct eigenvalues? Feng, Li and Huang answer the question in the affirmative for matrices of small size, and later that same year M.-D. Choi, Z. Huang, C.-K. Li and N.-S. Sze [2] settle the general question, also in the affirmative. A formulation of their result states that complex square matrices not diagonally equivalent to a matrix with distinct eigenvalues are exactly the non-invertible matrices whose classical adjoint has zero diagonal.

In 2014, G. Cigler and M. Jerman [4] observed that the question posed by Feng, Li and Huang, and settled by Choi, Huang, Li and Sze, can be considered to be part of a more general inquiry. Given a group

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$\mathcal{G}$ of complex $n \times n$ matrices, and a matrix $A \in \mathbb{M}_n$, one can consider the set $\mathcal{G}A$ and ask whether this set contains a matrix with a maximal possible number (i.e., rank(A)) of distinct eigenvalues. When it does, Cigler and Jerman say that A is *$\mathcal{G}$-separable*.

Cigler and Jerman prove that every matrix is $\mathcal{G}$-separable when $\mathcal{G}$ is the group $\mathcal{U}_n$ of all unitary matrices, or the group $\mathcal{M}_n$ of all monomial matrices. We shall express this by saying that $\mathcal{U}_n$ and $\mathcal{M}_n$ are *eigenvalue separating groups*. *Monomial matrices*, also known as "weighted permutations", are the products of invertible diagonal matrices and permutation matrices.

Irreducible matrix groups are the groups that do not have common non-trivial invariant subspaces, where the trivial subspaces are $\{0_n\}$ and the whole space. By the celebrated Burnside's theorem (in the complex setting) these are exactly the groups that span $\mathbb{M}_n$. $\mathcal{U}_n$ and $\mathcal{M}_n$ are examples of irreducible subgroups of the general linear group $\mathbb{GL}_n$.

Cigler and Jerman give an example of an irreducible subgroup $\mathcal{G}$ of $\mathbb{GL}_4$ no member of which has four distinct eigenvalues. In particular, I_4 is not $\mathcal{G}$-separable, and so not all irreducible subgroups of $\mathbb{GL}_n$ are eigenvalue separating groups.

Consequently, it is natural to ask whether irreducible matrix groups can fail at separating eigenvalues in a big way. Is there an irreducible matrix group $\mathcal{G}$ in dimensions higher than 1, and a non-zero matrix A, such that every matrix in $\mathcal{G}A$ has a single eigenvalue? Cigler and Jerman prove in [4] that no such $\mathcal{G}$ and A exist. They express this by saying that A is *$\mathcal{G}$-semi-separable*, which we restate by saying that the irreducible matrix groups (in dimensions greater than 1) are *(eigenvalue) semi-separating*.

In their consequent paper [3] from the same year, Cigler and Jerman continue the exploration of eigenvalue semi-separating matrix groups by considering the group $\mathcal{P}_n$ of all $n \times n$ permutation matrices (i.e., the symmetric group). Note that $\mathcal{P}_n$ is not irreducible, but has only one pair of (complementary) non-trivial invariant subspaces: the span of the vector all of whose entries are 1, and the orthogonal complement of that span. Decomposing $\mathcal{P}_n$ along these subspaces gives a representation of $\mathcal{P}_n$ of the form $\mathcal{G} \oplus \{1\}$, where $\mathcal{G}$ is a finite irreducible group of unitary matrices.

Cigler and Jerman show that every complex invertible 3×3 matrix is $\mathcal{P}_3$-semi-separable, and determine the $\mathcal{P}_n$-semi-separable nilpotent matrices. They also show that if the modulus of the sum of the entries of A does not exceed $n\sqrt[n]{\det A}$, then A is $\mathcal{P}_n$-semi-separable.

In the present article, we continue this line of inquiry. We consider groups $\mathcal{G}$ of matrices of the form $\left[\begin{smallmatrix} G & 0 \\ 0 & z \end{smallmatrix}\right]$, where $z \in \mathbb{C}$ and the matrices G form an irreducible subgroup of $\mathbb{GL}_n$. We prove that $\mathcal{G}$ semi-separates the eigenvalues of any invertible A whenever $\mathcal{G}$ is not essentially finite. A matrix group $\mathcal{G}$ is *essentially finite* if there is a finite group $\mathcal{F}$ such that

$$\mathcal{G} \subset \mathbb{C}\mathcal{F}.$$

Furthermore, we consider groups $\mathcal{G}$ of matrices of the form $\left[\begin{smallmatrix} G & x \\ 0 & 1 \end{smallmatrix}\right]$, where matrices G comprise a bounded irreducible subgroup of $\mathbb{GL}_n$. We prove that $\mathcal{G}$ semi-separates the eigenvalues of any invertible A, whenever $\mathcal{G}$ is not finite.

Let us mention the following terminology that is used in this paper.

If $\mathcal{W}$ and $\mathcal{Z}$ are complementary invariant subspaces for a subgroup $\mathcal{G}$ of $\mathbb{GL}_n$, we say that $\mathcal{W}$ and $\mathcal{Z}$ are *reducing subspaces* for $\mathcal{G}$.

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Let $\mathbb{M}_n$ denote the algebra of $n \times n$ complex matrices. A matrix $A \in \mathbb{M}_n$ is *monopotent* if the (complex) spectrum of A is a singleton. A is monopotent exactly when $A = \alpha I + N$ where $\alpha \in \mathbb{C}$ and N is a nilpotent matrix.

We will say a subgroup $\mathcal{G}$ of $\mathbb{GL}_n$ *acts monopotently* on a matrix $A \in \mathbb{M}_n$ if GA is monopotent for all $G \in \mathcal{G}$. We also extend this definition to general collections of matrices, and not just groups.

We denote the multiplicative group of the non-zero complex numbers by $\mathbb{C}^*$. For $z \in \mathbb{C}$ we write $\Omega_n(z)$ for the set of n^{th} roots of z, and drop the reference to z when $z = 1$.

The normalized trace functional $\mathrm{Trace}\,(\cdot)$ on $\mathbb{M}_n$ is defined by:

$$\mathrm{Trace}\,(A) \stackrel{\text{def}}{=} \frac{\mathrm{trace}\,(A)}{n}.$$

We write $\|A\|$ for the ℓ^2-operator norm of a matrix A, and $\|A\|_{\text{t-n}}$ for the trace-norm of A, defined by

$$\|A\|_{\text{t-n}} \stackrel{\text{def}}{=} \mathrm{trace}\left(\sqrt{A^*A}\right),$$

where A^* is the conjugate-transpose of A.

2. Preliminary results.

Observation 2.1. Given matrices $A, B \in \mathbb{M}_n$ such that

$$\mathrm{trace}\,(A^p) = \mathrm{trace}\,(B^p), \quad \text{for } p = 1, 2, 3, \ldots, n,$$

it is well-known (see for example Theorem 2.1.16 of [10]) that A and B have the same eigenvalues, counting algebraic multiplicity. Consequently, $B \in \mathbb{M}_n$ is a monopotent matrix with an eigenvalue α if and only if

$$\mathrm{Trace}\,(B^p) = \alpha^p, \quad \text{for } p = 1, 2, 3, \ldots, n.$$

It follows that $B \in \mathbb{M}_n$ is monopotent if and only if

$$\Big(\mathrm{Trace}\,(B)\Big)^p = \mathrm{Trace}\,(B^p), \quad \text{for } p = 2, 3, \ldots, n.$$

It is also obvious that for a monopotent matrix $B \in \mathbb{M}_n$:

$$\Big(\mathrm{Trace}\,(B)\Big)^n = \mathrm{Trace}\,(B^n) = \det B. \tag{2.1}$$

LEMMA 2.2. [8, Proof of Theorem 5] *If $z_1, z_2, z_3, \ldots, z_n$ are complex numbers of modulus 1, and the sequence $\left[z_1^p + z_2^p + \cdots + z_n^p\right]_{p=1}^{\infty}$ converges, then the limit of this sequence is n.*

PROPOSITION 2.3. *For an invertible $A \in \mathbb{M}_n$, the following are equivalent:*

1. *For all $p \in \mathbb{N}$:* $\Big(\mathrm{Trace}(A^p)\Big)^n = \det A^p$.
2. *For all $p \in \mathbb{N}$:* $\Big(\mathrm{Trace}(A^p)\Big)^n = \Big(\mathrm{Trace}(A)\Big)^{pn}$.
3. *For $p = 1, 2, \ldots, n$:* $\Big(\mathrm{Trace}(A^p)\Big) = \Big(\mathrm{Trace}(A)\Big)^p$.
4. *A is monopotent.*

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Proof. We have established the equivalence of claims 3) and 4) already in Observation 2.1.

4) $\Longrightarrow$ 1) : Since A is monopotent, so is every natural power of A, and 1) follows from (2.1).

1) $\Longrightarrow$ 2) : When $p=1$, claim 1) states that $\Big(\text{Trace}\,(A)\Big)^n = \det A$. Thus, assuming 1) holds, we get:

$$\Big(\text{Trace}\,(A^p)\Big)^n = \det A^p = (\det A)^p = \Big(\text{Trace}\,(A)\Big)^{pn},$$

which yields 2).

2) $\Longrightarrow$ 4) : The argument is essentially that of [8, Proof of Theorem 5], with just a few modifications. After replacing A with a non-zero scalar multiple of A, which of course does not affect the equalities in 2), we may assume that $\text{Trace}\,(A)=1$.

Hence, we can rewrite the equalities in 2) as

$$\Big(\text{Trace}\,(A^p)\Big)^n = 1, \quad \text{for all} \quad p\in\mathbb{N};$$

from where we see that

$$\text{Trace}\,(A^p) \in \Omega_n\ , \quad \text{for all} \quad p\in\mathbb{N}. \tag{2.2}$$

Let us write $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ for the eigenvalues of A, listed with algebraic multiplicity and in order of the decreasing modulus. Then (2.2) states that

$$s(p) \overset{\text{def}}{=} \sum_{i=1}^{n} \lambda_i^{p} \quad \in n\cdot\Omega_n, \tag{2.3}$$

for all natural p, and obviously $|s(p)| = n$ for all such.

Suppose that $\rho = |\lambda_1| = \cdots = |\lambda_r|$, and all other λ_i have strictly smaller modulus than ρ. Then

$$\left(\frac{s(p)}{\rho^p} - \sum_{i=1}^{r}\left(\frac{\lambda_i}{\rho}\right)^p\right) \longrightarrow 0, \quad \text{as} \ \ p\longrightarrow\infty. \tag{2.4}$$

If $\rho > 1$, then we would have

$$\left(\frac{s(p)}{\rho^p}\right) \longrightarrow 0, \quad \text{as} \ \ p\longrightarrow\infty,$$

and therefore,

$$\sum_{i=1}^{r}\left(\frac{\lambda_i}{\rho}\right)^p \longrightarrow 0, \quad \text{as} \ \ p\longrightarrow\infty,$$

which would dictate (via Lemma 2.2) that $r=0$, leading to a contradiction.

If $\rho < 1$, then we would have

$$n = |s(p)| \le \sum_{i=1}^{n} |\lambda_i|^p \le n\cdot\rho^p < n,$$

causing a contradiction again. Therefore, it must be that $\rho = 1$, and (2.4) becomes:

$$s(p) - \sum_{i=1}^{r}\lambda_i^p \longrightarrow 0, \quad \text{as} \ \ p\longrightarrow\infty. \tag{2.5}$$

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Yet $|s(p)| = n$ and $\left|\sum_{i=1}^{r} \lambda_i^p\right| \leq r$, and so, we can deduce from (2.5) that $r = n$. This shows that all eigenvalues of A have modulus 1.

Yet we also have

$$n = |s(p)| = \left|\sum_{i=1}^{n} \lambda_i^p\right| \leq \sum_{i=1}^{n} \left|\lambda_i^p\right| = n.$$

The equality in the complex triangle inequality holds exactly when all of the summands are positive multiples of a fixed complex number of modulus 1. Since all λ_i have modulus 1, this means that they are all equal 1, and so A is monopotent. ☐

PROPOSITION 2.4. *If $\mathcal{G}$ is a subgroup of $\mathbb{C}^*$ and $I_n \oplus \mathcal{G}$ acts monopotently on an invertible matrix $A \in \mathbb{M}_{n+1}$, then $\mathcal{G} \subset \{-1, 1\}$.*

Proof. Let us write

$$\mathcal{H} = I_n \oplus \mathcal{G} = \left\{ \begin{bmatrix} I_n & 0 \\ 0 & \alpha \end{bmatrix} \;\middle|\; \alpha \in \mathcal{G} \right\},$$

and express A as $\left[\begin{smallmatrix} B_\circ & x_\circ \\ y_\circ & a_\circ \end{smallmatrix}\right]$ with respect to the same direct sum decomposition of the underlying space.

Let us prove that $\mathcal{G}$ has at most two elements. Suppose, for the sake of contradiction, that $\mathcal{G}$ has at least three distinct elements.

By Proposition 2.3,

$$(n+1)\,\text{trace}\left(C_\alpha^2\right) = \Big(\text{trace}\,(C_\alpha)\Big)^2,$$

where $\alpha \in \mathcal{G}$ and

$$C_\alpha \stackrel{\text{def}}{=} \begin{bmatrix} I_n & 0 \\ 0 & \alpha \end{bmatrix} A = \begin{bmatrix} B_\circ & x_\circ \\ \alpha y_\circ & \alpha a_\circ \end{bmatrix}.$$

This leads to the equality

$$(n+1)\left(\text{trace}\left(B_\circ^2\right) + 2\alpha y_\circ x_\circ + \alpha^2 a_\circ^2\right) = \left(\text{trace}\,(B_\circ) + \alpha a_\circ\right)^2, \tag{2.6}$$

for all $\alpha \in \mathcal{G}$. Since $\mathcal{G}$ contains at least three distinct α's, the equality (2.6) is equivalent to the equality of the corresponding coefficients of the powers of α :

$$\begin{cases} (n+1)\text{trace}\left(B_\circ^2\right) = \left(\text{trace}\,(B_\circ)\right)^2; \\ (n+1)y_\circ x_\circ = a_\circ\,\text{trace}\,(B_\circ)\,; \\ (n+1)a_\circ^2 = a_\circ^2. \end{cases}$$

The third equality yields $a_\circ = 0$, which leads to a contradiction, because in such a case, according to (2.1), for any $\alpha \in \mathcal{G}$:

$$\alpha \det A = \det C_\alpha = \left(\frac{\text{trace}\,(C_\alpha)}{n+1}\right)^{n+1} = \left(\frac{\text{trace}\,(B_\circ)}{n+1}\right)^{n+1} = \left(\frac{\text{trace}\,(A)}{n+1}\right)^{n+1} = \det A, \tag{2.7}$$

which indicates that $\mathcal{G} = \{1\}$, contradictng our hypothesis that $\mathcal{G}$ has at least three elements. ☐

EXAMPLE 2.5. Proposition 2.4 is not vacuous. For example, the group

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

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acts monopotently on the invertible matrix $\begin{bmatrix} 1+1 & 1 \\ 1 & 1-i \end{bmatrix}$, and the group

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

acts monopotently on the invertible matrix $\begin{bmatrix} 3 & 3 & 2 \\ -4 & -3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

Observation 2.6. If a collection $\mathcal{F}$ in $\mathbb{M}_n$ acts monopotently on a matrix A, then so does the closure $\overline{\mathbb{C}\mathcal{F}}$ of $\mathbb{C}\mathcal{F}$. Indeed, it follows immediately from Proposition 2.3, and from the continuity of power functions and of trace, that the set of monopotent matrices is closed. Hence,

$$\left(\overline{\mathbb{C}\mathcal{F}}\right) A \subset \overline{\mathbb{C}\mathcal{F}A} \subset \overline{\{\text{monopotent matrices}\}} = \{\text{monopotent matrices}\},$$

and the claim follows.

3. Main results.

THEOREM 3.1. *If $\mathcal{G}$ is an irreducible subgroup of $\mathbb{GL}_n$ and $\mathcal{G} \oplus 1$ acts monopotently on some $A \in \mathbb{GL}_{n+1}$, then $\mathcal{G}$ is finite.*

Proof. Let us write

$$\mathcal{H} = \mathcal{G} \oplus 1 = \left\{ \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; G \in \mathcal{G} \right\},$$

and express A as $\left[\begin{smallmatrix} B_\circ & x_\circ \\ y_\circ & a_\circ \end{smallmatrix}\right]$ with respect to the same direct sum decomposition of the underlying space. After multiplying A by a scalar, we can assume without loss of generality that $\det A = 1$.

Since $\mathbb{C}^*\mathcal{H}$ acts monopotently on A, the same can be said of the group $\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1}$.

Now

$$\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1} = \left\{ \begin{bmatrix} T & 0 \\ 0 & \alpha \end{bmatrix} \;\middle|\; \alpha \in \mathbb{C}^*, \quad T \in \alpha\mathcal{G}, \quad \alpha \det T = 1 \right\}$$

$$= \left\{ \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} \;\middle|\; \det(T)\, T \in \mathcal{G} \right\}.$$

Let

$$\widetilde{\mathcal{G}} \stackrel{\text{def}}{=} \{\ T \in \mathbb{M}_n \mid \det(T)T \in \mathcal{G}\ \}, \tag{3.8}$$

so that

$$\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1} = \left\{ \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} \;\middle|\; T \in \widetilde{\mathcal{G}} \right\}.$$

It is easy to check that $\widetilde{\mathcal{G}}$ is a group. Let us demonstrate that

$$\widetilde{\mathcal{G}} = \left\{ \frac{G}{\alpha} \;\middle|\; G \in \mathcal{G}, \quad \alpha \in \Omega_{n+1}(\det G) \right\}. \tag{3.9}$$

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$\langle\supset\rangle$ Suppose $G \in \mathcal{G}$ and $\alpha \in \Omega_{n+1}(\det G)$. Then

$$\det\left(\frac{G}{\alpha}\right)\ \frac{G}{\alpha} = \frac{\det G}{\alpha^{n+1}}\ G = G \in \mathcal{G}. \tag{3.10}$$

$\langle\subset\rangle$ If $T \in \widetilde{\mathcal{G}}$, then $T = \frac{1}{\det T} G$ for some $G \in \mathcal{G}$. Thus,

$$G = \det(T)\, T = \det\left(\frac{G}{\det T}\right)\ \frac{G}{\det T} = \frac{\det G}{(\det T)^{n+1}}\ G,$$

from where it follows that

$$(\det T)^{n+1} = \det G.$$

Therefore, $T = \frac{1}{\alpha} G$, where $\alpha \in \Omega_{n+1}(\det G)$.

Using equalities (3.8)–(3.10), we can now write:

$$\mathcal{G} = \left\{\ (\det T)T \ \mid\ T \in \widetilde{\mathcal{G}}\ \right\},$$

and conclude that $\mathbb{C}^*\mathcal{G} = \mathbb{C}^*\widetilde{\mathcal{G}}$, and that $\mathcal{G}$ is finite if and only if $\widetilde{\mathcal{G}}$ is finite. Since a group $\mathcal{K}$ is irreducible if and only if $\mathbb{C}^*K$ is irreducible, it must be that $\widetilde{\mathcal{G}}$ is an irreducible group.

By Burnside's theorem $\mathbb{M}_n$ has no proper irreducible subalgebras, and hence, $Span\left(\widetilde{\mathcal{G}}\right)$, being an irreducible subalgebra of $\mathbb{M}_n$, must equal $\mathbb{M}_n$. Thus, $\widetilde{\mathcal{G}}$ contains a basis $T_1, \ldots, T_{n^2}$ of $\mathbb{M}_n$.

Recall that $\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1}$ acts monopotently on A, and

$$(\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1})\, A = \left\{ \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} A \ \middle|\ T \in \widetilde{\mathcal{G}} \right\} = \left\{\ C_T \ \mid\ T \in \widetilde{\mathcal{G}}\ \right\},$$

where

$$C_T \stackrel{\text{def}}{=} \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} A.$$

Clearly, for every $i \le n^2$:

$$\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} C_T \in (\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1})\, A,$$

and so, each $\left[\begin{smallmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{smallmatrix}\right] C_T$ has a sole eigenvalue which we denote by $\omega_i(T)$, and which is an $(n+1)$-st root of unity, since both $\left[\begin{smallmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{smallmatrix}\right] C_T$ and C_T have determinant 1.

Consequently, for every $i = 1, \ldots, n^2$,

$$\text{Trace}\left(\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} C_T\right) = \omega_i(T) \quad \in \quad \Omega_{n+1}.$$

Therefore, for every $T \in \widetilde{\mathcal{G}}$, C_T can be interpreted as a solution of a system

$$\left\{ \text{Trace}\left(\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} Z\right) = \omega_i(T)\ ; \quad i = 1, \ldots, n^2, \right. \tag{3.11}$$

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of n^2 linearly independent equations in $(n+1)^2$ variable entries of $Z \in \mathbb{M}_{n+1}$.

Writing Z as $\left[\begin{smallmatrix} K & u \\ v & t \end{smallmatrix}\right]$ allows us to express the linear system (3.11) as

$$\left\{\operatorname{trace}(T_i\, K) = (n+1)\omega_i(T) - \frac{t}{\det T_i}\ ; \quad i = 1, \dots, n^2, \right. \tag{3.12}$$

making it apparent that the only relevant variables are the complex scalar t and the n^2 entries of K.

Since $T_1, \dots, T_{n^2}$ are linearly independent, for each choice of a complex t there is a unique $K_{(T,t)} \in \mathbb{M}_n$ that is a solution to system (3.12). We shall write K_T for $K_{(T,0)}$, and observe that $\left[\begin{smallmatrix} K_T & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a solution to system (3.11).

Since $\left\{ \ (n+1)\omega_i(T) \ \middle| \ T \in \widetilde{\mathcal{G}} \ \right\}$ is a subset of a finite set $(n+1)\Omega_{n+1}$, the set $\left\{ \ K_T \ \middle| \ T \in \widetilde{\mathcal{G}} \ \right\}$ must be finite.

Furthermore, $C_T - \left[\begin{smallmatrix} K_T & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a solution of the homogenious system

$$\left\{\operatorname{trace}\left(\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} Z\right) = 0\ ; \quad i = 1, \dots, n^2, \right. \tag{3.13}$$

of n^2 linearly independent equations in $(n+1)^2$ variable entries of $Z \in \mathbb{M}_{n+1}$.

Let us compute:

$$C_T - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} A - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix} \tag{3.14}$$

$$= \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} \begin{bmatrix} B_\circ & x_\circ \\ y_\circ & a_\circ \end{bmatrix} - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix} \tag{3.15}$$

$$= \begin{bmatrix} TB_\circ - K_T & Tx_\circ \\ \frac{y_\circ}{\det T} & \frac{a_\circ}{\det T} \end{bmatrix}. \tag{3.16}$$

Using the matricial form of Z, system (3.13) can be expressed as

$$\left\{\frac{t}{\det T_i} + \operatorname{trace}(T_i\, K) = 0\ ; \quad i = 1, \dots, n^2. \right. \tag{3.17}$$

The dimension of the solution space of system (3.13) is $(n+1)^2 - n^2$, i.e., $2n+1$. Using (3.17), we can see that

$$\left\{ \ \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix} \ \middle| \ u, v^T \in \mathbb{M}_{n\times 1} \ \right\}$$

is a $2n$-dimensional subspace of the solution space of system (3.13), and therefore, the system has a non-zero solution of the form $\left[\begin{smallmatrix} K_* & 0 \\ 0 & t_* \end{smallmatrix}\right]$.

If $t_* = 0$, then K_* satisfies the equations

$$\operatorname{trace}(T_i\, K_*) = 0; \quad i = 1, \dots, n^2,$$

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which can only happen if $K_* = 0$, because $T_1, \ldots, T_{n^2}$ are linearly independent. Therefore, $t_* \neq 0$, and so after scaling we see that system (3.13) has a solution of the form $\left[\begin{smallmatrix} K_\circ & 0 \\ 0 & 1 \end{smallmatrix}\right]$, so that

$$\left\{ \begin{bmatrix} \gamma K_\circ & u \\ v & \gamma \end{bmatrix} \;\middle|\; \gamma \in \mathbb{C}, \quad u, v^T \in \mathbb{M}_{n\times 1} \right\}$$

is the general solution of system (3.13).

It follows from (3.16) that

$$\begin{bmatrix} TB_\circ - K_T & Tx_\circ \\ \frac{y_\circ}{\det T} & \frac{a_\circ}{\det T} \end{bmatrix} \in \left\{ \begin{bmatrix} \gamma K_\circ & u \\ v & \gamma \end{bmatrix} \;\middle|\; \gamma \in \mathbb{C}, \quad u, v^T \in \mathbb{M}_{n\times 1} \right\},$$

so that

$$TB_\circ = \frac{a_\circ}{\det T} K_\circ + K_T, \quad \text{for every} \quad T \in \widetilde{\mathcal{G}}. \tag{3.18}$$

Letting $T = I$, we obtain

$$B_\circ = a_\circ K_\circ + K_I,$$

so that

$$a_\circ K_\circ = B_\circ - K_I.$$

Thus, (3.18) can be rewritten as:

$$TB_\circ = K_T + \frac{B_\circ - K_I}{\det T}, \quad \text{for every} \quad T \in \widetilde{\mathcal{G}}. \tag{3.19}$$

Using equation (3.19) we arrive at the following identity for all $S, T \in \widetilde{\mathcal{G}}$:

$$\begin{aligned} K_{ST} + \frac{B_\circ - K_I}{\det ST} &= (ST)B_\circ = S(TB_\circ) = SK_T + \frac{SB_\circ - SK_I}{\det T} \\ &= SK_T + \frac{1}{\det T}\left(K_S + \frac{B_\circ - K_I}{\det S} - SK_I\right) \\ &= SK_T + \frac{K_S - SK_I}{\det T} + \frac{B_\circ - K_I}{\det ST} \end{aligned}$$

from which it follows that

$$K_{ST} = SK_T + \frac{K_S - SK_I}{\det T}, \tag{3.20}$$

for all $S, T \in \widetilde{\mathcal{G}}$. There are two possibilities.

Case 1. There exists an $S_\circ \in \widetilde{\mathcal{G}}$ such that $K_{S_\circ} \neq S_\circ K_I$.

In this case, for all $T \in \widetilde{\mathcal{G}}$:

$$K_{S_\circ T} - S_\circ K_T = \frac{K_{S_\circ} - S_\circ K_I}{\det T} \neq 0. \tag{3.21}$$

Recall that the set $\left\{ K_P \mid P \in \widetilde{\mathcal{G}} \right\}$ is finite, and consequently, so is the set

$$\left\{ K_{S_\circ T} - S_\circ K_T \mid T \in \widetilde{\mathcal{G}} \right\}.$$

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Thus, from (3.21) we see that $\left\{ \det T \mid T \in \widetilde{\mathcal{G}} \right\}$ is also finite, and the same is true for $\left\{ TB_\circ \mid T \in \widetilde{\mathcal{G}} \right\}$, since

$$TB_\circ = K_T + \frac{B_\circ - K_I}{\det T},$$

by (3.19).

When $B_\circ \neq 0$, which certainly holds true when $n > 1$ (since A is invertible), $\widetilde{\mathcal{G}}$ is finite by Corollary 5 in [10], and so $\mathcal{G}$ is finite as well.

If $n = 1$ and $B_\circ = 0$, then (since $\det A = 1$):

$$A = \begin{bmatrix} 0 & x_\circ \\ -\frac{1}{x_\circ} & a_\circ \end{bmatrix},$$

and the characteristic polynomial of A is

$$p(z) = z^2 - \operatorname{trace}(A)\, z + \det A = z^2 - a_\circ z + 1.$$

Since A is monopotent, $a_\circ = \pm 2$. The characteristic polynomial of $\left[\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right] \begin{bmatrix} 0 & x_\circ \\ -\frac{1}{x_\circ} & a_\circ \end{bmatrix}$ is

$$p_g(z) = z^2 - (\pm 2) z + g,$$

which is a perfect square (for the sake of monopotency of the matrix) if and only if $g = 1$. Thus, $\mathcal{G} = \{1\}$, and we are done with the first case.

Case 2. For all $S \in \widetilde{\mathcal{G}}$: $K_S = SK_I$.

In this case, (3.19) states that for all $T \in \widetilde{\mathcal{G}}$:

$$TB_\circ = TK_I + \frac{B_\circ - K_I}{\det T},$$

and therefore,

$$(\det(T)\, T)\,(B_\circ - K_I) = (B_\circ - K_I)\,. \tag{3.22}$$

Since $\mathcal{G} = \left\{ \det(T)\, T \mid T \in \widetilde{\mathcal{G}} \right\}$, equation (3.22) states that

$$G(B_\circ - K_I) = (B_\circ - K_I), \quad \text{for all} \quad G \in \mathcal{G}.$$

Thus, the non-zero columns of $B_\circ - K_I$ are common eigenvectors of the elements of the irreducible group $\mathcal{G}$, whose elements obviously have no common eigenvectors. Thus, $B_\circ = K_I$, so that for all $T \in \widetilde{\mathcal{G}}$:

$$TB_\circ = K_T \in \left\{ K_P \mid P \in \widetilde{\mathcal{G}} \right\},$$

and the rightmost set is finite. Again, when $B_\circ \neq 0$, which holds true when $n > 1$, $\widetilde{\mathcal{G}}$ is finite by Corollary 5 in [10], so that $\mathcal{G}$ is finite as well, and we have already dealt with the case of $n = 1$ and $B_\circ = 0$ above. This completes the proof of the present case and hence of the theorem. ∎

COROLLARY 3.2. *Suppose that all elements of a subgroup $\mathcal{H}$ of $\mathbb{GL}_{n+1}$ have the form $\left[\begin{smallmatrix} G & 0 \\ 0 & * \end{smallmatrix}\right]$, where the matrices G comprise an irereducible subgroup $\mathcal{G}$ of $\mathbb{GL}_n$.*

If $\mathcal{H}$ acts monopotently on an invertible matrix A, then $\mathcal{H}$ is essentially finite.

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Proof. By our hypothesis,

$$\mathcal{G} \overset{\text{def}}{=} \left\{ G \;\middle|\; \begin{bmatrix} G & 0 \\ 0 & \alpha \end{bmatrix} \in \mathcal{H}, \quad \text{for some} \quad \alpha \in \mathbb{C} \right\}$$

is an irreducible group.

Let us define

$$\mathcal{H}_1 \overset{\text{def}}{=} \left\{ \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; \begin{bmatrix} \alpha T & 0 \\ 0 & \alpha \end{bmatrix} \in \mathcal{H}, \quad \text{for some} \quad \alpha \in \mathbb{C}^* \right\}$$

It is easy to check that $\mathcal{H}_1$ is a group, and that $\mathbb{C}^*\mathcal{H} = \mathbb{C}^*\mathcal{H}_1$. In particular, $\mathcal{H}_1$ acts monopotently on A.

Let $\mathcal{G}_1$ be the compression of $\mathcal{H}_1$ to $\mathcal{M}$; i.e.,

$$\mathcal{G}_1 = \left\{ T \;\middle|\; \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}_1 \right\}.$$

Then $\mathbb{C}^*\mathcal{G} = \mathbb{C}^*\mathcal{G}_1$, and consequently, $\mathcal{G}_1$ is an irreducible group.

By Theorem 3.1, $\mathcal{G}_1$ is finite, and so $\mathcal{H}_1$ is finite. Since

$$\mathcal{H} \subset \mathbb{C}^*\mathcal{H} = \mathbb{C}^*\mathcal{H}_1,$$

our proof is complete. ∎

4. A case of block-upper triangular groups.

LEMMA 4.1. *Suppose that $\mathcal{G}$ is a subgroup of $\mathbb{GL}_n$ and $\varphi : \mathcal{G} \longrightarrow \mathbb{C}^n$. Then the following are equivalent:*

1. $\mathcal{H} \overset{\text{def}}{=} \left\{ \begin{bmatrix} G & \varphi(G) \\ 0 & 1 \end{bmatrix} \;\middle|\; G \in \mathcal{G} \right\}$ *is a subgroup of* $\mathbb{GL}_{n+1}$*;*
2. $\varphi(AB) = A\,\varphi(B) + \varphi(A)$*, for any* $A, B \in \mathcal{G}$*.*

Proof. The forward implication is trivial. To see the reverse implication, observe that under hypothesis (2) $\mathcal{H}$ is a semigroup of invertible matrices, and hence, it is sufficient to show that $\mathcal{H}$ is closed under inversion. Clearly,

$$\begin{bmatrix} G & \varphi(G) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G^{-1} & -G^{-1}\varphi(G) \\ 0 & 1 \end{bmatrix},$$

and we would like to demonstrate that

$$\varphi\left(G^{-1}\right) = -G^{-1}\varphi(G),$$

for $G \in \mathcal{G}$. Let $A = G^{-1}$ and $B = G$ in our hypothesis (2), and arrive at

$$\varphi(I) = G^{-1}\varphi(G) + \varphi\left(G^{-1}\right).$$

The proof will be complete as soon as we show that $\varphi(I) = 0$. This is accomplished by letting $A = I = B$ in our hypothesis (2). ∎

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Lemma 4.2. *Suppose that all elements of a subgroup $\mathcal{H}$ of $\mathbb{GL}_{n+1}$ have the form $\left[\begin{smallmatrix} G & x \\ 0 & 1\end{smallmatrix}\right]$, where the matrices G comprise a subgroup $\mathcal{G}$ of $\mathbb{GL}_n$. Let*

$$\mathcal{N}_\circ = \left\{ x \in \mathbb{C}^n \;\middle|\; \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} \in \mathcal{H} \right\}.$$

Then $\mathcal{N}_\circ$ contains zero, is closed under addition and subtraction, and is invariant (as a set) under $\mathcal{G}$.

Consequently, since the span of $\mathcal{N}_\circ$ is invariant under $\mathcal{G}$, if $\mathcal{G}$ is irreducible, then either $\mathcal{N}_\circ = \{0\}$, or $\mathcal{N}_\circ$ spans $\mathbb{C}^n$.

Proof. The first part of the claim follows immediately from the observations that

$$\begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & x+y \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} I & -x \\ 0 & 1 \end{bmatrix}.$$

Let us also note that

$$\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G & Gy+x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} G^{-1} & -G^{-1}x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & Gy \\ 0 & 1 \end{bmatrix},$$

which shows that $\mathcal{N}_\circ$ is invariant under $\mathcal{G}$. ∎

Lemma 4.3. *Suppose that a group $\mathcal{H}$ satisfying the hypotheses of Lemma 4.2 acts monopotently on an invertible matrix $A \in \mathbb{M}_{n+1}$, where $n \geq 2$.*

If $\mathcal{G}$ is irreducible then there is a function $\varphi : \mathcal{G} \longrightarrow \mathbb{C}^n$ such that

$$\mathcal{H} = \left\{ \begin{bmatrix} G & \varphi(G) \\ 0 & 1 \end{bmatrix} \;\middle|\; G \in \mathcal{G} \right\}.$$

Proof. Scaling A if necessary, we can assume without loss of generality that $\det \mathcal{A} = 1$. Thus, $A = \omega I + N$, for some nilpotent N and some $\omega \in \Omega_{n+1}$. The same holds true for every $\left[\begin{smallmatrix} I & x \\ 0 & 1\end{smallmatrix}\right] A$, where $x \in \mathcal{N}_\circ$. Let us write

$$A = \begin{bmatrix} B_\circ & z_\circ \\ v_\circ & a_\circ \end{bmatrix},$$

so that

$$\operatorname{trace}\left(\begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} A\right) = \operatorname{trace}(A) + \operatorname{trace}(x v_\circ) = \operatorname{trace}(A) + v_\circ x.$$

It follows that for every $x \in \mathcal{N}_\circ$:

$$v_\circ x \in (n+1)(\Omega_{n+1} - \Omega_{n+1}),$$

and the set on the right is finite and independent of x.

Yet as we have seen in Lemma 4.2, since $\mathcal{N}_\circ$ is closed under addition, either $\mathcal{N}_\circ = \{0\}$ or $\mathcal{N}_\circ$ spans $\mathbb{C}^n$. Hence, either $\mathcal{N}_\circ = \{0\}$ or $v_\circ = 0$.

If $v_\circ = 0$, then the irreducible group $\mathcal{G}$ acts monopotently on the invertible matrix $B_\circ$, which is not possible by Corollary 4.6 of [4]. Hence, it must be that $\mathcal{N}_\circ = \{0\}$.

In this case, the identity

$$\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} G & y \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} G^{-1} & -G^{-1}y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & x-y \\ 0 & 1 \end{bmatrix}$$

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demonstrates that for each $G \in \mathcal{G}$ there is a unique x such that $\left[\begin{smallmatrix} G & x \\ 0 & 1 \end{smallmatrix}\right] \in \mathcal{H}$, which is equivalent to the desired conclusion. ∎

THEOREM 4.4. *Suppose that all elements of a subgroup $\mathcal{H}$ of $\mathbb{GL}_{n+1}$ have the form $\left[\begin{smallmatrix} G & x \\ 0 & 1 \end{smallmatrix}\right]$, where the matrices G comprise a bounded irreducible subgroup $\mathcal{G}$ of $\mathbb{GL}_n$, and $n \geq 2$.*

If $\mathcal{H}$ acts monopotently on an invertible matrix, then $\mathcal{H}$ is finite; (equivalently: a simultaneous similarity applied to $\mathcal{H}$ produces $\mathcal{U} \oplus 1$, where $\mathcal{U}$ is a finite unitary group).

Proof. A well-known theorem of Auerbach (see, for example, Theorem 3.1.5 in [9]) states that every bounded subgroup of $\mathbb{M}_n(\mathbb{C})$ is simultaneously similar to a group of unitary matrices. After applying to $\mathcal{H}$ a similarity of the form

$$\begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}^{-1} (\quad) \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix},$$

we may assume that $\mathcal{G}$ is an irreducible unitary group.

Suppose that $\mathcal{H}$ acts monopotently on a matrix $A_\circ$, and we express $A_\circ$ as $\left[\begin{smallmatrix} B_\circ & z_\circ \\ v_\circ & a_\circ \end{smallmatrix}\right]$, assuming without loss of generality that $\det A_\circ = 1$.

Since we know from [4] that irreducible subgroups of $\mathbb{GL}_n$ do not act monopotently on invertible matrices (for $n \geq 2$), we see that $v_\circ \neq 0$ in our case.

By Lemmas 4.1 and 4.3, there is a function $\varphi : \mathcal{G} \longrightarrow \mathbb{C}^n$ such that

$$\mathcal{H} = \left\{ \begin{bmatrix} U & \varphi(U) \\ 0 & 1 \end{bmatrix} \;\middle|\; U \in \mathcal{G} \right\}, \tag{4.23}$$

and

$$\varphi(UW) = U\,\varphi(W) + \varphi(U), \tag{4.24}$$

for any $U, W \in \mathcal{G}$.

Let us write

$$H_U \stackrel{\text{def}}{=} \begin{bmatrix} U & \varphi(U) \\ 0 & 1 \end{bmatrix}, \quad D_U \stackrel{\text{def}}{=} \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad N_U \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \varphi(U) \\ 0 & 0 \end{bmatrix}.$$

Clearly,

$$\operatorname{trace}(H_U A_\circ) - \operatorname{trace}(D_U A_\circ) = \operatorname{trace}(N_U A_\circ) = \operatorname{trace}(\varphi(U) v_\circ) = v_\circ \varphi(U). \tag{4.25}$$

Let us denote by λ_U the eigenvalue of the monopotent matrix $H_U A_\circ$. Since

$$|\det H_U| = |\det U| = 1 = \det A_\circ,$$

we see that

$$|\lambda_U| = 1 \quad \text{and} \quad |\operatorname{trace}(H_U A_\circ)| \leq n + 1.$$

Since

$$|\operatorname{trace}(D_U A_\circ)| \leq \|D_U\| \cdot \|A_\circ\|_{\text{t-n}} = \operatorname{trace}\left(\sqrt{A_\circ^* A_\circ}\right) \leq (n+1),$$

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we see that $\{\ v_\circ \varphi(U) \mid U \in \mathcal{G}\ \}$ is a bounded set. Of course

$$\begin{aligned}\{\ v_\circ\varphi(U) \mid U \in \mathcal{G}\ \} &= \{\ v_\circ\varphi(WU) \mid U, W \in \mathcal{G}\ \} \\ &\overset{via\ (4.24)}{=} \left\{\ v_\circ\Big(W\varphi(U) + \varphi(W)\Big)\ \Big|\ U, W \in \mathcal{G}\ \right\} \\ &= \{\ (v_\circ W)\varphi(U) + v_\circ\varphi(W) \mid U, W \in \mathcal{G}\ \},\end{aligned}$$

and since $\{\ v_\circ\varphi(W) \mid W \in \mathcal{G}\ \}$ is bounded, we conclude that

$$\{\ (v_\circ W)\varphi(U) \mid U, W \in \mathcal{G}\ \} \quad \text{is a bounded set.} \tag{4.26}$$

Since $\mathcal{G}$ is irreducible and $v_\circ \neq 0$, and

$$\operatorname{span}\{(v_\circ W)^*\} = \operatorname{span}\{W^* v_\circ^*\} = \operatorname{span}\left\{W^{-1} v_\circ^*\right\},$$

which is a non-zero invariant subspace of $\mathcal{G}$, we can conclude that

$$\operatorname{span}\{(v_\circ W)^*\} = \mathbb{C}^n,$$

and therefore, $\{\ v_\circ W \mid W \in \mathcal{G}\ \}$ contains a basis $R_1, \dots, R_n$ of $\mathbb{M}_{1\times n}$. Let $T \in \mathbb{M}_n$ be the invertible matrix with rows $R_1, \dots, R_n$. Then, by (4.26),

$$\{\ T\varphi(U) \mid U \in \mathcal{G}\ \} \quad \text{is a bounded set.}$$

Since

$$\|\varphi(U)\| = \|T^{-1}T\varphi(U)\| \leq \|T^{-1}\| \cdot \|T\varphi(U)\|,$$

it follows that $\{\ \varphi(U) \mid U \in \mathcal{G}\ \}$ is a bounded set, and therefore, $\mathcal{H}$ is bounded (see (4.23)).

Applying Auerbach's theorem to $\mathcal{H}$, we see that after an application of a similarity, we can take $\mathcal{H}$ to be a group of some unitary matrices in $\mathbb{M}_{n+1}$. On the other hand, an application of a similarity does not change the fact that $\mathcal{H}$ has an invariant subspace $\mathcal{M}$ of dimension n that is minimal among non-trivial invariant subspaces of $\mathcal{H}$. Since $\mathcal{H}$ is a group of unitaries, every invariant subspace $\mathcal{L}$ of $\mathcal{H}$ is reducing (with $\mathcal{L}^\perp$ also being invariant). It follows that we can take $\mathcal{H}$ to have a block-diagonal form with respect to the decomposition

$$\mathbb{C}^{n+1} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Recalling that $\mathcal{H}$ has a common eigenvector for the eigenvalue 1, i.e., a "fixed" non-zero vector, and that $\mathcal{H}_{|_\mathcal{M}}$ is irreducible, and that $n \geq 2$, we deduce that $\mathcal{H} = \mathcal{H}_{|_\mathcal{M}} \oplus 1$.

To complete the proof we now simply apply Theorem 3.1 to $\mathcal{H}$. ∎

Notice that the example of $\mathcal{G} = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \ \middle|\ x \in \mathbb{C} \right\}$ and $A = I_2$ shows that the hypothesis "$n \geq 2$" cannot be removed from Lemma 4.3 and Theorem 4.4.

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