



# PERMUTATIVE NONNEGATIVE MATRICES WITH PRESCRIBED SPECTRUM\*

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**Abstract.** An  $n \times n$  permutative matrix is a matrix in which every row is a permutation of the first row. In this paper, the result given by Paparella in [P. Paparella. Realizing Suleimanova spectra via permutative matrices. *Electron. J. Linear Algebra*, 31:306–312, 2016.] is extended to a more general lists of real and complex numbers, and a negative partial answer to a question posed by him is given.

**Key words.** Permutative matrices, Nonnegative matrices, Inverse problems.

**AMS subject classifications.** 15A18.

**1. Introduction.** The *nonnegative inverse eigenvalue problem (NIEP)* is the problem of characterizing all posible spectra of entrywise nonnegative matrices. This problem remains unsolved. A complete solution is known only for  $n \leq 4$ . A list  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  of complex numbers is it realizable if  $\Lambda$  is the spectrum of an  $n \times n$  nonnegative matrix  $A$ . In this case,  $A$  is said to be a realizing matrix. From the Perron-Frobenius theory, we have that if  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is the spectrum of an  $n \times n$  nonnegative matrix  $A$  then  $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$  is an eigenvalue of  $A$ . This eigenvalue is called the *Perron eigenvalue* of  $A$  and we shall assume, in this paper, that  $\rho(A) = \lambda_1$ . A matrix  $A = [a_{ij}]$  is said to have *constant row sums* if each of its row sums is equal to the same constant, say  $\alpha$ , i.e.,  $\sum_{j=1}^n a_{ij} = \alpha$ ,  $i = 1, \dots, n$ . The set of all matrices with constant row sums equal to  $\alpha$  will be denoted by  $\mathcal{CS}_\alpha$ . It is clear that any matrix in  $\mathcal{CS}_\alpha$  has the eigenvector  $\mathbf{e} = (1, 1, \dots, 1)^T$  corresponding to the eigenvalue  $\alpha$ . We shall denote by  $\mathbf{e}_k$  the  $n$ -dimensional vector with one in the  $k$ -th position and zeros elsewhere. The real matrices with constant row sums are important because it is known that the problem of finding a nonnegative matrix with spectrum  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  is equivalent to the problem of finding a nonnegative matrix in  $\mathcal{CS}_{\lambda_1}$  with spectrum  $\Lambda$ .

The following definition, given in [5], is due to Charles Johnson:

**DEFINITION 1.1.** Let  $\mathbf{x} \in \mathbb{C}^n$  and let  $P_2, \dots, P_n$  be  $n \times n$  permutation matrices. A permutative matrix is any matrix of the form

$$P = \begin{bmatrix} \mathbf{x}^T \\ (P_2 \mathbf{x})^T \\ \vdots \\ (P_n \mathbf{x})^T \end{bmatrix}.$$

It is clear that  $P \in \mathcal{CS}_S$ , where  $S$  is the sum of the entries of the vector  $\mathbf{x}$ . A list of real numbers  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 > 0 \geq \lambda_2 \geq \dots \geq \lambda_n$  is called list of Suleimanova type. The following result was announced by Suleimanova [12] and proved by Perfect [6]:

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**THEOREM 1.2.** *Let  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be a list of real numbers with  $\lambda_i < 0$ ,  $i = 2, \dots, n$ . Then  $\Lambda$  is the spectrum of an  $n \times n$  nonnegative matrix if and only if  $\sum_{i=1}^n \lambda_i \geq 0$ .*

In [5], the author proves that each list  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  of real numbers of Suleimanova type [12] is realizable by a permutative nonnegative matrix. The author in [5] also poses the question: can all realizable lists of real numbers be realized by a permutative nonnegative matrix? The following two results, which we give here for completeness, have been shown to be very useful, not only to derive sufficient conditions for the realizability of the NIEP, but for constructing a realizing matrix as well. The first result, due to Brauer [2], shows how to modify a single eigenvalue of a matrix, via a rank-one perturbation, without changing any of its remaining eigenvalues (see [6, 8, 11] and the references therein to see how Brauer's result has been applied to the NIEP). The second result, due to Rado and introduced by Perfect in [7] is an extension of Brauer's result and it shows how to change  $r$  eigenvalues of an  $n \times n$  matrix ( $r < n$ ), via a perturbation of rank  $r$ , without changing any of its remaining  $n - r$  eigenvalues (see [7, 9] to see how Rado's result has been applied to the NIEP). Both results will be employed here to obtain conditions for lists of real and complex numbers to be the spectrum of a permutative nonnegative matrix.

**THEOREM 1.3.** (Brauer, [2]) *Let  $A$  be an  $n \times n$  arbitrary matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Let  $\mathbf{v} = (v_1, \dots, v_n)^T$  be an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda_k$  and let  $\mathbf{q}$  be any  $n$ -dimensional vector. Then the matrix  $A + \mathbf{v}\mathbf{q}^T$  has eigenvalues  $\lambda_1, \dots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \dots, \lambda_n$ .*

**THEOREM 1.4.** (Rado, [7]) *Let  $A$  be an  $n \times n$  arbitrary matrix with spectrum  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ . Let  $X = [\mathbf{x}_1 \mid \dots \mid \mathbf{x}_r]$  be such that  $\text{rank}(X) = r$  and  $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ ,  $i = 1, \dots, r$ ,  $r \leq n$ . Let  $C$  be an  $r \times n$  arbitrary matrix. Then  $A + XC$  has eigenvalues  $\mu_1, \dots, \mu_r, \lambda_{r+1}, \dots, \lambda_n$ , where  $\mu_1, \dots, \mu_r$  are eigenvalues of the matrix  $\Omega + CX$  with  $\Omega = \text{diag}\{\lambda_1, \dots, \lambda_r\}$ .*

A simple proof of Theorem 1.2 was given in [8] by applying Brauer's result. The following result in [10], is a symmetric version of the Rado's result, which we shall use to obtain some of the results in this paper:

**THEOREM 1.5.** [10] *Let  $A$  be an  $n \times n$  real symmetric matrix with spectrum  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , and for some  $r \leq n$ , let  $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$  be an orthonormal set of eigenvectors of  $A$  spanning the invariant subspace associated with  $\lambda_1, \lambda_2, \dots, \lambda_r$ . Let  $X$  be the  $n \times r$  matrix with  $i$ -th column  $\mathbf{x}_i$ , let  $\Omega = \text{diag}\{\lambda_1, \dots, \lambda_r\}$ , and let  $C$  be any  $r \times r$  symmetric matrix. Then the symmetric matrix  $A + XCX^T$  has eigenvalues  $\mu_1, \dots, \mu_r, \lambda_{r+1}, \dots, \lambda_n$ , where  $\mu_1, \dots, \mu_r$  are eigenvalues of the matrix  $\Omega + C$ .*

In this paper, we give very simple, short proofs to show that both, a list of real numbers of Suleimanova type and a list of complex numbers of Suleimanova type, that is,  $\Lambda = \overline{\Lambda}$ ,  $\text{Re}\lambda_k \leq 0$ ,  $|\text{Re}\lambda_k| \geq |\text{Im}\lambda_k|$ ,  $k = 2, \dots, n$ , are realizable by a permutative nonnegative matrix. We use Theorems 1.4 and 1.5 to obtain sufficient conditions for more general lists to be the spectrum of a permutative nonnegative matrix and the spectrum of a symmetric permutative nonnegative matrix. Our results generate an algorithmic procedure to compute a realizing matrix. The paper is organized as follows: In Section 2, we show that a list of real numbers of Suleimanova type is always the spectrum of a permutative nonnegative matrix, and we give sufficient conditions for the problem to have a solution in the case of more general lists of real numbers. We also explore the existence and construction of symmetric permutative nonnegative matrices with prescribed spectrum. We show that the question in [5] has a negative answer, that is, there are realizable lists of real numbers which are not the spectrum of a permutative nonnegative matrix. In Section 3, we consider the case of realizable lists of complex numbers  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  of Suleimanova type, with the condition  $\lambda_{n-j+2} = \overline{\lambda_j}$ ,  $j = 2, 3, \dots, \lceil \frac{n+1}{2} \rceil$ , and we show that they are also realizable by permutative nonnegative

matrices. We also give some examples to illustrate the results.

**2. Permutative matrices with prescribed real spectrum.** In this section, we give a simple, short proof of Theorem 3.3 in [5], and we prove sufficient conditions for the existence of a (symmetric) permutative nonnegative matrix with prescribed real spectrum. We also give a response to the question in [5].

**THEOREM 2.1.** *Let  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be a list of real numbers with  $\lambda_1 > 0$ ,  $\lambda_i < 0$ ,  $i = 2, \dots, n$ . Then  $\Lambda$  is the spectrum of an  $n \times n$  permutative nonnegative matrix if and only if  $\sum_{i=1}^n \lambda_i \geq 0$ .*

*Proof.* The necessity is clear. Suppose that  $\alpha = \sum_{i=1}^n \lambda_i \geq 0$ . Then we take the list  $\Lambda_\alpha = \{\lambda_1 - \alpha, \lambda_2, \dots, \lambda_n\}$  and consider the matrix

$$C = \begin{bmatrix} \lambda_1 - \alpha & 0 & \cdots & \cdots & 0 \\ \lambda_1 - \alpha - \lambda_2 & \lambda_2 & \ddots & & \vdots \\ \lambda_1 - \alpha - \lambda_3 & 0 & \lambda_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \lambda_1 - \alpha - \lambda_n & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \in \mathcal{CS}_{\lambda_1 - \alpha}.$$

From Brauer's result, for  $\mathbf{q} = [\alpha - \lambda_1, -\lambda_2, -\lambda_3, \dots, -\lambda_n]$ , we have that  $B = C + \mathbf{e}\mathbf{q}^T$  is a permutative nonnegative matrix with spectrum  $\Lambda_\alpha$ , and  $B \in \mathcal{CS}_{\lambda_1 - \alpha}$ . Now,  $A = B + \mathbf{e}\mathbf{r}^T$ , where  $\mathbf{r} = [\frac{\alpha}{n}, \frac{\alpha}{n}, \dots, \frac{\alpha}{n}]$ , is the desired permutative nonnegative matrix with spectrum  $\Lambda$  (we may also take  $A = C + \mathbf{e}(\mathbf{q}^T + \mathbf{r}^T)$ ).  $\square$

Now we give sufficient conditions for more general lists of real numbers:

**LEMMA 2.2.** *The matrix*

$$(2.1) \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{11} - \lambda_2 & a_{12} + \lambda_2 & a_{13} & \cdots & a_{1n} \\ a_{11} - \lambda_3 & a_{12} & a_{13} + \lambda_3 & \cdots & a_{1n} \\ \vdots & \vdots & a_{13} & \ddots & \vdots \\ a_{11} - \lambda_n & a_{12} & \cdots & \cdots & a_{1n} + \lambda_n \end{bmatrix}$$

has eigenvalues  $\lambda_1 = \sum_{j=1}^n a_{1j}$ ,  $\lambda_2, \dots, \lambda_n$ .

*Proof.* Since  $A$  has constant row sums equal to  $\sum_{j=1}^n a_{1j}$ , it follows  $\lambda_1 = \sum_{j=1}^n a_{1j}$ . Moreover, it is clear that  $\det(\lambda I - A) = 0$  for  $\lambda = \lambda_i$ ,  $i = 2, \dots, n$ .  $\square$

We point out that Lemma 2.2 can also be easily proved from Brauer's result Theorem 1.3.

**THEOREM 2.3.** *Let  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be a list of real numbers and let  $a_{11}, a_{12}, \dots, a_{1n}$  be real non-negative numbers. If*

$$a_{11} = \frac{1}{n} \sum_{k=1}^n \lambda_k, \quad a_{11} - \lambda_k \geq 0, \quad k = 2, \dots, n,$$

then the matrix  $A$  in (2.1) is permutative nonnegative. If  $\sum_{k=1}^n \lambda_k > 0$  and  $a_{11} - \lambda_k > 0$ , then  $A$  in (2.1) becomes permutative positive.

*Proof.* It is enough to take  $a_{1k} = a_{11} - \lambda_k$ ,  $k = 2, 3, \dots, n$ . Then  $\sum_{k=1}^n a_{1k} = na_{11} - \sum_{k=2}^n \lambda_k = \lambda_1$  and  $a_{11} = \frac{1}{n} \sum_{k=1}^n \lambda_k$ . Thus, the  $k$ -th row of  $A$ ,  $k = 2, \dots, n$ , is a permutation of the first row and  $A$  is an  $n \times n$  permutative nonnegative matrix with spectrum  $\Lambda$ . It is clear that if  $\sum_{k=1}^n \lambda_k > 0$  and  $a_{11} - \lambda_k > 0$ , then  $A$  in (2.1) is positive.  $\square$

THEOREM 2.4. Let  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be a list of real numbers. Suppose that:

(i) There exist a partition  $\Lambda = \Lambda_0 \cup \underbrace{\Lambda_1 \cup \dots \cup \Lambda_1}_{r \text{ times}}$ , where

$$\Lambda_0 = \{\lambda_{01}, \lambda_{02}, \dots, \lambda_{0r}\}, \quad \Lambda_1 = \{\lambda_{11}, \lambda_{12}, \dots, \lambda_{1p}\},$$

such that  $\Gamma_1 = \{\lambda, \lambda_{11}, \lambda_{12}, \dots, \lambda_{1p}\}$ ,  $0 \leq \lambda \leq \lambda_1$ , is the spectrum of a  $(p+1) \times (p+1)$  permutative nonnegative matrix.

(ii) There exists an  $r \times r$  permutative nonnegative matrix with spectrum  $\Lambda_0$  and diagonal entries  $\lambda, \lambda, \dots, \lambda$  ( $r$  times).

Then, there exists a permutative nonnegative matrix  $P$  with spectrum  $\Lambda$ .

*Proof.* From (i), there exists a  $(p+1) \times (p+1)$  permutative nonnegative matrix  $P_1$  with spectrum  $\Gamma_1$ . Let

$$A = \begin{bmatrix} P_1 & & & \\ & P_1 & & \\ & & \ddots & \\ & & & P_1 \end{bmatrix},$$

with  $r$  blocks  $P_1$ . Then  $P_1 \mathbf{x} = \lambda \mathbf{x}$  with  $\mathbf{x} = \frac{1}{\sqrt{p+1}} \mathbf{e}$  (that is,  $\|\mathbf{x}\| = 1$ ), where  $\lambda$  and  $\mathbf{x}$  are the Perron eigenvalue and the Perron eigenvector of  $P_1$ , respectively.

From (ii), there exists an  $r \times r$  permutative nonnegative matrix  $B$  with spectrum  $\Lambda_0$  and diagonal entries  $\lambda, \lambda, \dots, \lambda$  ( $r$  times). Let  $\Omega$  be the  $r \times r$  diagonal matrix  $\Omega = \text{diag}\{\lambda, \lambda, \dots, \lambda\}$ . Then for

$$C = B - \Omega, \quad X = \begin{bmatrix} \mathbf{x} & 0 & \dots & 0 \\ 0 & \mathbf{x} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{x} \end{bmatrix},$$

where  $X$  is the  $r(p+1) \times r$  matrix of eigenvectors of  $A$ , it follows that  $XCX^T$  is a permutative nonnegative matrix, and from Theorem 1.4, with  $C = CX^T$ ,  $P = A + XCX^T$  is a permutative nonnegative matrix with spectrum  $\Lambda$ . Observe that  $P$  is also an  $r \times r$  block permutative nonnegative matrix.  $\square$

EXAMPLE 2.5. Let  $\Lambda = \{10, 4, 2, 0, -1, -1, -1, -1, -3, -3, -3, -3\}$ . We take the partition

$$\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_1 \cup \Lambda_1 \cup \Lambda$$

with

$$\Lambda_0 = \{10, 4, 2, 0\}, \quad \Lambda_1 = \{-1, -3\}, \quad \Gamma_1 = \{4, -1, -3\}.$$

Then we look for a permutative nonnegative matrix  $P_1$  with spectrum  $\Gamma_1$ , and a permutative nonnegative matrix  $B$  with spectrum  $\Lambda_0$  and diagonal entries 4, 4, 4, 4. Such matrices are

$$P_1 = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 2 & 4 \\ 0 & 4 & 2 & 4 \\ 2 & 0 & 4 & 4 \\ 4 & 0 & 2 & 4 \end{bmatrix},$$

obtained from Theorem 1.3 and Theorem 2.3, respectively. Observe that  $B$  cannot be obtained from Theorem 2.1. Let

$$A = \begin{bmatrix} P_1 & & & \\ & P_1 & & \\ & & P_1 & \\ & & & P_1 \end{bmatrix}, \quad X = \begin{bmatrix} \mathbf{x} & 0 & 0 & 0 \\ 0 & \mathbf{x} & 0 & 0 \\ 0 & 0 & \mathbf{x} & 0 \\ 0 & 0 & 0 & \mathbf{x} \end{bmatrix}, \quad \text{with } \mathbf{x} = \frac{1}{\sqrt{3}}\mathbf{e}.$$

Then for  $C = B - \text{diag}\{4, 4, 4, 4\}$ , we have that

$$P = A + XCX^T = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ 1 & 0 & 3 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ 3 & 1 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 & 3 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 1 & 0 & 3 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 3 & 1 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 1 & 3 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 1 & 0 & 3 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 3 & 1 & 0 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 1 & 3 \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 1 & 0 & 3 \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 3 & 1 & 0 \end{bmatrix}$$

is a permutative nonnegative matrix with spectrum  $\Lambda$ . Observe that  $P$  is also a  $4 \times 4$  block permutative nonnegative matrix with permutative blocks.

REMARK 2.6. If, in the proof of Theorem 2.4, the matrices  $P_1$  and  $B$  can be chosen as symmetric permutative nonnegative, then  $A + XCX^T$  becomes symmetric permutative nonnegative. In fact, if  $r = 3$ ,

for instance, we have that

$$\begin{aligned} & \begin{bmatrix} P_1 & & \\ & P_1 & \\ & & P_1 \end{bmatrix} + \begin{bmatrix} \mathbf{x} & 0 & 0 \\ 0 & \mathbf{x} & 0 \\ 0 & 0 & \mathbf{x} \end{bmatrix} \begin{bmatrix} 0 & c & c \\ c & 0 & c \\ c & c & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}^T & 0 & 0 \\ 0 & \mathbf{x}^T & 0 \\ 0 & 0 & \mathbf{x}^T \end{bmatrix} \\ &= \begin{bmatrix} P_1 & & \\ & P_1 & \\ & & P_1 \end{bmatrix} + \begin{bmatrix} 0 & c\mathbf{x}\mathbf{x}^T & c\mathbf{x}\mathbf{x}^T \\ c\mathbf{x}\mathbf{x}^T & 0 & c\mathbf{x}\mathbf{x}^T \\ c\mathbf{x}\mathbf{x}^T & c\mathbf{x}\mathbf{x}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} P_1 & c\mathbf{x}\mathbf{x}^T & c\mathbf{x}\mathbf{x}^T \\ c\mathbf{x}\mathbf{x}^T & P_1 & c\mathbf{x}\mathbf{x}^T \\ c\mathbf{x}\mathbf{x}^T & c\mathbf{x}\mathbf{x}^T & P_1 \end{bmatrix} \end{aligned}$$

is symmetric permutative nonnegative.

In particular, for  $n = 3$ , we have the following pattern of symmetric permutative nonnegative matrices  $B$ :

$$i) \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}, \quad ii) \begin{bmatrix} a & a & b \\ a & b & a \\ b & a & a \end{bmatrix}, \quad iii) \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix},$$

with eigenvalues of the form  $\lambda_1, \lambda_2, -\lambda_2$ , and  $\lambda_1, \lambda_2, \lambda_2$ .

Conditions for cases *ii*) and *iii*) are

$$\begin{aligned} ii) \quad a &= \frac{1}{3}(\lambda_1 + \lambda_2), \quad b = \frac{1}{3}(\lambda_1 - 2\lambda_2), \\ iii) \quad a &= \frac{1}{3}(\lambda_1 + 2\lambda_2), \quad b = \frac{1}{3}(\lambda_1 - \lambda_2). \end{aligned}$$

However, except for the case *iii*),  $C = B - \text{diag}B$  need not to be permutative. Consider the following example:

EXAMPLE 2.7. Let  $\Lambda = \{8, 6, 3, 3, -5, -5, -5, -5\}$  with the partition

$$\Lambda_0 = \{8, 6, 3, 3\}, \quad \Lambda_1 = \{-5\}, \quad \Gamma_1 = \{5, -5\}.$$

We compute the matrices

$$P_1 = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & \frac{1}{2} & \frac{1}{2} \\ 2 & 5 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 5 & 2 \\ \frac{1}{2} & \frac{1}{2} & 2 & 5 \end{bmatrix},$$

with spectrum  $\Gamma_1$  and spectrum  $\Lambda_0$  and diagonal entries 5, 5, 5, 5, respectively. Then

$$A = \begin{bmatrix} P_1 & & & \\ & P_1 & & \\ & & P_1 & \\ & & & P_1 \end{bmatrix} + XCX^T = \begin{bmatrix} 0 & 5 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 5 & 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 0 & 5 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & 5 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 5 & 1 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 5 & 0 & 1 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 & 1 & 0 & 5 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 & 1 & 5 & 0 \end{bmatrix}$$

is symmetric permutative nonnegative with spectrum  $\Lambda$ .

We finish this section by given a negative answer the question in [5], that is, we show that there are lists of real numbers which are the spectrum of a nonnegative matrix, but not the spectrum of a permutative nonnegative matrix. We shall need the following result, due to Perfect [7]:

**THEOREM 2.8.** (Perfect, [7]) *The real numbers  $\lambda_1, \lambda_2, \lambda_3$  and  $\omega_1, \omega_2, \omega_3$ , are respectively, the eigenvalues and the diagonal entries of a  $3 \times 3$  nonnegative matrix with Perron eigenvalue  $\lambda_1$ , if and only if*

- i)  $0 \leq \omega_i \leq \lambda_1, \quad i = 1, 2, 3,$
- ii)  $\lambda_1 + \lambda_2 + \lambda_3 = \omega_1 + \omega_2 + \omega_3,$
- iii)  $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \leq \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3,$
- iv)  $\max_{1 \leq i \leq 3} \omega_i \geq \lambda_2.$

Perfect proposes the following matrix with the required properties

$$\begin{bmatrix} \omega_1 & 0 & \lambda_1 - \omega_1 \\ \lambda_1 - \omega_2 - p & \omega_2 & p \\ 0 & \lambda_1 - \omega_3 & \omega_3 \end{bmatrix},$$

where  $p = \frac{1}{\lambda_1 - \omega_3} (\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3).$

**LEMMA 2.9.** *There is no permutative nonnegative matrix with spectrum  $\Lambda = \{6, 5, 1\}.$*

*Proof.* It is clear that  $\Lambda = \{6, 5, 1\}$  is realizable (trivially by  $A = \text{diag}\{6, 5, 1\}$ ). Suppose  $P$  is a permutative nonnegative matrix with spectrum  $\Lambda$  and first row  $(a, b, c)$ . Then  $a + b + c = 6$ ,  $P \in \mathcal{CS}_6$ ,  $\text{tr}(P) = 12$ . We have three cases:

i) The entries of the main diagonal of  $P$  are of the form  $x, x, x$ . Then  $x = 4$  and the condition iv) in Theorem 2.8 is not satisfied.

ii) The entries of the main diagonal of  $P$  are of the form  $x, x, y$ . Then  $2x + y = 12$ , and from Perfect conditions of Theorem 2.8,  $x \geq 5$ ,  $y \leq 2$  or  $y \geq 5$ ,  $x \leq \frac{7}{2}$ , which contradicts  $x + y + z = 6$ , except for  $x = 6$ ,  $y = z = 0$ . In this last case however, the conditions in Theorem 2.8 are not satisfied either.

iii) The main diagonal entries of  $P$  are distinct. Then  $x + y + z = 12$  contradicts  $x + y + z = 6$ .

Thus,  $\Lambda$  cannot be the spectrum of a permutative nonnegative matrix.  $\square$

Observe, however that  $\Lambda = \{6, 5, 1\}$  is the spectrum of the direct sum of permutative nonnegative matrices

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

After this paper was submitted, R. Loewy [4] showed that a realizable list of real numbers need not to be the spectrum of a permutative nonnegative matrix nor the spectrum of a direct sum of permutative nonnegative matrices.

**3. Permutative matrices with prescribed complex spectrum.** In this section, we show that certain lists of complex numbers are realizable by permutative nonnegative matrices. First we recall some basic facts about circulant matrices. An  $n \times n$  circulant matrix is a matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ddots & \vdots \\ \vdots & c_{n-1} & \ddots & \ddots & c_2 \\ c_2 & \vdots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix},$$

and it is uniquely determined by the entries of its first row, which we denoted by  $\mathbf{c} = (c_0, c_1, \dots, c_{n-1})$ . It is clear that  $C$  is also permutative. Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  with

$$\begin{aligned} \lambda_1 &= c_0 + c_1 + \cdots + c_{n-1}, \\ \lambda_j &= c_0 + c_1 \omega^{j-1} + c_2 \omega^{2(j-1)} + \cdots + c_{n-1} \omega^{(n-1)(j-1)}, \quad j = 2, \dots, n, \\ \omega &= \exp\left(\frac{2\pi i}{n}\right), \end{aligned}$$

being the eigenvalues of the circulant matrix  $C = \text{circ}(c_0, c_1, \dots, c_{n-1})$ . Then

$$(3.2) \quad \lambda_{n-j+2} = \overline{\lambda_j}, \quad j = 2, 3, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Let

$$F = (f_{kj}) = [\mathbf{1} \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n]$$

with

$$\mathbf{1} = (1, 1, \dots, 1)^T, \quad \mathbf{v}_j = \left(1, \omega^{j-1}, \omega^{2(j-1)}, \dots, \omega^{(n-1)(j-1)}\right)^T, \quad j = 2, \dots, n.$$

Then

$$f_{kj} = \omega^{(k-1)(j-1)}, \quad 1 \leq k, j \leq n, \quad F\overline{F} = \overline{F}F = nI,$$

and

$$(3.3) \quad F\mathbf{c} = \lambda \quad \text{and} \quad \mathbf{c} = \frac{1}{n}\overline{F}\lambda.$$

We also recall a result of Loewy and London [3], which solves the NIEP for  $n = 3$ :



**THEOREM 3.1.** *Let  $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$  be a list of complex numbers. Then  $\Lambda$  is the spectrum of a nonnegative matrix if and only if*

$$\Lambda = \overline{\Lambda}, \quad \lambda_1 \geq |\lambda_j|, \quad j = 2, 3, \quad \lambda_1 + \lambda_2 + \lambda_3 \geq 0, \quad (\lambda_1 + \lambda_2 + \lambda_3)^2 \leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

Then, we have the following:

**COROLLARY 3.2.** *Every realizable list  $\Lambda = \{\lambda_1, a+bi, a-bi\}$  of complex numbers is in particular realizable by a permutative nonnegative matrix.*

*Proof.* The realizing matrix in the proof of Theorem 3.1 is circulant. Since circulant matrices are permutative the result follows.  $\square$

**REMARK 3.3.** Theorem 2.4 can also be applied to a list of complex numbers, as the following example shows:

**EXAMPLE 3.4.** Let  $\Lambda = \{3, 2, 1, -1 \pm i, -1 \pm i, -1 \pm i\}$  with

$$\Lambda_0 = \{3, 2, 1\}, \quad \Gamma_1 = \{2, -1 + i, -1 - i\}.$$

The matrices

$$P_1 = \begin{bmatrix} 0 & 1 - \frac{\sqrt{3}}{3} & 1 + \frac{\sqrt{3}}{3} \\ 1 + \frac{\sqrt{3}}{3} & 0 & 1 - \frac{\sqrt{3}}{3} \\ 1 - \frac{\sqrt{3}}{3} & 1 + \frac{\sqrt{3}}{3} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

are permutative with spectrum  $\Gamma_1$  and  $\Lambda_0$ , respectively, with  $B$  being computed from Theorem 2.3. Moreover,  $B$  has the required diagonal entries. Then

$$P = \begin{bmatrix} P_1 & & \\ & P_1 & \\ & & P_1 \end{bmatrix} + X C X^T = \begin{bmatrix} P_1 & 0 & \frac{1}{3} \mathbf{e} \mathbf{e}^T \\ 0 & P_1 & \frac{1}{3} \mathbf{e} \mathbf{e}^T \\ 0 & \frac{1}{3} \mathbf{e} \mathbf{e}^T & P_1 \end{bmatrix}$$

is permutative nonnegative with spectrum  $\Lambda$ .

In [1], the authors proved that a list of complex numbers of Suleimanova type, that is, a list  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  with  $\lambda_1 > 0$ ,  $\lambda_i \geq |\lambda_i|$ ,  $i = 2, \dots, n$ , and

$$\lambda_j \in \{z \in \mathbb{C} : \operatorname{Re} z \leq 0, |\operatorname{Re} z| \geq |\operatorname{Im} z|\}, \quad j = 2, 3, \dots, n,$$

is realizable by a nonnegative matrix if and only if  $\sum_{i=1}^n \lambda_i \geq 0$ .

The following result shows that a list of complex numbers of Suleimanova type, with the property (3.2), is also realizable by a permutative nonnegative matrix, indeed by a circulant nonnegative matrix.

**THEOREM 3.5.** *Let  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  be a list of complex numbers with  $\Lambda = \overline{\Lambda}$ , and*

$$\lambda_j \in \{z \in \mathbb{C} : \operatorname{Re} z \leq 0, |\operatorname{Re} z| \geq |\operatorname{Im} z|\}, \quad j = 2, 3, \dots, n,$$

*satisfying  $\lambda_{n-j+2} = \overline{\lambda_j}$ ,  $j = 2, 3, \dots, \lfloor \frac{n+1}{2} \rfloor$ . Then  $\Lambda$  is the spectrum of a permutative nonnegative matrix if and only if  $\sum_{j=1}^n \lambda_j \geq 0$ .*

*Proof.* The condition is necessary. Suppose  $\sum_{j=1}^n \lambda_j \geq 0$ . We shall prove that  $\Lambda$  is the spectrum of a nonnegative circulant matrix, and therefore  $\Lambda$  is the spectrum of a nonnegative permutative matrix. From (3.3) the explicit formulas for the  $c_k$  are

$$(3.4) \quad c_k = \frac{1}{2m+1} \left( \lambda_1 + 2 \sum_{j=2}^{m+1} \operatorname{Re} \lambda_j \cos \frac{2k(j-1)\pi}{2m+1} + 2 \sum_{j=2}^{m+1} \operatorname{Im} \lambda_j \sin \frac{2k(j-1)\pi}{2m+1} \right),$$

$k = 0, 1, \dots, 2m$ , if  $n = 2m + 1$ , and

$$c_k = \frac{1}{2m+2} \left( \lambda_1 + 2 \sum_{j=2}^{m+1} \operatorname{Re} \lambda_j \cos \frac{k(j-1)\pi}{m+1} + (-1)^k \lambda_{m+2} + 2 \sum_{j=2}^{m+1} \operatorname{Im} \lambda_j \sin \frac{k(j-1)\pi}{m+1} \right),$$

$k = 0, 1, \dots, 2m + 1$ , if  $n = 2m + 2$ . Now, in order to show the nonnegativity of  $c_k$  we take  $\mu = -\sum_{j=2}^n \lambda_j$ .

Then the list  $\Lambda_\mu = \{\mu, \lambda_2, \dots, \lambda_n\}$ , with  $\mu + \sum_{j=2}^n \lambda_j = 0$  is realizable. Then, for  $n = 2m + 1$ , we have

$$(3.5) \quad c_k = \frac{1}{2m+1} \left( \mu + 2 \sum_{j=2}^{m+1} \operatorname{Re} \lambda_j \cos \frac{2k(j-1)\pi}{2m+1} + 2 \sum_{j=2}^{m+1} \operatorname{Im} \lambda_j \sin \frac{2k(j-1)\pi}{2m+1} \right)$$

and

$$(3.6) \quad c_k = \frac{1}{2m+1} \left( 2 \sum_{j=2}^{m+1} \left( \cos \frac{2k(j-1)\pi}{2m+1} - 1 \right) \operatorname{Re} \lambda_j + 2 \sum_{j=2}^{m+1} \operatorname{Im} \lambda_j \sin \frac{2k(j-1)\pi}{2m+1} \right),$$

$k = 0, 1, \dots, 2m$ . Since  $\operatorname{Re} \lambda_j \leq 0$ ,  $\left( \cos \frac{2k(j-1)\pi}{2m+1} - 1 \right) \operatorname{Re} \lambda_j \geq 0$ ,  $j = 2, \dots, m + 1$ . Moreover, if the angle  $\theta$  is located in quadrant *I* or quadrant *II*, then  $\sin \theta \geq 0$  and the sum in (3.6) is nonnegative. If  $\theta$  is located in quadrant *III*, then  $\sin \theta, \cos \theta \leq 0$ . However,  $|\cos \theta - 1| \geq 1 \geq |\sin \theta|$  and since  $|\operatorname{Re} \lambda_j| \geq |\operatorname{Im} \lambda_j|$ , we have

$$(\cos \theta - 1) \operatorname{Re} \lambda_j \geq |\operatorname{Im} \lambda_j \sin \theta|.$$

Then the sum in (3.6) is also nonnegative. In quadrant *IV*, the negativity of  $\sin \theta$  in some terms in the sum  $\sum_{j=2}^{m+1} \operatorname{Im} \lambda_j \sin \frac{2k(j-1)\pi}{2m+1}$  is compensated by the positivity of  $\sum_{j=2}^{m+1} \left( \cos \frac{2k(j-1)\pi}{2m+1} - 1 \right) \operatorname{Re} \lambda_j$ . Thus,  $c_k \geq 0$ ,  $k = 0, 1, \dots, 2m$ , and  $C = \operatorname{circ}(c_0, c_1, \dots, c_{n-1})$  is a circulant nonnegative matrix with spectrum  $\Lambda_\mu$ , which is also permutative nonnegative. The proof is similar for  $n = 2m + 2$ . If  $\sum_{j=1}^n \lambda_j = \alpha > 0$ , then  $\alpha = \lambda_1 - \mu$ , and the matrix  $C' = C + \frac{\alpha}{n} \mathbf{e} \mathbf{e}^T$  is circulant nonnegative (permutative nonnegative) with spectrum  $\Lambda$ .  $\square$

**REMARK 3.6.** We observe that circulant matrices are permutative, but permutative matrices are not circulant. Moreover, if  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  is the spectrum of a circulant matrix, then  $\lambda_1, \dots, \lambda_n$  must satisfy (3.2), that is, the vector  $\lambda = (\lambda_1, \dots, \lambda_n)$  must be conjugate pair. This condition is not necessary for the spectrum of a permutative matrix. Thus, both problems, the realizability by circulant matrices and the realizability by permutative matrices, are different.

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