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# A NEW KIND OF COMPANION MATRIX*

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**Abstract.** A new kind of companion matrix is introduced, for polynomials of the form $c(\lambda) = \lambda a(\lambda)b(\lambda) + c_0$, where upper Hessenberg companions are known for the polynomials $a(\lambda)$ and $b(\lambda)$. This construction can generate companion matrices with smaller entries than the Fiedler or Frobenius forms. This generalizes Piers Lawrence's Mandelbrot companion matrix. The construction was motivated by use of Narayana-Mandelbrot polynomials, which are also new to this paper.

**Key words.** Companion matrix, Eigenvalue, Narayana's cows sequence, Mandelbrot polynomials, Mandelbrot matrices.

**AMS subject classifications.** 15A18, 15A23, 65F15, 65F50.

## 1. Introduction.

Recently, we generalized the Mandelbrot polynomials

$$p_{n+1} = zp_n^2 + 1, \quad p_0 = 0$$

to the Fibonacci-Mandelbrot polynomials

$$q_{n+1} = zq_nq_{n-1} + 1, \quad q_0 = 0, q_1 = 1$$

and generalized Piers Lawrence's supersparse[1] companion matrix for $p_n$ [8] to an analogous one for $q_n$. See [4], [5] and [7] for details, though we summarize these constructions below.

If $p_n = \det(z\mathbf{I} - \mathbf{M}_n)$ for the Mandelbrot polynomials, then the subdiagonals of $\mathbf{M}_n$ are all $-1$ and the matrices are the same size, which gives

$$\mathbf{M}_{n+1} = \begin{bmatrix} \mathbf{M}_n & & -\mathbf{c}_n\mathbf{r}_n \\ -\mathbf{r}_n & 0 & \\ & -\mathbf{c}_n & \mathbf{M}_n \end{bmatrix}, \tag{1.1}$$

where $\mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ are both of length $d_n$. This is Piers Lawrence's original construction [8]. These are remarkable matrices: They contain only $-1$ or 0, and therefore are Bohemian matrices[2]; yet the characteristic polynomial contains coefficients that grow exponentially in the degree $d_n$ (doubly exponentially in $n$).

For the Fibonacci-Mandelbrot polynomials, the degree of $q_n = F_n - 1$ and the construction contains matrices of different size. We begin with

$$\mathbf{M}_3 = \begin{bmatrix} -1 \end{bmatrix}$$

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[1] A matrix is supersparse if it is sparse and its nonzero elements are drawn from a small set, e.g. $\{-1, 1\}$.

[2] The name "Bohemian" is an acronym for Bounded height matrix of integers. See example OEIS A272658.

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and

$$\mathbf{M}_4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

to construct our recursive companion matrix:

$$\mathbf{M}_{n+1} = \begin{bmatrix} \mathbf{M}_n & & (-1)^{d_{n+1}}\mathbf{c}_n\mathbf{r}_{n-1} \\ -\mathbf{r}_n & 0 & \\ & -\mathbf{c}_{n-1} & \mathbf{M}_{n-1} \end{bmatrix},$$

where $\mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $\mathbf{c}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ are, as before, the row and column vectors of length $d_n$. This gives a matrix of slightly greater height than (1.1) because the entries may be $\{-1, 0, 1\}$.

The surprising analogy between these two families of supersparse companions led us to conjecture and prove the following.

## 2. Main result.

THEOREM 2.1. *Suppose $a(z) = \det(z\mathbf{I}-\mathbf{A})$, $b(z) = \det(z\mathbf{I}-\mathbf{B})$, and both $\mathbf{A}$ and $\mathbf{B}$ are upper Hessenberg matrices with nonzero subdiagonal entries, and*

$$\alpha = \frac{1}{\left(\prod_{j=1}^{d_a-1} a_{j+1,j}\right)\left(\prod_{j=1}^{d_b-1} b_{j+1,j}\right)}$$

*is the reciprocal of the product of the subdiagonal entries of $\mathbf{A}$ and $\mathbf{B}$, and $d_a = \deg_z a$ and $d_b = \deg_z b$, so the dimension of $\mathbf{A}$ is $d_a \times d_a$ and the dimension of $\mathbf{B}$ is $d_b \times d_b$. Suppose both $d_a$ and $d_b$ are at least 1. Then if*

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & & -\alpha c_0 \mathbf{c_a r_b} \\ -\mathbf{r_a} & 0 & \\ & -\mathbf{c_b} & \mathbf{B} \end{bmatrix},$$

*where $\mathbf{r_a} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ of length $d_a$ and $\mathbf{c_b} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ of length $d_b$, we have*

$$c(z) = \det(z\mathbf{I} - \mathbf{C}) = z \cdot a(z)b(z) + c_0.$$

REMARK 2.2. Proving this theorem automatically proves the validity of the constructions of the supersparse companion matrices for $p_n$, $q_n$, and $r_n$.

REMARK 2.3. Starting with a polynomial $c(z)$, we see that there are potentially many such $a(z)$ and $b(z)$. This freedom may be quite valuable or, it may be an obstacle.

*Proof.* Partition

$$z\mathbf{I} - \mathbf{C} = \left[\begin{array}{c:c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hdashline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right],$$

where $\mathbf{C}_{22} = z\mathbf{I} - \mathbf{B}$ is nonsingular if $z$ is not an eigenvalue of $\mathbf{B}$, i.e., $b(z) \neq 0$. Later we will remove this

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restriction. Also,

$$\mathbf{C}_{21} = \begin{bmatrix} & 1 \\ & \\ & \end{bmatrix}$$

is $d_b \times (d_a + 1)$ and has only one nonzero element, which is a 1 in the upper right corner. Next,

$$\mathbf{C}_{12} = \begin{bmatrix} & \alpha c_0 \\ & \\ & \end{bmatrix}$$

is $(1 + d_a) \times d_b$ and again has only one nonzero element, $\alpha c_0$ in the upper right corner. (In fact, $c_0$ can be zero.) This leaves

$$\mathbf{C}_{11} = \left[\begin{array}{c:c} & 0 \\ z\mathbf{I} - \mathbf{A} & \vdots \\ & 0 \\ & 0 \\ \hdashline \qquad\quad 1 & z \end{array}\right],$$

which is $d_a + 1$ by $d_a + 1$.

The Schur factoring is

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{C}_{12} \\ 0 & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} & 0 \\ \mathbf{C}_{22}^{-1}\mathbf{C}_{21} & \mathbf{I} \end{bmatrix},$$

with the computation of the Schur complement $\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$ going to do most of the work in the proof. The Schur determinantal formula [10, Chapter 12] is then

$$\det \mathbf{C} = \det(\mathbf{C}_{22}) \det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right).$$

We have the following propositions.

1. $z\mathbf{I} - \mathbf{A}$ and $z\mathbf{I} - \mathbf{B}$ are upper Hessenberg because $\mathbf{A}$ and $\mathbf{B}$ are.
2. The first $d_a$ columns of $\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$ are zero.
3. The final column of $\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$ is the solution, say $\vec{v}$, of $(z\mathbf{I} - \mathbf{B})\,\vec{v} = \mathbf{e}_1$. Again, $z\mathbf{I} - \mathbf{B}$ is nonsingular.
4. By Cramer's rule, the final entry in $\vec{v}$, say $v$, is

$$v = \frac{\det\left(\mathbf{C}_{22} \underset{d_b}{\leftarrow} \mathbf{e}_1\right)}{\det(\mathbf{C}_{22})},$$

where the notation $\mathbf{M} \underset{k}{\leftarrow} \vec{v}$ means replace the $k$th column of $\mathbf{M}$ with the vector $\vec{v}$ [3].

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5. Since $\mathbf{C}_{22} = z\mathbf{I} - \mathbf{B}$ is upper Hessenberg,

$$\mathbf{C}_{22} \underset{d_b}{\longleftarrow} e_1 = \begin{bmatrix} * & * & * & \cdots & * & 1 \\ -b_{21} & * & * & \cdots & * & 0 \\ & -b_{32} & * & & \vdots & \vdots \\ & & -b_{43} & \ddots & & \\ & & & \ddots & & \\ & & & & * & 0 \\ & & & & -b_{d_b,d_b-1} & 0 \end{bmatrix}.$$

Laplace expansion about the final column gives

$$\begin{aligned} \det\left(\mathbf{C}_{22} \underset{d_b}{\longleftarrow} \mathbf{e}_1\right) &= (-1)^{d_b-1}(-1)^{d_b-1}\prod_{j=1}^{d_b-1} b_{j+1,j} \\ &= \prod_{j=1}^{d_b-1} b_{j+1,j}. \end{aligned}$$

Therefore,

$$v = \frac{\prod_{j=1}^{d_b-1} b_{j+1,j}}{b(z)}$$

because $\det \mathbf{C}_{22} = \det(z\mathbf{I} - \mathbf{B}) = b(z)$ by hypothesis.

6. Now

$$\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21} = \begin{bmatrix} & & \alpha c_0 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} * \\ \vdots \\ * \\ v \end{bmatrix} = \begin{bmatrix} & & \alpha c_0 v \\ & & \\ & & \end{bmatrix}$$

is $d_a + 1$ by $d_a + 1$ and has its only nonzero entry, $\alpha c_0 v$, in the upper right corner.

7. The Schur complement is therefore

$$\left[\begin{array}{cccc:c} & & & & -\alpha c_0 v \\ & & & & 0 \\ & z\mathbf{I} - \mathbf{A} & & & \vdots \\ & & & & 0 \\ \hdashline 0 & \cdots & 0 & 1 & z \end{array}\right],$$

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and we compute $\det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right)$ by Laplace expansion on the last column:

$$\begin{aligned}
\det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right) &= -(-1)^{d_a}\alpha c_0 v \det\begin{bmatrix} -a_{21} & * & * & \cdot & * \\ & -a_{32} & * & & * \\ & & -a_{43} & & \vdots \\ & & & \ddots & \\ & & & & -a_{d_a,d_a-1} \end{bmatrix} \\
&\quad + z\det(z\mathbf{I} - \mathbf{A}) \\
&= -(-1)^{d_a}\alpha c_0 v \prod_{j=1}^{d_a-1}(-a_{j+1,j}) + z\cdot a(z) \\
&= \alpha v \prod_{j=1}^{d_a-1} a_{j+1,j}\cdot c_0 + z\cdot a(z) \\
&= \alpha\cdot\frac{\left(\prod_{j=1}^{d_b-1} b_{j+1,j}\right)}{b(z)}\cdot\left(\prod_{j=1}^{d_a-1} a_{j+1,j}\right)\cdot c_0 + z\cdot a(z) \\
&= \frac{c_0}{b(z)} + z\cdot a(z)
\end{aligned}$$

by the definition of $\alpha$.

Therefore, by the Schur determinantal formula,

$$\begin{aligned}
\det(z\mathbf{I} - \mathbf{C}) &= \det(\mathbf{C}_{22})\det\left(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\right) \\
&= b(z)\left(\frac{c_0}{b(z)} + z\cdot a(z)\right) \\
&= z\cdot a(z)b(z) + c_0.
\end{aligned}$$

Since the left hand side is a polynomial as is the right hand side, the formula will be true even if $b(z) = 0$, by continuity. ◻

**3. Applications and examples.** Sequence A000930 of the Online Encyclopedia of Integer Sequences, Narayana's cows sequence, begins

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, \ldots$$

and is generated by $R_n = R_{n-1} + R_{n-3}$ [13]. The connection to cows is that an ideal cow produces a calf every year, starting in its fourth year. Narayana was a mathematician in 14th century India. Various facts are known for this sequence, which is similar to the Fibonacci sequence: For instance, the generating function is $1/(1 - x - x^3)$. Many references are given in the OEIS, but see also [12].

We define the Narayana-Mandelbrot polynomials by $r_0 = 1, r_1 = r_2 = 1$ and

$$r_{n+1} = zr_nr_{n-2} + 1$$

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for $n \geq 2$. We construct a recursive family of companion matrices $\mathbf{R}_n$, i.e., such that

$$r_n(z) = \det(z\mathbf{I} - \mathbf{R}_n).$$

Just as the Fibonacci-Mandelbrot polynomials, the construction contains matrices of different sizes. However, for this family, we start with

$$\mathbf{R}_3 = \left[\begin{array}{c} -1 \end{array}\right],$$

$$\mathbf{R}_4 = \left[\begin{array}{rr} 0 & 1 \\ -1 & -1 \end{array}\right],$$

and

$$\mathbf{R}_5 = \left[\begin{array}{rrr} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{array}\right].$$

Our construction is then

$$\mathbf{R}_{n+1} = \left[\begin{array}{ccc} \mathbf{R}_n & & (-1)^{d_{n+1}}\mathbf{c}_n\mathbf{r}_{n-2} \\ -\mathbf{r}_n & 0 & \\ & -\mathbf{c}_{n-2} & \mathbf{R}_{n-2} \end{array}\right],$$

where $\mathbf{r}_n = \left[\begin{array}{cccc} 0 & 0 & \cdots & 1 \end{array}\right]$ and $\mathbf{c}_n = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]^T$ are, as before, the row and column vectors of length $d_n = \deg r_n = R_{n+1} - 1$.

This construction also allows new matrix families. For instance, suppose $s_0 = 0$, $s_{n+1} = z^3 s_n^4 + 1$. Then if $\mathbf{S}_n$ is an upper Hessenberg companion for $s_n$ (with all $-1$ on the subdiagonal) the matrix

$$\mathbf{S_{n+1}} = \left[\begin{array}{ccccccc} \mathbf{S}_n & & & & & & -\mathbf{c}_n\mathbf{r}_n \\ -\mathbf{r}_n & 0 & & & & & \\ & -\mathbf{c}_n & \mathbf{S}_n & & & & \\ & & -\mathbf{r}_n & 0 & & & \\ & & & -\mathbf{c}_n & \mathbf{S}_n & & \\ & & & & -\mathbf{r}_n & 0 & \\ & & & & & -\mathbf{c}_n & \mathbf{S}_n \end{array}\right]$$

is an upper Hessenberg companion for $s_{n+1}$.

**4. Concluding remarks.** This is a genuinely new kind of companion matrix. We demonstrate this on Newton's example polynomial $x^3 - 2x - 5$. We see that $x^3 - 2x - 5 = x(x^2 - 2) - 5 = x(x - \sqrt{2})(x + \sqrt{2}) - 5$, and companion matrices for $x - \sqrt{2}$ and $x + \sqrt{2}$ are just $[+\sqrt{2}]$ and $[-\sqrt{2}]$ respectively. Thus, a companion matrix for Newton's polynomial is

$$\left[\begin{array}{ccc} \sqrt{2} & & 5 \\ -1 & & \\ & -1 & -\sqrt{2} \end{array}\right].$$

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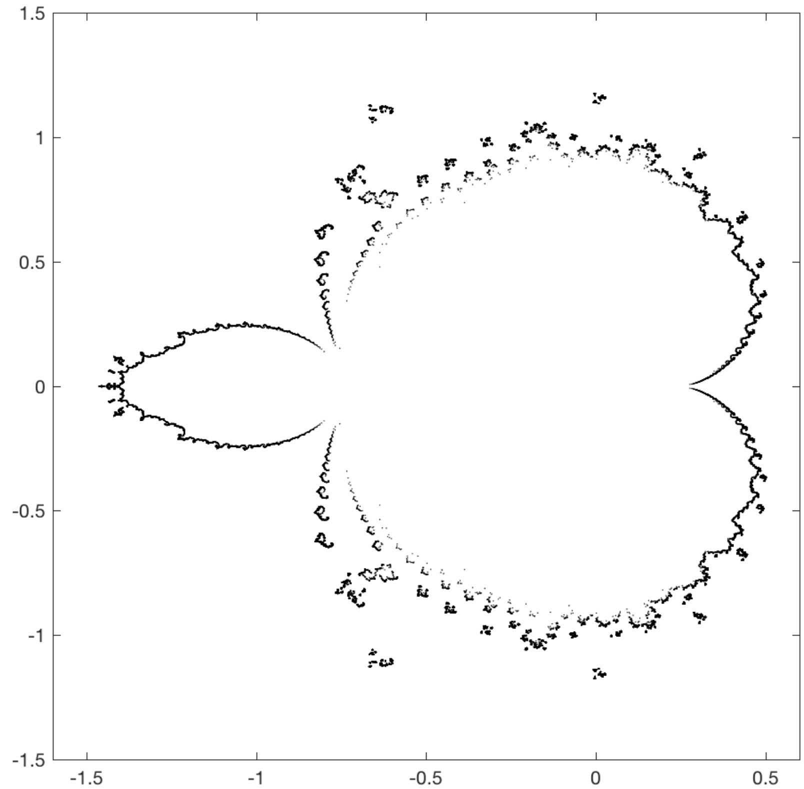


FIGURE 1. *Roots of Narayana-Mandelbrot polynomial, $r_{36}(z)$. The degree of $r_{36}(z)$ is 578,948.*

This matrix contains $\sqrt{5}$, unlike any previously recorded companion matrix. For unimodular polynomials, such companion matrices may be of lower height than the Frobenius or Fiedler [9] companions, and may offer better numerical condition.

We have now established that if $c(z) = z \cdot a(z)b(z) + c_0$ and $\mathbf{A}$ and $\mathbf{B}$ are upper Hessenberg companion matrices for the polynomials $a(z)$ and $b(z)$ respectively, then

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & & -\alpha c_0 \mathbf{c_a} \mathbf{r_b} \\ -\mathbf{r_a} & 0 & \\ & -\mathbf{c_b} & \mathbf{B} \end{bmatrix}$$

is a companion matrix for $c(z)$. One wonders immediately about a corresponding linearization, $\mathbf{L_C}$, strong or otherwise, for the matrix polynomial

$$\mathbf{C}(z) = z\mathbf{A}(z)\mathbf{B}(z) + \mathbf{C_0} \,,$$

if $\mathbf{L_A}$ is a linearization for $\mathbf{A}$, $\mathbf{L_B}$ for $\mathbf{B}$. Some very preliminary experiments, where $\mathbf{L_A}$ and $\mathbf{L_B}$ were block

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upper Hessenberg with all blocks $\mathbf{I}$, so $\alpha = 1$, find that indeed

$$\mathbf{L_C} = \begin{bmatrix} \mathbf{L_A} & & & -\mathbf{C}_0 \\ & -\mathbf{I} & 0 & \\ & & -\mathbf{I} & \\ & & & \mathbf{L_B} \end{bmatrix}$$

is a (strong) linearization for $c(z)$, in the examples we tried.

In a paper to be submitted soon, we have now proved that this construction can be extended to matrix polynomials; see [6].

A referee pointed out that Robol et al. [11] use a similar construction to linearize polynomials of the form $p(z) = a(z)b(z) + zc(z)d(z)$ to find the roots of rational functions, which can also be applied to matrix polynomials.

We leave these extensions to future work.

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