Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.

# THE ENHANCED PRINCIPAL RANK CHARACTERISTIC SEQUENCE OVER A FIELD OF CHARACTERISTIC 2*

XAVIER MARTÍNEZ-RIVERA†

**Abstract.** The enhanced principal rank characteristic sequence (epr-sequence) of an $n \times n$ symmetric matrix over a field $\mathbb{F}$ was recently defined as $\ell_1\ell_2\cdots\ell_n$, where $\ell_k$ is either $\mathtt{A}$, $\mathtt{S}$, or $\mathtt{N}$ based on whether all, some (but not all), or none of the order-$k$ principal minors of the matrix are nonzero. Here, a complete characterization of the epr-sequences that are attainable by symmetric matrices over the field $\mathbb{Z}_2$, the integers modulo 2, is established. Contrary to the attainable epr-sequences over a field of characteristic 0, this characterization reveals that the attainable epr-sequences over $\mathbb{Z}_2$ possess very special structures. For more general fields of characteristic 2, some restrictions on attainable epr-sequences are obtained.

**Key words.** Principal rank characteristic sequence, Enhanced principal rank characteristic sequence, Minor, Rank, Symmetric matrix, Finite field.

**AMS subject classifications.** 15A15, 15A03.

**1. Introduction.** For an $n \times n$ real symmetric matrix $B$, Brualdi et al. [2] introduced the *principal rank characteristic sequence* (abbreviated pr-sequence), which was defined as $\operatorname{pr}(B) = r_0]r_1\cdots r_n$, where, for $k \geq 1$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero principal minor of order } k\text{, and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if $B$ has a 0 diagonal entry. This definition was generalized for symmetric matrices over any field by Barrett et al. [1].

Our focus will be studying a sequence that was introduced by Butler et al. [4] as a refinement of the pr-sequence of an $n \times n$ symmetric matrix $B$ over a field $\mathbb{F}$, which they called the *enhanced principal rank characteristic sequence* (abbreviated epr-sequence), and which was defined as $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all the principal minors of order } k \text{ are nonzero;} \\ \mathtt{S} & \text{if some but not all the principal minors of order } k \text{ are nonzero;} \\ \mathtt{N} & \text{if none of the principal minors of order } k \text{ are nonzero, i.e., all are zero.} \end{cases}$$

The definition of the epr-sequence was later extended to the class of real skew-symmetric matrices in [6], where a complete characterization of the epr-sequences realized by this class was presented. However, things are more subtle for the class of symmetric matrices over a field $\mathbb{F}$, and thus obtaining a similar characterization presents a difficult problem. When $\mathbb{F}$ is of characteristic 0, it is known that any epr-sequence of the form $\ell_1\cdots\ell_{n-k}\overline{\mathtt{N}}$, with $\ell_i \in \{\mathtt{A},\mathtt{S}\}$, is attainable by an $n \times n$ symmetric matrix over $\mathbb{F}$, where $\overline{\mathtt{N}}$ (which may be empty) is the sequence consisting of $k$ consecutive $\mathtt{N}$s [4] – if $\overline{\mathtt{N}}$ is empty, note that we

*Received by the editors on August 28, 2016. Accepted for publication on July 4, 2017. Handling Editor: Michael Tsatsomeros.

†Department of Mathematics, Iowa State University, Ames, IA 50011, USA (xaviermr@iastate.edu).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



must have $\ell_n = \mathtt{A}$. In general, the subtlety for symmetric matrices becomes evident once the Ns are not restricted to occur consecutively at the end of the sequence: Sequences such as NSA, NNA and NNS can never occur as a subsequence of the epr-sequence of a symmetric matrix over any field [4]; the same holds for the sequences NAN and NAS when the field is of characteristic not 2 [4]. Moreover, over fields of characteristic not 2, the sequence ANS can only occur at the start of the sequence [4]. Over the real field, SNA can only occur as a terminal subsequence, or in the terminal subsequence SNAA [10]. Furthermore, over the real field, we also know that when the subsequence ANA occurs as a non-terminal subsequence, it forces every other term of the sequence to be A [10]. However, it is unknown what kind of restrictions a subsequence such as SNS imposes on an attainable sequence (over any field); this is one of the difficulties in arriving at a complete characterization of the epr-sequences attainable by a symmetric matrix over a field $\mathbb{F}$. In order to simplify this problem, it is natural to consider the case when $\mathbb{F}$ is of characteristic 2. The analogous problem for pr-sequences was already settled in [1]:

THEOREM 1.1. [1, Theorem 3.1] *A pr-sequence of order $n \geq 2$ is attainable by a symmetric matrix over a field of characteristic 2 if and only if it has one of the following forms:*

$$0]1\ \overline{1}\ \overline{0}, \qquad 1]\overline{01}\ \overline{0}, \qquad 1]1\ \overline{1}\ \overline{0}.$$

We see that for any two fields of characteristic 2, the class of pr-sequences attainable by symmetric matrices over each of the two fields is the same. This is not true in the case of epr-sequences: Consider an epr-sequence starting with AA over the field $\mathbb{Z}_2 = \{0, 1\}$, the integers modulo 2; over this field, any such sequence must be $\mathtt{AA}\overline{\mathtt{A}}$, since any symmetric matrix attaining this sequence must be the identity matrix. However, in Example 2.5 below, it is shown that the epr-sequence AAN is attainable over a field of characteristic 2, implying that not all fields of characteristic 2 give rise to the same class of attainable epr-sequences. In light of this difficulty, our main focus here will be on the field $\mathbb{F} = \mathbb{Z}_2$; after establishing some restrictions for the attainability of epr-sequences over a field of characteristic 2 at the beginning of Section 2, our main objective is a complete characterization of the epr-sequences that are attainable by symmetric matrices over $\mathbb{Z}_2$ (see Theorems 3.2, 3.8 and 3.11). We find that the attainable epr-sequences over $\mathbb{Z}_2$ possess very special structures, which is in contrast to the family of attainable epr-sequences over a field of characteristic 0 described above.

Another motivating factor for considering this problem is that it is a simplification of the *principal minor assignment problem* as stated in [8], which also served as motivation for the introduction of the pr-sequence in [2]. Note that epr-sequences provide more information than pr-sequences, and thus are a step closer to the principal minor assignment problem.

Extra motivation for this problem comes from the observation that there is a one-to-one correspondence between adjacency matrices of simple graphs and symmetric matrices over $\mathbb{Z}_2$ with zero diagonal, and, more generally, between adjacency matrices of loop graphs and symmetric matrices over $\mathbb{Z}_2$.

It should be noted that, although epr-sequences have received attention after their introduction in [4] (see [5], [6] and [10], for example), very little is known about epr-sequences of symmetric matrices over a field of characteristic 2, since the vast majority of what has appeared in the literature regarding epr-sequences has been focused on fields of characteristic *not* 2.

Although Theorem 1.1 sheds some light towards settling the problem under consideration, it does not render it trivial by any means; one reason is the observation that two symmetric matrices may have distinct

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



epr-sequences while having the same pr-sequence. As it is shown in Theorem 3.8 below, the epr-sequences $\mathtt{ASAA}$ and $\mathtt{ASSA}$, which are associated with the pr-sequence $0]1111$, are both attainable over $\mathbb{Z}_2$.

To highlight a second reason, we state the two results upon which Barrett et al. [1] relied in order to obtain Theorem 1.1 (the latter is a variation of a result of Friedland [7, p. 426]).

Lemma 1.2. [1, Lemma 3.2] *Let $\mathbb{F}$ be a field of characteristic 2, let $B$ be a symmetric matrix over $\mathbb{F}$ with $\operatorname{pr}(B) = r_0]r_1 \cdots r_n$, and let $E$ be an $n \times n$ invertible matrix over $\mathbb{F}$. Then $\operatorname{pr}(EBE^T) = r_0']r_1 r_2 \cdots r_n$ for some $r_0' \in \{0, 1\}$.*

In what follows, $K_n$ denotes the complete graph on $n$ vertices, and $A(K_n)$ denotes its adjacency matrix.

Lemma 1.3. [1, Lemma 3.3] *Let $B$ be a symmetric matrix over a field $\mathbb{F}$ with characteristic 2. Then $B$ is congruent to the direct sum of a (possibly empty) invertible diagonal matrix $D$, and a (possibly empty) direct sum of $A(K_2)$ matrices, and a (possibly empty) zero matrix.*

The two lemmas above permitted Barrett et al. [1] to arrive at their characterization for pr-sequences in Theorem 1.1 by restricting themselves to symmetric matrices that are in the canonical form described in Lemma 1.3. We cannot use this approach to obtain our desired characterization for epr-sequences: Suppose one tries to apply the congruence described in Lemma 1.2 to a symmetric matrix $B$ with $\operatorname{epr}(B) = \mathtt{ASAN}$, which is shown to be attainable in Theorem 3.8. Then, because $B$ is singular, and because multiplication by an invertible matrix preserves the rank of the original matrix, once $B$ has been transformed into the canonical form described in Lemma 1.3, it must be the case that in this resulting matrix the zero summand is non-empty. Thus, the resulting matrix has a zero row (and zero column), which implies that it contains a principal minor of order 3 that is zero. Then, as the principal minors of order 3 of the original matrix $B$ were all nonzero, the congruence performed did not preserve the third term of $\operatorname{epr}(B)$, which is in contrast to what happens to $\operatorname{pr}(B)$, which, with the exception of the zeroth term, must be preserved completely by Lemma 1.2.

We say that a (pr- or epr-) sequence is *attainable* over a field $\mathbb{F}$ provided that there exists a symmetric matrix $B \in \mathbb{F}^{n \times n}$ that attains it. A pr-sequence and an epr-sequence are *associated* with each other if a matrix (which may not exist) attaining the epr-sequence also attains the pr-sequence. A subsequence that does not appear in any attainable sequence is *prohibited*. We say that a sequence has *order n* if it corresponds to a matrix of order $n$. Let $B$ be an $n \times n$ matrix, and let $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$; then the submatrix lying in rows indexed by $\alpha$ and columns indexed by $\beta$ is denoted by $B[\alpha, \beta]$. The matrix obtained by deleting the rows indexed by $\alpha$ and columns indexed by $\beta$ is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while $B(\alpha, \alpha)$ is abbreviated to $B(\alpha)$. The matrices $O_n$ and $I_n$ denote, respectively, the zero and identity matrix of order $n$. We denote by $J_{m,n}$ the $m \times n$ all-1s matrix, and, when $m = n$, $J_{n,n}$ is abbreviated to $J_n$. The block diagonal matrix formed from two square matrices $B$ and $C$ is denoted by $B \oplus C$. The matrices $B$ and $C$ are *permutationally* similar if there exists a permutation matrix $P$ such that $C = P^T BP$. Given a graph $G$, $A(G)$ denotes the adjacency matrix of $G$.

### 1.1. Results cited.

This section lists results that will be cited frequently, with some of them being assigned abbreviated nomenclature.

Theorem 1.4. [4, Theorem 2.3] ($\mathtt{NN}$ Theorem) *Suppose $B$ is a symmetric matrix over a field $\mathbb{F}$,* $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, *and $\ell_k = \ell_{k+1} = \mathtt{N}$ for some $k$. Then $\ell_i = \mathtt{N}$ for all $i \geq k$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



THEOREM 1.5. [4, Theorem 2.4] (Inverse Theorem) *Suppose $B$ is a nonsingular symmetric matrix over a field $\mathbb{F}$. If* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_{n-1}\mathtt{A}$, *then* $\operatorname{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2}\cdots\ell_1\mathtt{A}$.

Given a matrix $B$, the $i$th term in its epr-sequence is denoted by $[\operatorname{epr}(B)]_i$.

THEOREM 1.6. [4, Theorem 2.6] (Inheritance Theorem) *Suppose that $B$ is a symmetric matrix over a field $\mathbb{F}$, $m \leq n$, and $1 \leq i \leq m$.*

1. *If* $[\operatorname{epr}(B)]_i = \mathtt{N}$, *then* $[\operatorname{epr}(C)]_i = \mathtt{N}$ *for all $m \times m$ principal submatrices $C$.*
2. *If* $[\operatorname{epr}(B)]_i = \mathtt{A}$, *then* $[\operatorname{epr}(C)]_i = \mathtt{A}$ *for all $m \times m$ principal submatrices $C$.*
3. *If* $[\operatorname{epr}(B)]_m = \mathtt{S}$, *then there exist $m \times m$ principal submatrices $C_A$ and $C_N$ of $B$ such that* $[\operatorname{epr}(C_A)]_m = \mathtt{A}$ *and* $[\operatorname{epr}(C_N)]_m = \mathtt{N}$.
4. *If $i < m$ and* $[\operatorname{epr}(B)]_i = \mathtt{S}$, *then there exists an $m \times m$ principal submatrix $C_S$ such that* $[\operatorname{epr}(C_S)]_i = \mathtt{S}$.

In the rest of this paper, each instance of $\cdots$ is permitted to be empty.

COROLLARY 1.7. [4, Corollary 2.7] ($\mathtt{NSA}$ Theorem) *No symmetric matrix over any field can have* $\mathtt{NSA}$ *in its epr-sequence. Further, no symmetric matrix over any field can have the epr-sequence* $\cdots\mathtt{ASN}\cdots\mathtt{A}\cdots$.

Given a matrix $B$ with a nonsingular principal submatrix $B[\alpha]$, we denote by $B/B[\alpha]$ the Schur complement of $B[\alpha]$ in $B$ [12]. The next fact is a generalization of [4, Proposition 2.13] to any field; the proof is exactly the same, and is omitted here (we note that the proof was also omitted in [4]).

THEOREM 1.8. (Schur Complement Theorem) *Suppose $B$ is an $n \times n$ symmetric matrix over a field $\mathbb{F}$, with* $\operatorname{rank} B = r$. *Let $B[\alpha]$ be a nonsingular principal submatrix of $B$ with $|\alpha| = k \leq r$, and let $C = B/B[\alpha]$. Then the following results hold.*

(*i*) *$C$ is an $(n-k) \times (n-k)$ symmetric matrix.*
(*ii*) *Assuming the indexing of $C$ is inherited from $B$, any principal minor of $C$ is given by*

$$\det C[\gamma] = \det B[\gamma \cup \alpha]/\det B[\alpha].$$

(*iii*) $\operatorname{rank} C = r - k$.

The next result, which is immediate from the Schur Complement Theorem, has been used implicitly in [4] and [10], but we state it here in the interest of clarity (it should be noted that this result appeared in [5] for Hermitian matrices).

COROLLARY 1.9. (Schur Complement Corollary) *Let $B$ be a symmetric matrix over a field $\mathbb{F}$,* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, *and let $B[\alpha]$ be a nonsingular principal submatrix of $B$, with* $|\alpha| = k \leq \operatorname{rank} B$. *Let $C = B/B[\alpha]$ and* $\operatorname{epr}(C) = \ell'_1\ell'_2\cdots\ell'_{n-k}$. *Then, for $j = 1, \ldots, n-k$, $\ell'_j = \ell_{j+k}$ if $\ell_{j+k} \in \{\mathtt{A}, \mathtt{N}\}$.*

OBSERVATION 1.10. [4, Observation 2.19] Let $B$ be a symmetric matrix over a field $\mathbb{F}$, with epr-sequence $\ell_1\ell_2\cdots\ell_n$.

1. Form a matrix $B'$ from $B$ by copying the last row down and then the last column across. Then the epr-sequence of $B'$ is $\ell_1\ell'_2\cdots\ell'_n\mathtt{N}$ with $\ell'_i = \mathtt{N}$ if $\ell_i = \mathtt{N}$ and $\ell'_i = \mathtt{S}$ otherwise for $2 \leq i \leq n$.
2. Form a matrix $B''$ from $B$ by taking the direct sum with $[0]$. Then the epr-sequence of $B''$ is $\ell''_1\ell''_2\cdots\ell''_n\mathtt{N}$ with $\ell''_i = \mathtt{N}$ if $\ell_i = \mathtt{N}$ and $\ell''_i = \mathtt{S}$ otherwise for $1 \leq i \leq n$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



**2. Restrictions on attainable epr-sequences over a field of characteristic 2.** Before stating our main results in Section 3, we devote this section towards establishing restrictions for the attainability of epr-sequences over a field of characteristic 2.

Observation 2.1. ($\mathtt{NA}$-$\mathtt{NS}$ Observation.) Let $B$ be a symmetric matrix over a field of characteristic 2, with $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. If $\ell_k\ell_{k+1} = \mathtt{NA}$ or $\ell_k\ell_{k+1} = \mathtt{NS}$ for some $k$, then $k$ is odd and $\ell_j = \mathtt{N}$ when $j$ is odd.

*Proof.* Let $\text{pr}(B) = r_0]r_1\cdots r_n$. Suppose $\ell_k\ell_{k+1} = \mathtt{NA}$ or $\ell_k\ell_{k+1} = \mathtt{NS}$. Then $r_kr_{k+1} = 01$. Since $k \geq 1$, Theorem 1.1 implies that $\text{pr}(B) = 1]01\ \overline{01}\ \overline{0}$, and therefore that $k$ is odd and that $\ell_j = \mathtt{N}$ when $j$ is odd. ∎

Over a field of characteristic 2, the $\mathtt{NN}$ Theorem admits a generalization when the first $\mathtt{N}$ occurs in an even position of the epr-sequence, which is immediate from the $\mathtt{NA}$-$\mathtt{NS}$ Observation and the $\mathtt{NN}$ Theorem.

Observation 2.2. ($\mathtt{N}$-Even Observation.) Let $B$ be a symmetric matrix over a field of characteristic 2, with $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k = \mathtt{N}$ with $k$ even. Then $\ell_j = \mathtt{N}$ for all $j \geq k$.

The next observation establishes another generalization of the $\mathtt{NN}$ Theorem for epr-sequences beginning with $\mathtt{S}$ or $\mathtt{A}$, and it is immediate from Theorem 1.1.

Observation 2.3. Let $B$ be a symmetric matrix over a field of characteristic 2, with $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_1 \neq \mathtt{N}$. If $\ell_k = \mathtt{N}$ for some $k$, then $\ell_j = \mathtt{N}$ for all $j \geq k$.

In the interest of brevity, adopting the notation in [2], the principal minor $\det(B[I])$ is denoted by $B_I$ (when $I = \emptyset$, $B_\emptyset$ is defined to have the value 1). Moreover, when $I = \{i_1, i_2, \ldots, i_k\}$, $B_I$ is written as $B_{i_1i_2\cdots i_k}$.

The next result will be of particular relevance later in this section, and its proof resorts to Muir's law of extensible minors [11]; for a more recent treatment of this law, the reader is referred to [3].

Lemma 2.4. *Let $n \geq 2$, and let $B$ be a symmetric matrix over a field of characteristic 2, with $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose that $\ell_{n-1}\ell_n = \mathtt{AN}$. Then every minor of $B$ of order $n-1$ is nonzero.*

*Proof.* Since the desired conclusion is obvious when $n = 2$, we assume that $n \geq 3$. By hypothesis, every principal minor of $B$ of order $n-1$ is nonzero. Let $i, j \subseteq \{1, 2, \ldots, n\}$ be distinct, and let $I = \{1, 2, \ldots, n\} \setminus \{i, j\}$. Consider the $(n-1) \times (n-1)$ *non-principal* submatrix resulting from deleting row $i$ and column $j$, i.e., the submatrix $B[I \cup \{j\}|I \cup \{i\}]$. Since $I$ does not contain $i$ and $j$, using Muir's law of extensible minors (see [11] or [3]), one may extend the homogenous polynomial identity

$$B_\emptyset B_{ij} = B_iB_j - \det(B[\{i\}|\{j\}])\det(B[\{j\}|\{i\}]),$$

to obtain the identity

$$B_IB_{I\cup\{i,j\}} = B_{I\cup\{i\}}B_{I\cup\{j\}} - \det(B[I\cup\{i\}|I\cup\{j\}])\det(B[I\cup\{j\}|I\cup\{i\}]).$$

Since $B_{I\cup\{i,j\}} = \det(B)$, and because $\ell_n = \mathtt{N}$, we must have

$$\det(B[I\cup\{i\}|I\cup\{j\}])\det(B[I\cup\{j\}|I\cup\{i\}]) = B_{I\cup\{i\}}B_{I\cup\{j\}}.$$

Then, as $\ell_{n-1} = \mathtt{A}$, $B_{I\cup\{i\}}B_{I\cup\{j\}} \neq 0$, implying that $\det(B[I\cup\{j\}|I\cup\{i\}]) \neq 0$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



**2.1. Restrictions on attainable epr-sequences over $\mathbb{Z}_2$.** This section focuses on establishing restrictions for epr-sequences over $\mathbb{Z}_2$.

With the purpose of establishing a contrast between the attainable epr-sequences over $\mathbb{Z}_2$ and those over other fields of characteristic 2, the next example exhibits matrices over a particular field of characteristic 2 attaining epr-sequences that are not attainable over $\mathbb{Z}_2$ (their unattainability over $\mathbb{Z}_2$ is established in this section).

EXAMPLE 2.5. Let $\mathbb{F} = \mathbb{Z}_2$. Consider the field $\mathbb{F}[z] = \{0, 1, z, z+1\}$, where $z^2 = z + 1$. For each of the following (symmetric) matrices over the field $\mathbb{F}[z]$, $\mathrm{epr}(M_\sigma) = \sigma$, where $\sigma$ is an epr-sequence.

$$M_{\mathtt{AAN}} = \begin{bmatrix} 1 & z & z+1 \\ z & 1 & 0 \\ z+1 & 0 & 1 \end{bmatrix}, \; M_{\mathtt{ASSAN}} = \begin{bmatrix} z & 1 & z & z+1 & 0 \\ 1 & z & z+1 & 0 & 1 \\ z & z+1 & z & 1 & z \\ z+1 & 0 & 1 & z & z+1 \\ 0 & 1 & z & z+1 & z \end{bmatrix},$$

$$M_{\mathtt{NANSNN}} = \begin{bmatrix} 0 & z & z+1 & 1 & 1 & 1 \\ z & 0 & 1 & 1 & 1 & 1 \\ z+1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \; M_{\mathtt{SAAA}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & z & z \\ 1 & z & 0 & 1 \\ 1 & z & 1 & 0 \end{bmatrix},$$

$$M_{\mathtt{SASN}} = \begin{bmatrix} 1 & z & z & 1 \\ z & 1 & 1 & 1 \\ z & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \; M_{\mathtt{SASSA}} = \begin{bmatrix} 1 & z & z & z & 1 \\ z & 1 & 0 & 1 & 1 \\ z & 0 & 1 & 1 & 1 \\ z & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

REMARK 2.6.

1. If $B$ is an $n \times n$ symmetric matrix over $\mathbb{Z}_2$ having an epr-sequence starting with $\mathtt{AA}$, then $B = I_n$. This is because a symmetric matrix with nonzero diagonal must have each of its off-diagonal entries equal to zero in order to have all of its order-2 principal minors be nonzero.
2. A similar argument shows that if an $n \times n$ symmetric matrix $B$ over $\mathbb{Z}_2$ has an epr-sequence starting with $\mathtt{NA}$, then $B = A(K_n)$.

Given a sequence $t_{i_1} t_{i_2} \cdots t_{i_k}$, the notation $\overline{t_{i_1} t_{i_2} \cdots t_{i_k}}$ indicates that the sequence may be repeated as many times as desired (or it may be omitted entirely).

PROPOSITION 2.7. *Let $n \geq 2$. Then, over $\mathbb{Z}_2$,* $\mathrm{epr}(A(K_n)) = \mathtt{NA}\overline{\mathtt{NA}}$ *when $n$ is even, and* $\mathrm{epr}(A(K_n)) = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$ *when $n$ is odd.*

*Proof.* Let $\mathrm{epr}(A(K_n)) = \ell_1 \ell_2 \cdots \ell_n$. Obviously, $\ell_1 = \mathtt{N}$. Observe that, for $2 \leq q \leq n$, every $q \times q$ principal submatrix of $B$ is equal to $A(K_q)$. Since $A(K_q) = J_q - I_q$, $\det(A(K_q)) = (-1)^{q-1}(q-1) = q - 1$ (in characteristic 2). Hence, $\ell_q = \mathtt{N}$ when $q$ is odd and $\ell_q = \mathtt{A}$ when $q$ is even. ∎

LEMMA 2.8. ($\mathtt{NA}$ Lemma) *Let $B$ be a symmetric matrix over $\mathbb{Z}_2$, with* $\mathrm{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$. *If $\ell_k \ell_{k+1} = \mathtt{NA}$, then $\ell_k \cdots \ell_n = \mathtt{NA}\overline{\mathtt{NA}}$ or $\ell_k \cdots \ell_n = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



*Proof.* Suppose $\ell_k\ell_{k+1} = \mathtt{NA}$. If $k = 1$, then Remark 2.6 implies that $B = A(K_n)$, and therefore that $\operatorname{epr}(B) = \mathtt{NA}\overline{\mathtt{NA}}$ or $\operatorname{epr}(B) = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$ (by Proposition 2.7). Now, suppose $k \geq 2$, and that $\ell_j \neq \mathtt{A}$ for some even integer $j > k+1$. By the Inheritance Theorem, $B$ contains a singular $j \times j$ principal submatrix, $B'$, whose epr-sequence $\ell'_1\ell'_2 \cdots \ell'_j$ has $\ell'_k\ell'_{k+1} = \mathtt{NA}$ and $\ell'_j = \mathtt{N}$. Since $k \geq 2$, the NN Theorem implies that $\ell'_{k-1} \neq \mathtt{N}$. Let $B'[\alpha]$ be a nonsingular $(k-1) \times (k-1)$ principal submatrix of $B'$. It follows from the Schur Complement Theorem that $B'/B'[\alpha]$ is a (symmetric) matrix of order $j-k+1$, and from the Schur Complement Corollary that $\operatorname{epr}(B'/B'[\alpha]) = \mathtt{NA}\cdots\mathtt{N}$. Since $\operatorname{epr}(B'/B'[\alpha])$ begins with NA, $B'/B'[\alpha] = A(K_{j-k+1})$ (by Remark 2.6). Then, as $\operatorname{epr}(B'/B'[\alpha])$ ends with N, Proposition 2.7 implies that $\operatorname{epr}(B'/B'[\alpha]) = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$; hence, $j-k+1$ is odd, which is a contradiction, since $j$ is even and $k$ is odd. ∎

The epr-sequence of the matrix $M_{\mathtt{NANSNN}}$ in Example 2.5 demonstrates that the NA Lemma cannot be generalized to all fields of characteristic 2.

THEOREM 2.9. (AA Theorem) *If an epr-sequence containing* AA *as a non-terminal subsequence is attainable over $\mathbb{Z}_2$, then it is the sequence* $\overline{\mathtt{A}}\mathtt{AAA}\overline{\mathtt{A}}$.

*Proof.* Let $B$ be an $n\times n$ symmetric matrix over $\mathbb{Z}_2$, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose that $\ell_k\ell_{k+1} = \mathtt{AA}$, where $k+1 < n$. We now show by contradiction that $\ell_{k+2} = \mathtt{A}$; thus, suppose $\ell_{k+2} \neq \mathtt{A}$. Hence, by the Inheritance Theorem, $B$ contains a $(k+2)\times(k+2)$ principal submatrix $C$ with $\operatorname{epr}(C) = \ell'_1\ell'_2\cdots\ell'_{k+2}$ having $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{AAN}$. Note that $C$ is singular. By Remark 2.6, $k \geq 2$ (otherwise, $C = I_3$, which is nonsingular). Let $I = \{1,2,\ldots,k+2\} \setminus \{1,2,3\}$. By [9, Theorem 2], and because $C$ is over a field of characteristic 2, the following equation holds:

$$C_I^2 C_{I\cup\{1,2,3\}}^2 + C_{I\cup\{1\}}^2 C_{I\cup\{2,3\}}^2 + C_{I\cup\{2\}}^2 C_{I\cup\{1,3\}}^2 + C_{I\cup\{3\}}^2 C_{I\cup\{1,2\}}^2 = 0, \tag{2.1}$$

which is the hyperdeterminantal relation obtained from the relation (2) appearing in [9, p. 635]. Then, as $|I| = k-1$, the fact that $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{AAN}$ leads to a contradiction, since the quantity on the left side of (2.1) must be nonzero. Hence, it must be the case that $\ell_{k+2} = \mathtt{A}$. It now follows inductively that $\ell_k\cdots\ell_n = \mathtt{AAA}\overline{\mathtt{A}}$.

Now, suppose that $\ell_j \neq \mathtt{A}$ for some $j < k$. Then, as $k+1 < n$, the Inverse Theorem implies that $\operatorname{epr}(B^{-1})$ starts with AA, and that $\operatorname{epr}(B^{-1}) \neq \mathtt{AA}\overline{\mathtt{A}}\mathtt{A}$. But, by Remark 2.6, $B^{-1} = I_n$, implying that $\operatorname{epr}(B^{-1}) = \mathtt{AA}\overline{\mathtt{A}}\mathtt{A}$, a contradiction.

Since $\overline{\mathtt{A}}\mathtt{AAA}\overline{\mathtt{A}}$ is attained by $I_n$, the desired conclusion follows. ∎

The epr-sequence of the matrix $M_{\mathtt{AAN}}$ in Example 2.5 shows that the AA Theorem does not hold for all fields of characteristic 2.

THEOREM 2.10. *Let $n \geq 3$, and let $B$ be a symmetric matrix over $\mathbb{Z}_2$, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose that $\ell_1 = \mathtt{A}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$. Then $n$ is even.*

*Proof.* By Lemma 2.4, every minor of $B$ of order $n-1$ is nonzero. We claim that each row of $B$ contains an even number of nonzero entries; to see this, let $k$ be the number of nonzero entries of $B$ in row $i$, and consider a calculation of $\det(B)$ via a Laplace expansion along row $i$. Because in the field $\mathbb{Z}_2$ every number is equal to its negative, this expansion calculates $\det(B)$ by adding $k$ minors of $B$ of order $n-1$; since each of these $k$ minors is nonzero, and because $\det(B) = 0$, it follows that $k$ must be even, as claimed. Hence, the total number of nonzero entries of $B$ must also be even. Then, as the number of nonzero off-diagonal entries of a symmetric matrix is always even, it is immediate that the number of nonzero diagonal entries of $B$ must also be even. Finally, since the number of nonzero diagonal entries of $B$ is $n$ (because $\ell_1 = \mathtt{A}$), $n$ is even, as desired. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



We note that the sequence $\mathtt{AS\overline{S}AN}$ is attainable over $\mathbb{Z}_2$ when its order is even (see Theorem 3.8), implying that a sequence of the form $\mathtt{A}\cdots\mathtt{AN}$ is not completely prohibited. Moreover, Theorem 2.10 does not hold for all fields of characterisitic 2 (see Example 2.5).

In the interest of brevity when proving the next result, define the $n \times n$ matrix $R_{n,k}$ as follows: For $n \geq 2$, let

$$R_{n,k} := \begin{bmatrix} I_k & J_{k,n-k} \\ J_{n-k,k} & A(K_{n-k}) \end{bmatrix},$$

where $0 \leq k \leq n$ (we assume that $R_{n,k} = I_n$ when $k = n$, and that $R_{n,k} = A(K_n)$ when $k = 0$).

Proposition 2.11. *An epr-sequence starting with* $\mathtt{SA}$ *is attainable by a symmetric matrix over* $\mathbb{Z}_2$ *if and only if it has one of the following forms:*

$$\mathtt{SA\overline{SA}}, \qquad \mathtt{SA\overline{SA}A}, \qquad \mathtt{SA\overline{SA}N}.$$

*Proof.* Let $0 \leq k \leq n$ be integers. We begin by showing that $\det(R_{n,k}) = 0$ only when $n$ is odd and $k$ is even. The desired conclusion is immediate for the case with $k = 0$ (by Proposition 2.7), and, for the case with $k = n$, it is obvious (since $R_{n,k} = I_n$). Now, suppose $0 < k < n$, and let $C = R_{n,k}$. Note that $\det(C) = \det(I_k)\det(C/I_k) = \det(C/I_k)$, where $C/I_k$ is the Schur complement of $I_k$ in $C$. Then, as

$$C/I_k = A(K_{n-k}) - J_{n-k,k} \cdot J_{k,n-k} = (1-k)J_{n-k} - I_{n-k},$$

$\det(C) = ((1-k)(n-k)-1)(-1)^{n-k-1} = (k+1)n+1$ (in characteristic 2). It follows that $\det(C) = 1$ when $n$ is even, and that $\det(C) = k$ when $n$ is odd. We can now conclude that $\det(R_{n,k}) = 0$ only when $n$ is odd and $k$ is even, as desired.

Let $\sigma$ be an epr-sequence starting with $\mathtt{SA}$. For the first direction, suppose that $\sigma = \mathrm{epr}(B)$ for some symmetric matrix $B$ over $\mathbb{Z}_2$. Let $\sigma = \ell_1\ell_2\cdots\ell_n$. By hypothesis, $\ell_1\ell_2 = \mathtt{SA}$. Without loss of generality, suppose that the first $k$ diagonal entries of $B$ are nonzero, and suppose that the remaining $n-k$ diagonal entries are zero. Note that, since $\ell_1 = \mathtt{S}$, $1 \leq k \leq n-1$. It is easy to verify that the condition that $\ell_2 = \mathtt{A}$ implies that $B = R_{n,k}$. It is also easy to see that for any integer $m$ with $3 \leq m \leq n$, any $m \times m$ principal submatrix of $R_{n,k}$ is of the form $R_{m,p}$, where $0 \leq p \leq k$ (and $0 \leq m-p \leq n-k$). The above argument implies that any principal minor of $B$ of order $m$ is nonzero when $m$ is even, implying that $\ell_j = \mathtt{A}$ whenever $j$ is even. Also, observe that for any odd integer $m$ with $3 \leq m < n$, there exists $0 \leq p \leq k$ even, and $0 \leq q \leq k$ odd, such that $R_{m,p}$ and $R_{m,q}$ are principal submatrices of $B$; then, as $\det(R_{m,p}) = 0$ and $\det(R_{m,q}) \neq 0$, $B$ contains both a zero and a nonzero principal minor of order $m$, implying that $\ell_j = \mathtt{S}$ whenever $j < n$ is odd. It now follows that $B$ must have one of the desired epr-sequences.

For the other direction, note that the order-$n$ sequence $\mathtt{SA\overline{SA}}$ is attained by the matrix $R_{n,1}$ when $n$ is even. Similarly, (when $n$ is odd) the order-$n$ sequences $\mathtt{SA\overline{SA}A}$ and $\mathtt{SA\overline{SA}N}$ are attained by $R_{n,1}$ and $R_{n,2}$, respectively. ∎

As with the previous results, Proposition 2.11 cannot be generalized either (see Example 2.5).

An observation following from the $\mathtt{NA}$ Lemma, the $\mathtt{AA}$ Theorem and Proposition 2.11 is in order:

Observation 2.12. Let $B$ be a symmetric matrix over $\mathbb{Z}_2$, with $\mathrm{epr}(B) = \ell_1\ell_2\cdots\ell_n$. If $\ell_2 = \mathtt{A}$, then $\ell_j = \mathtt{A}$ when $j$ is even.

The previous and the next result also do not hold for all fields of characteristic 2 (see Example 2.5).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



PROPOSITION 2.13. *For any* X*, the epr-sequence* SAXN *cannot occur as a subsequence of the epr-sequence of a symmetric matrix over* $\mathbb{Z}_2$.

*Proof.* Let $B$ be an $n\times n$ symmetric matrix over $\mathbb{Z}_2$, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k\cdots\ell_{k+3} = \texttt{SAXN}$ for some $1 \le k \le n-3$, where $\texttt{X} \in \{\texttt{A},\texttt{S},\texttt{N}\}$. By Proposition 2.11, $k \ge 2$. By the NSA Theorem, $\ell_{k-1} \ne \texttt{N}$. Let $B[\alpha]$ be a $(k-1)\times(k-1)$ nonsingular principal submatrix of $B$. By the Schur Complement Corollary, $\operatorname{epr}(B/B[\alpha]) = \texttt{YAZN}\cdots$, where $\texttt{Y},\texttt{Z} \in \{\texttt{A},\texttt{S},\texttt{N}\}$, which contradicts Observation 2.12. □

In the epr-sequence of a symmetric matrix over a field of characteristic *not* 2, [4, Theorem 2.15] asserts that ANS can only occur as the initial subsequence. Over $\mathbb{Z}_2$, the same restriction holds for ASS:

PROPOSITION 2.14. *In the epr-sequence of a symmetric matrix over* $\mathbb{Z}_2$*, the subsequence* ASS *can only occur as the initial subsequence.*

*Proof.* Let $B$ be an $n\times n$ symmetric matrix over $\mathbb{Z}_2$, with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose to the contrary that $\ell_k\ell_{k+1}\ell_{k+2} = \texttt{ASS}$ for some $2 \le k \le n-3$. By the Inheritance Theorem, $B$ contains a $(k+2)\times(k+2)$ principal submatrix $B'$ with $\operatorname{epr}(B') = \cdots\texttt{XAYN}$, where $\texttt{X},\texttt{Y} \in \{\texttt{A},\texttt{S},\texttt{N}\}$. By the NA Lemma, $\texttt{X} \ne \texttt{N}$, and, by the AA Theorem, $\texttt{X} \ne \texttt{A}$; hence, $\texttt{X} = \texttt{S}$, so that $\operatorname{epr}(B') = \cdots\texttt{SAYN}$, which contradicts Proposition 2.13. □

Once again, the previous result also cannot be generalized to all fields of characteristic 2 (see Example 2.5).

LEMMA 2.15. *Let $B$ be a symmetric matrix over $\mathbb{Z}_2$, with* $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. *Suppose* $\ell_k\ell_{k+1}\ell_{k+2} =$ ASA *for some* $1 \le k \le n-2$. *Then* $\ell_1 \ne \texttt{N}$ *and the following hold.*

1. *If* $\ell_1 = \texttt{A}$*, then $k$ is odd.*
2. *If* $\ell_1 = \texttt{S}$*, then $k$ is even.*

*Proof.* By the NN Theorem and the NA-NS Observation, $\operatorname{epr}(B)$ does not begin with NN, NA, nor NS; hence, $\ell_1 \ne \texttt{N}$.

(1): Suppose that $\ell_1 = \texttt{A}$ and that $k$ is even. Then, by the Inheritance Theorem, $B$ contains a $(k+2)\times(k+2)$ principal submatrix $B'$ with $\operatorname{epr}(B') = \texttt{A}\cdots\texttt{ASA}$. By the Inverse Theorem, $\operatorname{epr}(B^{-1}) = \texttt{SA}\cdots\texttt{AA}$. Since $B^{-1}$ is of order $k+2$, Proposition 2.11 implies that $k+2$ is odd, which is a contradiction to $k$ being even.

(2): Suppose that $\ell_1 = \texttt{S}$ and that $k$ is odd. Then, by the Inheritance Theorem, $B$ contains a $(k+2)\times(k+2)$ principal submatrix $B'$ with $\operatorname{epr}(B') = \texttt{S}\cdots\texttt{AXA}$, where $\texttt{X} \in \{\texttt{A},\texttt{S},\texttt{N}\}$. Since X occurs in an even position, the N-Even Observation implies that $\texttt{X} \ne \texttt{N}$; and, by the AA Theorem, $\texttt{X} \ne \texttt{A}$; hence, $\texttt{X} = \texttt{S}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \texttt{SA}\cdots\texttt{SA}$. Since $(B')^{-1}$ is of order $k+2$, Proposition 2.11 implies that $k+2$ is even, a contradiction. □

The inverse of the matrix $M_{\texttt{SASSA}}$ in Example 2.5, whose epr-sequence is SSASA, reveals that the previous result also cannot be generalized to all fields of characteristic 2; and, for the same reasons, the following theorem cannot be generalized either.

THEOREM 2.16. *Let $B$ be a symmetric matrix over $\mathbb{Z}_2$. Suppose* $\operatorname{epr}(B)$ *contains* ASA *as a subsequence. Then* $\operatorname{epr}(B)$ *is one of the following sequences.*

1. $\texttt{ASA}\overline{\texttt{SA}}$,

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



2. $\mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}$,
3. $\mathtt{ASA}\overline{\mathtt{SA}}\mathtt{N}$,
4. $\mathtt{SASA}\overline{\mathtt{SA}}$,
5. $\mathtt{SASA}\overline{\mathtt{SA}}\mathtt{A}$,
6. $\mathtt{SASA}\overline{\mathtt{SA}}\mathtt{N}$.

*Proof.* Suppose that $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$, and that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{ASA}$. By Lemma 2.15, $\ell_1 \neq \mathtt{N}$. We proceed by examining two cases.

Case 1: $\ell_1 = \mathtt{S}$. Because of Proposition 2.11, it suffices to show that $\ell_2 = \mathtt{A}$. By Lemma 2.15, $k$ is even. If $k = 2$, then, obviously, $\ell_2 = \mathtt{A}$. Now, suppose $k \geq 4$. By the Inheritance Theorem, $B$ contains a $(k+2)\times(k+2)$ principal submatrix, $B'$, whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_{k+2}$ has $\ell'_2 = \ell_2$ and $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{A}\ell'_{k+1}\mathtt{A}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \ell'_{k+1}\mathtt{A}\cdots\ell_2\ell'_1\mathtt{A}$. It follows from Observation 2.12 that $[\operatorname{epr}((B')^{-1})]_j = \mathtt{A}$ when $j$ is even. Then, as $k$ is even, and because $[\operatorname{epr}((B')^{-1})]_k = \ell_2$, we must have $\ell_2 = \mathtt{A}$.

Case 2: $\ell_1 = \mathtt{A}$. By Lemma 2.15, $k$ is odd. Let $1 < j < k$ be an odd integer. By the Inheritance Theorem, $B$ contains a $(k+2)\times(k+2)$ principal submatrix, $B'$, whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_{k+2}$ has $\ell'_j = \ell_j$ and $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{A}\ell'_{k+1}\mathtt{A}$. By the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \ell'_{k+1}\mathtt{A}\cdots\ell_j\cdots$. It follows from Observation 2.12 that $[\operatorname{epr}((B')^{-1})]_i = \mathtt{A}$ when $i$ is even. Since $k+2-j$ is even, $[\operatorname{epr}((B')^{-1})]_{k+2-j} = \mathtt{A}$. Then, as $[\operatorname{epr}((B')^{-1})]_{k+2-j} = \ell'_j = \ell_j$, we have $\ell_j = \mathtt{A}$. We conclude that $\ell_i = \mathtt{A}$ when $i$ is an odd integer with $1 < i < k$. Then, as $\ell_{k+1} = \mathtt{S}$, the $\mathtt{AA}$ Theorem implies that $\ell_i \neq \mathtt{A}$ when $i$ is an even integer with $1 < i < k$; and, since $\ell_k = \mathtt{A}$, the $\mathtt{N}$-Even Observation implies that $\ell_i \neq \mathtt{N}$ when $i$ is an even integer with $1 < i < k$. Hence, $\operatorname{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}\ell_{k+3}\cdots\ell_n$.

If $n = k+2$, then we are done; thus, suppose $n \geq k+3$. Suppose to the contrary that $\ell_q \neq \mathtt{A}$ for some odd integer $q$ with $k+3 \leq q \leq n$. By the Inheritance Theorem, $B$ contains a singular $q \times q$ principal submatrix, $B'$, whose epr-sequence $\ell'_1\ell'_2\cdots\ell'_q$ has $\ell'_i = \ell_i = \mathtt{A}$ when $i \leq k+2$ is odd, and, obviously, $\ell'_q = \mathtt{N}$. Let $B'[\alpha]$ be a (necessarily nonsingular) $1\times 1$ principal submatrix of $B'$. By the Schur Complement Theorem, $B'/B'[\alpha]$ is a $(q-1)\times(q-1)$ (symmetric) matrix, and, by the Schur Complement Corollary, $[\operatorname{epr}(B'/B'[\alpha])]_2 = \mathtt{A}$, and $[\operatorname{epr}(B'/B'[\alpha])]_{q-1} = \mathtt{N}$. It follows from Observation 2.12 that $[\operatorname{epr}(B'/B'[\alpha])]_i = \mathtt{A}$ when $i$ is even. Then, as $[\operatorname{epr}(B'/B'[\alpha])]_{q-1} = \mathtt{N}$, $q-1$ is odd, which is a contradiction to the fact that $q$ is odd. We conclude that $\ell_i = \mathtt{A}$ for all odd $i$ with $k+3 \leq i \leq n$.

Then, as $\ell_{k+1} = \mathtt{S}$, the $\mathtt{AA}$ Theorem implies that $\ell_i \neq \mathtt{A}$ when $i$ is an even integer with $k+3 \leq i \leq n-1$; and, since at least one of $\ell_{n-1}$ and $\ell_n$ must be $\mathtt{A}$ (because one of $n-1$ and $n$ must be even) the $\mathtt{N}$-Even Observation implies that $\ell_i \neq \mathtt{N}$ when $i$ is an even integer with $k+3 \leq i \leq n-1$. It follows that $\operatorname{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}$ when $n$ is odd, and that either $\operatorname{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}$ or $\operatorname{epr}(B) = \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{N}$ when $n$ is even. □

**3. Main results.** In this section, a complete characterization of the epr-sequences that are attainable by a symmetric matrix over $\mathbb{Z}_2$ is established. We start by characterizing those that begin with $\mathtt{N}$.

Lemma 3.1. *Let* $M_1 = A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2)$ *be* $n \times n$ *and*

$$M_2 = \begin{bmatrix} M_1 & \mathbb{1}_n \\ \mathbb{1}_n^T & O_1 \end{bmatrix},$$

*with both matrices being over* $\mathbb{Z}_2$. *Then* $\operatorname{epr}(M_1) = \overline{\mathtt{NS}}\mathtt{NA}$ *and* $\operatorname{epr}(M_2) = \overline{\mathtt{NS}}\mathtt{NAN}$.

*Proof.* Let $\operatorname{epr}(M_1) = \ell_1\ell_2\cdots\ell_n$. Note that $n$ is even. The desired conclusion is obvious when $n = 2$; hence, suppose $n \geq 4$. It is clear that $\ell_1\ell_2 = \mathtt{NS}$; thus, by the $\mathtt{NA}$-$\mathtt{NS}$ Observation, $\operatorname{epr}(M_1)$ has $\mathtt{N}$ in every odd

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



position. Clearly, $M_1$ is nonsingular, implying that $\ell_n = \mathtt{A}$. It remains to show that $\ell_j = \mathtt{S}$ when $j \le n-1$ is even. Since $\ell_n = \mathtt{A}$, by the NN Theorem, $\ell_j \ne \mathtt{N}$ when $j \le n-1$ is even. Now, because of the NA Lemma, to show that $\ell_j \ne \mathtt{A}$ when $j \le n-1$ is even, it suffices to show that $\ell_{n-2} = \mathtt{S}$. Clearly, $M_1(\{2,4\})$ is singular (since it contains a zero row). Then, as $\ell_{n-2} \ne \mathtt{N}$ (because $n-2$ is even), $\ell_{n-2} = \mathtt{S}$.

Let $\operatorname{epr}(M_2) = \ell'_1\ell'_2 \cdots \ell'_{n+1}$. The assertion is clear when $n = 2$ (note that $n$ is even, and that $M_2$ is of order $n+1$, not $n$); hence, suppose $n \ge 4$. Since (clearly) $\ell'_1\ell'_2 = \mathtt{NS}$, the NA-NS Observation implies that $\operatorname{epr}(M_2)$ has N in every odd position. Since $M_1$ is a principal submatrix of $M_2$, and because $\operatorname{epr}(M_1) = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NA}$, it is immediate that $\operatorname{epr}(M_2) = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{N}\ell'_n\mathtt{N}$. We now show that $\ell'_n = \mathtt{A}$. Observe that any $n \times n$ principal submatrix of $M_2$ is either $M_1$, which is nonsingular, or is one that is permutationally similar to the matrix

$$C = \begin{bmatrix} C(\{n\}) & \mathbb{1}_{n-1} \\ \mathbb{1}_{n-1}^T & O_1 \end{bmatrix},$$

where $C(\{n\}) = O_1 \oplus A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2)$. Let $C'$ be the matrix obtained from $C$ by first subtracting its first row from rows $2, 3, \ldots, n-1$, and then subtracting the first column of the resulting matrix from columns $2, 3, \ldots, n-1$. Now observe that $\det(C') = -\det(C'(\{1,n\}))$, where $C'(\{1,n\}) = A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2)$, which is a nonsingular matrix (of order $(n-2)$). Hence, $\det(C') \ne 0$. Then, as $\det(C) = \det(C')$, $C$ is nonsingular. We conclude that $\ell'_n = \mathtt{A}$. ∎

Theorem 3.2. *An epr-sequence starting with* N *is attainable by a symmetric matrix over* $\mathbb{Z}_2$ *if and only if it has one of the following forms.*

1. $\mathtt{NA}\overline{\mathtt{NA}}$,
2. $\mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$,
3. $\overline{\mathtt{NS}}\mathtt{N}\overline{\mathtt{N}}$,
4. $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NA}$,
5. $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NAN}$.

*Proof.* Let $\sigma = \ell_1\ell_2 \cdots \ell_n$ be an epr-sequence with $\ell_1 = \mathtt{N}$. Suppose that $\sigma = \operatorname{epr}(B)$, where $B$ is a symmetric matrix over $\mathbb{Z}_2$. If $n = 1$, then $\sigma = \overline{\mathtt{NS}}\mathtt{N}\overline{\mathtt{N}}$ with $\overline{\mathtt{NS}}$ and $\overline{\mathtt{N}}$ empty. Suppose $n \ge 2$. If $\ell_2 = \mathtt{N}$, then, by the NN Theorem, $\sigma = \overline{\mathtt{NS}}\mathtt{NN}\overline{\mathtt{N}}$ with $\overline{\mathtt{NS}}$ empty. If $\ell_2 = \mathtt{A}$, then, by the NA Lemma, $\sigma = \mathtt{NA}\overline{\mathtt{NA}}$ or $\sigma = \mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$.

Finally, suppose $\ell_2 = \mathtt{S}$. Since an attainable epr-sequence cannot end in S, $n \ge 3$. By the NA-NS Observation, $\ell_j = \mathtt{N}$ when $j$ is odd. Hence, $\operatorname{rank}(B)$ is even. We now show that SNA cannot occur as a subsequence of $\ell_1\ell_2 \cdots \ell_{n-2}$. Suppose to the contrary that $\ell_{k-1}\ell_k\ell_{k+1} = \mathtt{SNA}$, where $3 \le k \le n-3$. Clearly, since $\ell_j = \mathtt{N}$ when $j$ is odd, $k$ is odd and $\ell_{k+2} = \mathtt{N}$. By the Inheritance Theorem, $B$ contains a $(k+3)\times(k+3)$ principal submatrix $B'$ with $\operatorname{epr}(B') = \cdots \mathtt{SNANX}$, where $\mathtt{X} \in \{\mathtt{A},\mathtt{N}\}$. If $\mathtt{X} = \mathtt{A}$, then, by the Inverse Theorem, $\operatorname{epr}((B')^{-1}) = \mathtt{NANS}\cdots$, which contradicts the NA Lemma. Hence, $\mathtt{X} = \mathtt{N}$, and therefore $\operatorname{epr}(B') = \cdots \mathtt{SNANN}$, which contradicts the NA Lemma. We conclude that SNA cannot occur as a subsequence of $\ell_1\ell_2 \cdots \ell_{n-2}$. Now, let $r = \operatorname{rank}(B)$; hence, $\ell_r \ne \mathtt{N}$. Then, as $r$ is even, $\ell_{r-1} = \mathtt{N}$ (because $r-1$ is odd). Since $\ell_j = \mathtt{N}$ when $j$ is odd, the NN Theorem implies that $\ell_i \ne \mathtt{N}$ when $i \le r-1$ is even. We proceed by considering two cases.

Case 1: $r \ge n-1$. First, suppose $r = n-1$. Since $r$ is even, $r+1 = n$ is odd, implying that $\ell_n = \mathtt{N}$. Hence, $\ell_{n-1}\ell_n = \mathtt{AN}$ or $\ell_{n-1}\ell_n = \mathtt{SN}$. Then, as $\ell_2 = \mathtt{S}$, and because SNA cannot occur as a subsequence of $\ell_1\ell_2 \cdots \ell_{n-2}$, it follows inductively that $\sigma = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NAN}$ or $\sigma = \overline{\mathtt{NS}}\mathtt{NSN}$. Now, suppose $r = n$; hence, $n$ is even and $\ell_n = \mathtt{A}$. Since $\ell_{r-1} = \mathtt{N}$, $\ell_{n-1}\ell_n = \mathtt{NA}$. Then, as $\ell_2 = \mathtt{S}$, and because SNA cannot occur as a subsequence of $\ell_1\ell_2 \cdots \ell_{n-2}$, it follows inductively that $\sigma = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NA}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



Case 2: $r \leq n-2$. Hence, $\ell_{r+1}\cdots\ell_n = \mathtt{NN}\overline{\mathtt{N}}$. Since $\ell_{r-1} = \mathtt{N}$ and $\ell_r \neq \mathtt{N}$, $\ell_{r-1}\cdots\ell_n = \mathtt{NANN}\overline{\mathtt{N}}$ or $\ell_{r-1}\cdots\ell_n = \mathtt{NSNN}\overline{\mathtt{N}}$; but the former case contradicts the NA Lemma, implying that $\ell_{r-1}\cdots\ell_n = \mathtt{NSNN}\overline{\mathtt{N}}$. Then, as $\ell_2 = \mathtt{S}$, and because $\mathtt{SNA}$ cannot occur as a subsequence of $\ell_1\ell_2\cdots\ell_{n-2}$, it follows inductively that $\sigma = \mathtt{NS}\overline{\mathtt{NS}}\mathtt{NN}\overline{\mathtt{N}}$.

For the other direction, we show that each of the sequences listed above is attainable. Assume that the sequence under consideration has order $n$. The sequences $\mathtt{NA}\overline{\mathtt{NA}}$ and $\mathtt{NA}\overline{\mathtt{NA}}\mathtt{N}$ are attainable by Proposition 2.7. When $\overline{\mathtt{NS}}$ is non-empty the sequence $\overline{\mathtt{NS}}\mathtt{N}\overline{\mathtt{N}}$ is attainable by applying Observation 1.10(2) to the sequence $\mathtt{NA}\overline{\mathtt{NA}}$; and, when $\overline{\mathtt{NS}}$ is empty, it is attained by $O_n$. Finally, the sequences $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NA}$ and $\mathtt{NS}\overline{\mathtt{NS}}\mathtt{NAN}$ are attainable by Lemma 3.1. ∎

Naturally, due to the dependence of Theorem 3.2 on the results of Section 2.1, this theorem does not hold for other fields.

Some lemmas are necessary before stating the second of our three main results in Theorem 3.8.

LEMMA 3.3. *Let $n \geq 4$, $m \geq 5$, and let*

$$M_{\mathtt{ASA}} = \begin{bmatrix} I_2 & \mathbb{1}_2 \\ \mathbb{1}_2^T & J_1 \end{bmatrix},\ M_{\mathtt{ASAA}} = \begin{bmatrix} I_2 & J_2 \\ J_2 & I_2 \end{bmatrix}$$

*be over $\mathbb{Z}_2$. Let $B = I_{n-3} \oplus M_{\mathtt{ASA}}$, $B' = I_{m-4} \oplus M_{\mathtt{ASAA}}$, $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ and $\operatorname{epr}(B') = \ell_1'\ell_2'\cdots\ell_m'$. Then $\operatorname{epr}(M_{\mathtt{ASA}}) = \mathtt{ASA}$, $\operatorname{epr}(M_{\mathtt{ASAA}}) = \mathtt{ASAA}$, $\ell_1\ell_2\ell_3 = \ell_1'\ell_2'\ell_3' = \mathtt{ASS}$, $\ell_{n-1}\ell_n = \mathtt{SA}$ and $\ell_{m-1}'\ell_m' = \mathtt{AA}$.*

*Proof.* All of the assertions above are easily verified, except $\ell_{m-1}' = \mathtt{A}$, which we now prove. The case with $m = 5$ is easy to check; thus, suppose $m \geq 6$. Note that, since every $3\times 3$ principal submatrix of the $(4\times 4)$ matrix $M_{\mathtt{ASAA}}$ is nonsingular, and because every $(m-5)\times(m-5)$ principal submatrix of $I_{m-4}$ is also nonsingular, deleting row $i$ and column $i$ of $B'$ results in a matrix that is a direct sum of two nonsingular matrices; hence, every $(m-1)\times(m-1)$ principal submatrix of $B'$ is nonsingular, implying that $\ell_{m-1}' = \mathtt{A}$. ∎

A matrix that will play an important role here is defined as follows: For $n \geq 2$, let $F_n$ be the $n\times n$ matrix resulting from replacing the first diagonal entry of $A(K_n)$ with 1.

LEMMA 3.4. *Let $n \geq 2$, and let $F_n$ be over $\mathbb{Z}_2$. Then $F_n$ is nonsingular.*

*Proof.* The assertion is obvious when $n = 2$; thus, assume $n \geq 3$. Observe that

$$\det(F_n) = \det(F_n[\{1\}])\det(F_n/F_n[\{1\}]) = \det(J_1)\det(F_n/J_1) = \det(F_n/J_1),$$

where

$$F_n/J_1 = F_n[\{2,\ldots,n\}] - \mathbb{1}_{n-1}\cdot(J_1)^{-1}\cdot\mathbb{1}_{n-1}^T = A(K_{n-1}) - J_{n-1} = -I_{n-1}.$$

Hence, $\det(F_n) = \det(-I_{n-1}) \neq 0$. ∎

LEMMA 3.5. *Let $n = 4k+2$, where $k \geq 1$ is an integer. Let $m = \frac{n}{2}$, let*

$$B = \begin{bmatrix} J_m & I_m \\ I_m & I_m \end{bmatrix}$$

*be over $\mathbb{Z}_2$, and let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Then $\ell_1\ell_2\ell_3 = \mathtt{ASS}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



*Proof.* It is easily seen that $\ell_1\ell_2\ell_3 = \mathtt{ASS}$. Next, we show that $\ell_n = \mathtt{N}$. Observe that $\det(B) = \det(I_m)\det(B/I_m)$, where

$$B/I_m = J_m - I_m \cdot (I_m)^{-1} \cdot I_m = A(K_m).$$

Since $m = \frac{n}{2} = 2k+1$ is odd, Proposition 2.7 implies that $A(K_m)$ is singular, implying that $B/I_m$ is singular, and therefore that $\det(B) = 0$; hence, $\ell_n = \mathtt{N}$.

Now, to see that $\ell_{n-1} = \mathtt{A}$, note that the $(n-1)\times(n-1)$ principal submatrix resulting from the deletion of the $i$th row and $i$th column of $B$ must be one of the following two matrices:

$$C_1 = \begin{bmatrix} J_{m-1} & X^T \\ X & I_m \end{bmatrix},\ C_2 = \begin{bmatrix} J_m & X \\ X^T & I_{m-1} \end{bmatrix},$$

where $X = I_m(\emptyset, \{q\})$ and $q \in \{1, 2, \ldots, m\}$ is the unique integer such that $i = q$ or $i = m + q$ (that is, $q = i$ if $1 \le i \le m$, and $q = i - m$ if $m + 1 \le i \le n$). Observe that $\det(C_1) = \det(I_m)\det(C_1/I_m)$ and $\det(C_2) = \det(I_{m-1})\det(C_2/I_{m-1})$, where $C_1/I_m = J_{m-1} - X^T X$ and $C_2/I_{m-1} = J_m - XX^T$ are the Schur complements of $I_m$ and $I_{m-1}$ in $C_1$ and $C_2$, respectively. Since $X^T X = I_{m-1}$, $C_1/I_m = A(K_{m-1})$. Then, as $m - 1 = 2k$ is even, Proposition 2.7 implies that $C_1/I_m$ is nonsingular; hence, $\det(C_1) \neq 0$.

Finally, observe that $XX^T$ is the $m \times m$ matrix resulting from replacing the $q$th diagonal entry of $I_m$ with 0. Hence, $C_2/I_{m-1}$ is the matrix resulting from replacing the $q$th diagonal entry of $A(K_m)$ with 1. Then, as $C_2/I_{m-1}$ is permutationally similar to the nonsingular matrix $F_m$ (see Lemma 3.4), $C_2/I_{m-1}$ is nonsingular, implying that $\det(C_2) \neq 0$. ☐

A worthwhile observation is that the condition that $n$ is equal to 2 modulo 4 in Lemma 3.5 was of relevance when showing that $\det(B) = 0$, as it is consistent with the proof of Theorem 2.10, from which it can be deduced that, in order to have $\det(B) = 0$, it is necessary for $B$ to contain an even number of nonzero entries in each row (observe that $B$ contains $\frac{n}{2} + 1 = 2(k + 1)$ nonzero entries in each of the first $\frac{n}{2}$ rows, and 2 nonzero entries in each of the remaining rows). For the same reasons, the congruence modulo 4 of $n$ in the following lemma will once again be of relevance.

LEMMA 3.6. *Let $n = 4k$, where $k \ge 2$ is an integer. Let $m = \frac{n}{2}$, let*

$$B = \begin{bmatrix} J_{m-1} & W \\ W^T & I_{m+1} \end{bmatrix}$$

*be over $\mathbb{Z}_2$, where $W = [I_{m-1}, J_{m-1,2}]$, and let $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Then $\ell_1\ell_2\ell_3 = \mathtt{ASS}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$.*

*Proof.* It is easily verified that $\ell_1\ell_2\ell_3 = \mathtt{ASS}$. Now we verify that $\ell_n = \mathtt{N}$. Observe that $\det(B) = \det(I_{m+1})\det(B/I_{m+1})$, where $B/I_{m+1}$ is the Schur complement of $B[\{m, m+1, \ldots, n\}] = I_{m+1}$ in $B$. Note that

$$B/I_{m+1} = J_{m-1} - WW^T = J_{m-1} - I_{m-1} - 2J_{m-1} = A(K_{m-1}) - 2J_{m-1}.$$

Hence, $B/I_{m+1} = A(K_{m-1})$ (in characteristic 2). Then, as $m - 1 = 2k - 1$ is odd, Proposition 2.7 implies that $B/I_{m+1}$ is singular. It follows that $\det(B) = 0$, and therefore that $\ell_n = \mathtt{N}$.

Now we show that $\ell_{n-1} = \mathtt{A}$. Let $\alpha_1 = \{1, 2, \ldots, m-1\}$, $\alpha_2 = \{m, m+1, \ldots, n-2\}$ and $\alpha_3 = \{n-1, n\}$.

Let $B'$ be the matrix obtained from $B$ by deleting its $i$th row and $i$th column. Let $q = i - (m - 1)$. Suppose that $M_1 = B'$ if $i \in \alpha_1$, that $M_2 = B'$ if $i \in \alpha_2$, and that $M_3 = B'$ if $i \in \alpha_3$. It is easy to see that

$$C_1 = \begin{bmatrix} J_{m-2} & X \\ X^T & I_{m+1} \end{bmatrix},\ C_2 = \begin{bmatrix} J_{m-1} & Y \\ Y^T & I_m \end{bmatrix},\ C_3 = \begin{bmatrix} J_{m-1} & Z \\ Z^T & I_m \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



where

$$X = [I_{m-1}(\{i\}, \emptyset), J_{m-2,2}],\ Y = [I_{m-1}(\emptyset, \{q\}), J_{m-1,2}],\ Z = [I_{m-1}, \mathbb{1}_{m-1}].$$

We proceed to show that $B'$ is nonsingular by considering the three cases outlined above.

Case 1: $B' = C_1$. Note that $\det(C_1) = \det(I_{m+1}) \det(C_1/I_{m+1})$, where $C_1/I_{m+1}$ is the Schur Complement of $I_{m+1}$ in $C_1$, and that

$$C_1/I_{m+1} = J_{m-2} - XX^T = J_{m-2} - I_{m-2} - 2J_{m-2} = A(K_{m-2}) - 2J_{m-2}.$$

Hence, $C_1/I_{m+1} = A(K_{m-2})$ (in characteristic 2). Then, as $m - 2 = 2k - 2$ is even, Proposition 2.7 implies that $C_1/I_{m+1}$ is nonsingular, implying that $\det(C_1) \neq 0$.

Case 2: $B' = C_2$. Then $\det(C_2) = \det(I_m) \det(C_2/I_m)$, where $C_2/I_m$ is the Schur complement of $I_m$ in $C_2$, and

$$C_2/I_m = J_{m-1} - YY^T = J_{m-1} - I_{m-1}(\emptyset, \{q\}) \cdot I_{m-1}(\{q\}, \emptyset) - 2J_{m-1}.$$

Note that $I_{m-1}(\emptyset, \{q\}) \cdot I_{m-1}(\{q\}, \emptyset)$ is the matrix obtained from $I_{m-1}$ by replacing its $q$th diagonal entry with 0. Then, as $2J_{m-1} = O_{m-1}$ (in characteristic 2), $C_2/I_m$ is the matrix obtained from $A(K_{m-1})$ by replacing its $q$th diagonal entry with 1. Hence, $C_2/I_m$ is permutationally similar to the nonsingular matrix $F_{m-1}$ (see Lemma 3.4). It follows that $C_2/I_m$ is nonsingular, and therefore that $\det(C_2) \neq 0$.

Case 3: $B' = C_3$. Then $\det(C_3) = \det(I_m) \det(C_3/I_m)$, where $C_3/I_m$ is the Schur complement of $I_m$ in $C_3$, and

$$C_3/I_m = J_{m-1} - ZZ^T = J_{m-1} - I_{m-1} - J_{m-1} = -I_{m-1}.$$

It follows that $C_3/I_m$ is nonsingular, and therefore that $\det(C_3) \neq 0$. ☐

Lemma 3.7. *The following epr-sequences are attainable by a symmetric matrix over $\mathbb{Z}_2$:*

$$\mathtt{ASA}\overline{\mathtt{SA}},\qquad \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A},\qquad \mathtt{ASA}\overline{\mathtt{SA}}\mathtt{N}.$$

*Proof.* The attainability of $\mathtt{ASA}\overline{\mathtt{SA}}$ follows by observing that, by the Inverse Theorem, the inverse of any (symmetric) matrix attaining the sequence $\mathtt{S}\overline{\mathtt{SA}}\mathtt{A}$, which is attainable by Proposition 2.11, has epr-sequence $\mathtt{ASA}\overline{\mathtt{SA}}$.

Now, for $n \geq 4$ even, we show that the matrix

$$B = \begin{bmatrix} I_{n-2} & J_{n-2,2} \\ J_{2,n-2} & I_2 \end{bmatrix}$$

has epr-sequence $\mathtt{ASA}\overline{\mathtt{SA}}\mathtt{A}$. Because of Theorem 2.16, it suffices to show that $\mathrm{epr}(B)$ begins with $\mathtt{ASA}$, and that it ends with $\mathtt{A}$. Observe that $\det(B) = \det(I_{n-2}) \det(B/I_{n-2})$, where

$$B/I_{n-2} = I_2 - J_{2,n-2} \cdot (I_{n-2})^{-1} \cdot J_{n-2,2} = I_2 - (n-2)J_2.$$

Since $n$ is even, $B/I_{n-2} = I_2$ (in characteristic 2); hence, $B$ is nonsingular. It is clear that $\mathrm{epr}(B)$ begins with $\mathtt{AS}$. Finally, note that each $3 \times 3$ principal submatrix of $B$ must be $I_3$ or one of the following:

$$\begin{bmatrix} I_2 & \mathbb{1}_2 \\ \mathbb{1}_2^T & J_1 \end{bmatrix}, \qquad \begin{bmatrix} J_1 & \mathbb{1}_2^T \\ \mathbb{1}_2 & I_2 \end{bmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



Then, as each of these $3 \times 3$ matrices is nonsingular, $\operatorname{epr}(B)$ begins with $\mathtt{ASA}$, as desired.

With $n \geq 5$, let $B$ be an $n \times n$ symmetric matrix with epr-sequence $\mathtt{SASA\overline{SA}N}$, which is attainable by Proposition 2.11. Note that $n$ is odd. Let $\alpha \subseteq \{1, 2, \ldots, n\}$ with $|\alpha| = 1$ be such that $B[\alpha]$ is nonsingular. Let $\operatorname{epr}(B/B[\alpha]) = \ell'_1 \ell'_2 \cdots \ell'_{n-1}$. We now show that $\operatorname{epr}(B/B[\alpha]) = \mathtt{ASA\overline{SA}N}$. By the Schur Complement Corollary, $\ell'_j = \mathtt{A}$ when $j$ is odd, and $\ell'_{n-1} = \mathtt{N}$. Since $n-2$ is odd, $\ell'_{n-2} = \mathtt{A}$. Since $\ell'_{n-1} = \mathtt{N}$, the $\mathtt{AA}$ Theorem implies that $\ell'_j \neq \mathtt{A}$ when $j \leq n-3$ is even. Finally, as $\ell'_{n-2} = \mathtt{A}$, the $\mathtt{N}$-Even Observation implies that $\ell'_j = \mathtt{S}$ when $j \leq n-3$ is even. It follows that $\operatorname{epr}(B/B[\alpha]) = \mathtt{ASA\overline{SA}N}$, as desired. ◻

Before stating our characterization of the epr-sequences that begin with $\mathtt{A}$ in the next theorem, something needs to be clarified. Corollary 2.2 of [4] claims that the sequence $\mathtt{AS\overline{S}AAA\overline{A}}$ is attainable over $\mathbb{Z}_2$; this claim is false: Observe that it contradicts the $\mathtt{AA}$ Theorem. But it should be noted that [4, Corollary 2.22] becomes true once the field is restricted to be of characteristic 0, since it relies on [4, Proposition 2.18].

Theorem 3.8. *An epr-sequence of order $n$, and starting with $\mathtt{A}$, is attainable by a symmetric matrix over $\mathbb{Z}_2$ if and only if it has one of the following forms.*

1. $\mathtt{A\overline{A}}$,
2. $\mathtt{A\overline{S}N\overline{N}}$,
3. $\mathtt{ASS\overline{S}A}$,
4. $\mathtt{ASS\overline{S}AA}$,
5. $\mathtt{ASSS\overline{S}AN}$ *with $n$ even,*
6. $\mathtt{ASA\overline{SA}}$,
7. $\mathtt{ASA\overline{SA}A}$,
8. $\mathtt{ASA\overline{SA}N}$.

*Proof.* Let $\sigma = \ell_1 \ell_2 \cdots \ell_n$ be an epr-sequence with $\ell_1 = \mathtt{A}$. Suppose that $\sigma = \operatorname{epr}(B)$, where $B$ is a symmetric matrix over $\mathbb{Z}_2$. If $n = 1$ or $n = 2$, then $\sigma$ is $\mathtt{A}$, $\mathtt{AA}$, or $\mathtt{AN}$, all of which are listed above. Suppose $n \geq 3$. If $\ell_2 = \mathtt{A}$ or $\ell_2 = \mathtt{N}$, then the $\mathtt{AA}$ Theorem and the $\mathtt{N}$-Even Observation imply that $\sigma$ is either $\mathtt{AAA\overline{A}}$ or $\mathtt{ANN\overline{N}}$. Now, suppose $\ell_2 = \mathtt{S}$. If $\sigma$ contains the subsequence $\mathtt{ASA}$, then, by Theorem 2.16, $\sigma$ is either $\mathtt{ASA\overline{SA}}$, $\mathtt{ASA\overline{SA}A}$, or $\mathtt{ASA\overline{SA}N}$. Now, suppose $\sigma$ does not contain $\mathtt{ASA}$. Hence, $\ell_3 = \mathtt{N}$ or $\ell_3 = \mathtt{S}$, and $n \geq 4$. If $\ell_3 = \mathtt{N}$, then Observation 2.3 implies that $\sigma = \mathtt{ASNN\overline{N}}$. Now, assume that $\ell_3 = \mathtt{S}$. Let $k$ be a minimal integer with $3 \leq k \leq n-1$ such that $\ell_k \ell_{k+1} = \mathtt{SN}$ or $\ell_k \ell_{k+1} = \mathtt{SA}$. Hence, $\ell_1 \ell_2 \cdots \ell_k = \mathtt{ASS\overline{S}}$. If $\ell_{k+1} = \mathtt{N}$, then Observation 2.3 implies that $\sigma = \mathtt{ASS\overline{S}N\overline{N}}$. Now, assume that $\ell_{k+1} = \mathtt{A}$. If $n = k+1$, then $\sigma = \mathtt{ASS\overline{S}A}$. Thus, suppose $n \geq k+2$.

We now show that $n = k+2$. Suppose to the contrary that $n \geq k+3$. By the $\mathtt{AA}$ Theorem, $\ell_{k+2} \neq \mathtt{A}$. If $\ell_{k+2} = \mathtt{N}$, then Observation 2.3 implies that $\sigma$ contains $\mathtt{SANN}$, which is prohibited by Proposition 2.13; hence, $\ell_{k+2} = \mathtt{S}$, so that $\ell_k \ell_{k+1} \ell_{k+2} = \mathtt{SAS}$. Then, as $\sigma$ does not contain $\mathtt{ASA}$, and because $\mathtt{SASN}$ is prohibited by Proposition 2.13, $\ell_{k+3} = \mathtt{S}$, implying that $\sigma$ contains $\mathtt{ASS}$ as a non-initial subsequence, which contradicts Proposition 2.14. It follows that $n = k+2$, and therefore that $\sigma$ is either $\mathtt{ASS\overline{S}AA}$ or $\mathtt{ASS\overline{S}AN}$; in the case with $\sigma = \mathtt{ASS\overline{S}AN}$, Theorem 2.10 implies that $n$ is even, and therefore that $\sigma = \mathtt{ASSS\overline{S}AN}$.

Now, we establish the other direction. As before, we assume that the sequence under consideration has order $n$. First, the sequence $\mathtt{A\overline{A}}$ is attained by $I_n$. The sequence $\mathtt{A\overline{S}N\overline{N}}$ is attainable by applying Observation 1.10(1) to the sequence $\mathtt{A\overline{A}}$. To see that $\mathtt{ASS\overline{S}A}$ and $\mathtt{ASS\overline{S}AA}$ are attainable, observe that the matrices $B$ and $B'$ in Lemma 3.3 must attain these sequences, respectively, since the epr-sequence of these matrices must be one of those listed above. Similarly, when $n$ is even, one of the two matrices in the statements of Lemma

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



3.5 and Lemma 3.6 is required to attain the sequence $\mathtt{ASSS\overline{S}AN}$. Finally, the attainability of $\mathtt{ASA\overline{SA}}$, $\mathtt{ASA\overline{SA}A}$, and $\mathtt{ASA\overline{SA}N}$ follows from Lemma 3.7. □

The reader is once again referred to Example 2.5 to see why Theorem 3.8 cannot be generalized to other fields.

As before, we need more lemmas in order to prove the last of our three main results.

For an integer $n \geq 2$ and $k \in \{1, 2, \ldots, n\}$, we let $e_k^n$ denote the column vector of length $n$ with the $k$th entry equal to 1 and every other entry equal to zero; moreover, let

$$G_n := \begin{bmatrix} J_1 & (e_1^{n-1})^T \\ e_1^{n-1} & F_{n-1} \end{bmatrix}.$$

Lemma 3.9. *Let $n \geq 4$ be an even integer, let $G_n$ be over $\mathbb{Z}_2$, and let $\operatorname{epr}(G_n) = \ell_1\ell_2 \cdots \ell_n$. Then $\ell_1\ell_2 = \mathtt{SS}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$.*

*Proof.* It is easily verified that $\ell_1\ell_2 = \mathtt{SS}$. The final assertion is easy to check when $n = 4$; thus, suppose $n \geq 5$. Observe that any $(n-1) \times (n-1)$ principal submatrix of $G_n$ has one of the following forms: $G_{n-1}$, $F_{n-1}$ or $J_1 \oplus A(K_{n-2})$. Hence, to show that $\ell_{n-1}\ell_n = \mathtt{AN}$, it suffices to show that $G_{n-1}$, $F_{n-1}$ and $J_1 \oplus A(K_{n-2})$ are nonsingular, and that $G_n$ is singular. By Lemma 3.4, $F_{n-1}$ is nonsingular. By Proposition 2.7, and because $n-2$ is even, $J_1 \oplus A(K_{n-2})$ is also nonsingular.

Finally, we show that $\det(G_{n-1}) \neq 0$ and $\det(G_n) = 0$. Let $q \in \{n-1, n\}$. Observe that $\det(G_q) = \det(F_{q-1}) - \det(A(K_{q-2}))$. Since $\det(F_{q-1}) \neq 0$, $\det(G_q) = 1 - \det(A(K_{q-2}))$ (in characteristic 2). Hence, $\det(G_q) = 0$ if and only if $\det(A(K_{q-2})) \neq 0$. It follows from Proposition 2.7 that $\det(G_q) = 0$ if and only if $q$ is even. Then, as $n$ is even, $\det(G_{n-1}) \neq 0$ and $\det(G_n) = 0$. □

Lemma 3.10. *Let $n \geq 5$ be an odd integer. Then there exists a symmetric matrix over $\mathbb{Z}_2$ whose epr-sequence $\ell_1\ell_2 \cdots \ell_n$ has $\ell_1\ell_2 = \mathtt{SS}$ and $\ell_{n-1}\ell_n = \mathtt{AN}$.*

*Proof.* Clearly, $n+1$ is even and $n+1 \geq 6$. Let $m = \frac{n+1}{2}$, and let

$$B' = \begin{bmatrix} J_m & I_m \\ I_m & I_m \end{bmatrix},\ B'' = \begin{bmatrix} J_{m-1} & W \\ W^T & I_{m+1} \end{bmatrix},$$

where $W = [I_{m-1}, J_{m-1,2}]$. Observe that $B'$ and $B''$ are $(n+1) \times (n+1)$ symmetric matrices. Let $\operatorname{epr}(B') = \ell'_1\ell'_2 \cdots \ell'_{n+1}$ and $\operatorname{epr}(B'') = \ell''_1\ell''_2 \cdots \ell''_{n+1}$. We consider two cases:

Case 1: $n+1 = 4k+2$ for some integer $k \geq 1$. Observe that, by Lemma 3.5, $\ell'_n\ell'_{n+1} = \mathtt{AN}$. Let $\alpha = \{n+1\}$, let $C = B'/B'[\alpha]$, and let $\operatorname{epr}(C) = \ell_1\ell_2 \cdots \ell_n$. We now show that $C$ is a matrix with the desired properties. By the Schur Complement Corollary, $\ell_{n-1}\ell_n = \mathtt{AN}$. To show $\ell_1\ell_2 = \mathtt{SS}$, first, observe that by the Schur Complement Theorem and because $\det(B'[\alpha]) = 1$ (in characteristic 2),

$$\det(C[\{n\}]) = \det(B'[\{n\} \cup \alpha]), \quad \det(C[\{m\}]) = \det(B'[\{m\} \cup \alpha]),$$

$$\det(C[\{n-1, n\}]) = \det(B'[\{n-1, n\} \cup \alpha]), \quad \det(C[\{1, 2\}]) = \det(B'[\{1, 2\} \cup \alpha]).$$

Then, by observing that $B'[\{n\} \cup \alpha] = I_2$, that $B'[\{m\} \cup \alpha] = J_2$, that $B'[\{n-1, n\} \cup \alpha] = I_3$ and that $B'[\{1, 2\} \cup \alpha] = J_2 \oplus J_1$, we conclude that $\det(C[\{n\}])$ and $\det(C[\{n-1, n\}])$ are nonzero, and that $\det(C[\{m\}])$ and $\det(C[\{1, 2\}])$ are zero. Hence, $\ell_1\ell_2 = \mathtt{SS}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



Case 2: $n+1=4k$ for some integer $k \geq 2$. Observe that, by Lemma 3.6, $\ell''_n \ell''_{n+1} = \mathtt{AN}$. Let $\alpha = \{n+1\}$, let $C = B''/B''[\alpha]$, and let $\operatorname{epr}(C) = \ell_1 \ell_2 \cdots \ell_n$. As in Case 1, we show that $C$ is a matrix with the desired properties. By the Schur Complement Corollary, $\ell_{n-1}\ell_n = \mathtt{AN}$. To show that $\ell_1\ell_2 = \mathtt{SS}$, first, observe that, by the Schur Complement Theorem, and because $\det(B''[\alpha]) = 1$ (in characteristic 2),

$$\det(C[\{n\}]) = \det(B''[\{n\} \cup \alpha]), \quad \det(C[\{1\}]) = \det(B''[\{1\} \cup \alpha]),$$

$$\det(C[\{n-1,n\}]) = \det(B''[\{n-1,n\} \cup \alpha]), \quad \det(C[\{1,2\}]) = \det(B''[\{1,2\} \cup \alpha]).$$

Then, by observing that $B''[\{n\}\cup\alpha] = I_2$, $B''[\{1\}\cup\alpha] = J_2$, $B''[\{n-1,n\}\cup\alpha] = I_3$, and $B''[\{1,2\}\cup\alpha] = J_3$, we conclude that $\det(C[\{n\}])$ and $\det(C[\{n-1,n\}])$ are nonzero, and that $\det(C[\{1\}])$ and $\det(C[\{1,2\}])$ are zero. Hence, $\ell_1\ell_2 = \mathtt{SS}$. ∎

Together with Theorems 3.2 and 3.8, the next result completes the characterization of the attainable epr-sequences over $\mathbb{Z}_2$.

Theorem 3.11. *An epr-sequence starting with* $\mathtt{S}$ *is attainable by a symmetric matrix over* $\mathbb{Z}_2$ *if and only if it has one of the following forms.*

1. $\mathtt{S\overline{S}N\overline{N}}$,
2. $\mathtt{S\overline{S}A}$,
3. $\mathtt{S\overline{S}AA}$,
4. $\mathtt{SS\overline{S}AN}$,
5. $\mathtt{SASA\overline{SA}}$,
6. $\mathtt{SASA\overline{SA}A}$,
7. $\mathtt{SA\overline{SA}N}$.

*Proof.* Let $\sigma = \ell_1\ell_2\cdots\ell_n$ be an epr-sequence with $\ell_1 = \mathtt{S}$. Suppose that $\sigma = \operatorname{epr}(B)$, where $B$ is a symmetric matrix over $\mathbb{Z}_2$. Since an attainable epr-sequence cannot end with $\mathtt{S}$, $n \geq 2$. If $n=2$, then $\sigma$ is $\mathtt{SA}$ or $\mathtt{SN}$. Suppose $n \geq 3$. If $\ell_2 = \mathtt{A}$ or $\ell_2 = \mathtt{N}$, then Proposition 2.11 and the $\mathtt{N}$-Even Observation imply that $\sigma$ is either $\mathtt{SA\overline{SA}}$, $\mathtt{SA\overline{SA}A}$, $\mathtt{SA\overline{SA}N}$, or $\mathtt{SNN\overline{N}}$. Thus, suppose $\ell_2 = \mathtt{S}$. Hence, by Theorem 2.16, $\sigma$ does not contain $\mathtt{ASA}$. Let $k$ be a minimal integer with $2 \leq k \leq n-1$ such that $\ell_k\ell_{k+1} = \mathtt{SN}$ or $\ell_k\ell_{k+1} = \mathtt{SA}$; in the former case, Observation 2.3 implies that $\sigma = \mathtt{SS\overline{S}N\overline{N}}$. Now consider the latter case, namely $\ell_k\ell_{k+1} = \mathtt{SA}$. If $n = k+1$, then $\sigma = \mathtt{SS\overline{S}A}$. Thus, suppose $n \geq k+2$.

We now show that $n = k+2$. Suppose to the contrary that $n \geq k+3$. By the $\mathtt{AA}$ Theorem, $\ell_{k+2} \neq \mathtt{A}$. If $\ell_{k+2} = \mathtt{N}$, then Observation 2.3 implies that $\sigma$ contains $\mathtt{SANN}$, which is prohibited by Proposition 2.13; hence, $\ell_{k+2} = \mathtt{S}$, so that $\ell_k\ell_{k+1}\ell_{k+2} = \mathtt{SAS}$. Then, as $\sigma$ does not contain $\mathtt{ASA}$, and because $\mathtt{SASN}$ is prohibited by Proposition 2.13, $\ell_{k+3} = \mathtt{S}$, implying that $\sigma$ contains $\mathtt{ASS}$ as a non-initial subsequence, a contradiction to Proposition 2.14. It follows that $n = k+2$, and therefore that $\sigma$ is either $\mathtt{SS\overline{S}AA}$ or $\mathtt{SS\overline{S}AN}$.

Now, we establish the other direction. We assume that the sequence under consideration has order $n$. The sequence $\mathtt{S\overline{S}N\overline{N}}$ is attainable by applying Observation 1.10(2) to the attainable sequence $\mathtt{A\overline{A}}$. The sequence $\mathtt{S\overline{S}A}$ is attainable by [4, Observation 2.16]. The attainability of $\mathtt{S\overline{S}AA}$ follows by observing that, by the Inverse Theorem, the inverse of any symmetric matrix attaining the sequence $\mathtt{A\overline{S}SA}$, which is attainable by Theorem 3.8, has epr-sequence $\mathtt{S\overline{S}AA}$. To see that the sequence $\mathtt{SS\overline{S}AN}$ is attainable, observe that the argument above forces the matrix $G_n$ in Lemma 3.9 to attain this sequence when $n$ is even, and that it forces the matrix whose existence was established in Lemma 3.10 to attain this sequence when $n$ is odd. Finally, the sequences $\mathtt{SASA\overline{SA}}$, $\mathtt{SASA\overline{SA}A}$ and $\mathtt{SA\overline{SA}N}$ are attainable by Proposition 2.11. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 273-290, August 2017.



To conclude, we note that there is no known characterization of the epr-sequences that are attainable by symmetric matrices over the real field or any other field besides $\mathbb{Z}_2$. However, the results of Theorems 3.2, 3.8 and 3.11 provide such a characterization for symmetric matrices over $\mathbb{Z}_2$.

**Acknowledgement.** The author expresses his gratitude to Dr. Leslie Hogben, for introducing him to the topic of pr- and epr-sequences.

REFERENCES

[1] W. Barrett, S. Butler, M. Catral, S.M. Fallat, H.T. Hall, L. Hogben, P. van den Driessche, and M. Young. The principal rank characteristic sequence over various fields. *Linear Algebra and its Applications*, 459:222–236, 2014.

[2] R.A. Brualdi, L. Deaett, D.D. Olesky, and P. van den Driessche. The principal rank characteristic sequence of a real symmetric matrix. *Linear Algebra and its Applications*, 436:2137–2155, 2012.

[3] R.A. Brualdi and H. Schneider. Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley. *Linear Algebra and its Applications*, 52/53:769–791, 1983.

[4] S. Butler, M. Catral, S.M. Fallat, H.T. Hall, L. Hogben, P. van den Driessche, and M. Young. The enhanced principal rank characteristic sequence. *Linear Algebra and its Applications*, 498:181–200, 2016.

[5] S. Butler, M. Catral, H.T. Hall, L. Hogben, X. Martínez-Rivera, B. Shader, and P. van den Driessche. The enhanced principal rank characteristic sequence for Hermitian matrices. *Electronic Journal of Linear Algebra*, 32:58–75, 2017.

[6] S.M. Fallat, D.D. Olesky, and P. van den Driessche. The enhanced principal rank characteristic sequence for skew-symmetric matrices. *Linear Algebra and its Applications*, 498:366–377, 2016.

[7] S. Friedland. Quadratic forms and the graph isomorphism problem. *Linear Algebra and its Applications*, 150:423–442, 1991.

[8] O. Holtz and H. Schneider. Open problems on GKK $\tau$-matrices. *Linear Algebra and its Applications*, 345:263–267, 2002.

[9] O. Holtz and B. Sturmfels. Hyperdeterminantal relations among symmetric principal minors. *Journal of Algebra*, 316:634–648, 2007.

[10] X. Martínez-Rivera. Classification of families of pr- and epr-sequences. *Linear and Multilinear Algebra*, 65:1581–1599, 2017.

[11] T. Muir. The law of extensible minors in determinants. *Transactions of the Royal Society of Edinburgh*, 30:1–4, 1883.

[12] F. Zhang (editor). *The Schur Complement and its Applications*. Springer-Verlag, New York, 2005.