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INVERSES OF BICYCLIC GRAPHS*

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Abstract. A graph G is said to be nonsingular (resp., singular) if its adjacency matrix $A(G)$ is nonsingular (resp., singular). The inverse of a nonsingular graph G is the unique weighted graph whose adjacency matrix is similar to the inverse of the adjacency matrix $A(G)$ via a diagonal matrix of ± 1s. Consider connected bipartite graphs with unique perfect matchings such that the graph obtained by contracting all matching edges is also bipartite. In [C.D. Godsil. Inverses of trees. *Combinatorica*, 5(1):33–39, 1985.], Godsil proved that such graphs are invertible. He posed the question of characterizing the bipartite graphs with unique perfect matchings possessing inverses. In this article, Godsil's question for the class of bicyclic graphs is answered.

Key words. Adjacency matrix, Bicyclic graph, Perfect Matching, Alternating path, Inverse graph.

AMS subject classifications. 05C50, 15A09.

1. Introduction. Let G be a simple, undirected graph on vertices $1, 2, \ldots, n$. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $[i,j]$ to denote an edge between the vertices i and j. The *adjacency* matrix $A(G)$ of G is the square symmetric matrix of size n whose (i,j)th entry a_{ij} is given by

$$a_{ij} = \begin{cases} 1 & \text{if } [i,j] \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

A graph is called *nonsingular* (resp., *singular*) if $A(G)$ is nonsingular (resp., singular). A *perfect matching* in a graph G is a spanning forest whose components are paths on two vertices. If a graph has unique perfect matching, then we denote it by $\mathcal{M}$. Furthermore, when v is a vertex, we shall always use v' to denote the matching mate for v, that is, v' is the vertex for which the edge $[v, v'] \in \mathcal{M}$. Let $\mathcal{H}$ be the class of connected bipartite graphs with unique perfect matchings. Let $G \in \mathcal{H}$. In [3], the author used $G/\mathcal{M}$ to denote the graph which is obtained from G by contracting each matching edge to a single vertex. A *bicyclic* graph on n vertices is a connected graph with $n + 1$ edges. Let G be a graph and $K \subseteq E(G)$. By $G - K$ we denote the graph which is obtained from G by deleting all the edges in K.

In quantum chemistry, a graph known as the Hückel graph is used to model the molecular orbital energies of a hydrocarbon. Under some conditions a Hückle graph may be seen as a bipartite graph with a unique perfect matching, see [9]. This is one of the motives to consider bipartite graphs with unique perfect matchings.

DEFINITION 1.1. [3] Let $G \in \mathcal{H}$. We say G has an inverse G^+ if the matrix $A(G)^{-1}$ is signature similar to a nonnegative matrix, that is, $SA(G)^{-1}S \geq 0$ for some signature matrix S and G^+ is a weighted graph associated to the matrix $SA(G)^{-1}S$. Recall that, a signature matrix is a diagonal matrix with diagonal entries from $\{1, -1\}$. A graph G which is isomorphic to its own inverse, is called a *self-inverse* graph.

In [3], Godsil proved that each connected bipartite graph $G \in \mathcal{H}$ with a unique perfect matching such that $G/\mathcal{M}$ is bipartite is invertible. This raised the following interesting question: *Which graphs in $\mathcal{H}$ possess an inverse?* In [1], Akbari and Kirkland supplied a characterization of unicyclic graphs in $\mathcal{H}$ which possess inverses. In [7], Tifenbach and Kirkland supplied necessary and sufficient conditions for graphs in $\mathcal{H}$ to possess inverses, utilizing constructions derived from the graph itself. Below we mention that result recalling the necessary definitions and results while supplying an illustration.

The following lemma is essentially contained in [3].

LEMMA 1.2. *Let $G \in \mathcal{H}$. Then the adjacency matrix of G can be expressed as*

$$A(G) = \left[\begin{array}{cc} 0 & B_G \\ B_G^t & 0 \end{array} \right],$$

where B_G is a lower-triangular, square $(0,1)$-matrix with every diagonal entry equal to 1.

DEFINITION 1.3. [4] Consider a graph G with a unique perfect matching $\mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an *alternating path* if the edges on P are alternately matching and nonmatching edges, that is, for each i, if $[u_i, u_{i+1}]$ is a matching (resp., nonmatching) edge and $[u_{i+1}, u_{i+2}] \in E(G)$, then $[u_{i+1}, u_{i+2}]$ is a nonmatching (resp., matching) edge. Let $P = [u_1, u_2, \ldots, u_{2k}]$ be an alternating path. We say P is an *mm-alternating path* (matching-matching-alternating path) if $[u_1, u_2], [u_{2k-1}, u_{2k}] \in \mathcal{M}$. We say P is an *nn-alternating path* (nonmatching-nonmatching-alternating path) if $[u_1, u_2], [u_{2k-1}, u_{2k}] \notin \mathcal{M}$.

DEFINITION 1.4. [7] Let $G \in \mathcal{H}$ and B_G be the matrix as mentioned in Lemma 1.2. By D_G, denote the directed graph with adjacency matrix $B_G - I$. Let Γ_G be the directed subgraph of D_G such that the arc $x \to y$ is in Γ_G if $x \to y$ is in D_G and there is no directed x-y-path of length more than 1.

Fix vertices i and j. The undirected interval $G[i,j]$ is the subgraph of graph G induced by the vertices x such that there is an mm-alternating i-j-path in G which contains x.

THEOREM 1.5. [7, Theorem 2.6] *Let $G \in \mathcal{H}$. Then G^+ exists if and only if both the following conditions hold:*

(i) Each nonempty undirected interval $G[i,j]$ in G possesses an inverse.
(ii) The digraph Γ_G is bipartite.

REMARK 1.6. Let $G \in \mathcal{H}$ be a bicyclic graph such that $G[i,j] = G$ for some $i, j \in V(G)$. Even if we assume that the digraph Γ_G is bipartite and that each undirected interval $G[u,v] \neq G$ possesses an inverse, we cannot use Theorem 1.5 to show that G^+ exists. In fact, with these conditions G^+ may or may not exist, as shown below.

A) Consider the graph G shown in Figure 1. One can easily check that Γ_G is bipartite. The undirected interval $G[1, 4'] = G$. Any undirected interval $G[u,v] \neq G[1, 4']$ is a tree in $\mathcal{H}$. Hence $G[u,v]$ is invertible. By using Theorem 1.5, G is invertible if $G[1, 4'] = G$ is invertible. So Theorem 1.5 does not help to check whether G is invertible or not. Notice that G is invertible with the signature matrix $S = diag[1, 1, -1, -1, 1, 1, -1, -1]$.

B) Consider the graph H shown in Figure 1. One can easily check that Γ_H is bipartite. The undirected interval $H[1, 4'] = H$. Any undirected interval $H[u,v] \neq H[1, 4']$ is either a tree in $\mathcal{H}$ or a unicyclic graph in $\mathcal{H}$. By using Theorem 2.2 in [3] and Theorem 12 in [1], $H[u,v]$ is invertible. By using Theorem 1.5, H is invertible if $H[1, 4'] = H$ is invertible. So Theorem 1.5 does not help to check whether H is invertible or not. By using Example 25 in [4], the graph H is not invertible.
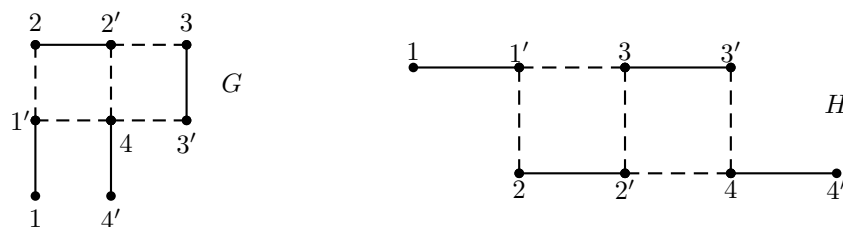
FIGURE 1. *Here, the solid edges are the matching edges.*

In [4], the authors introduced the 'even'ness of a nonmatching edge of a connected bipartite graph with a unique perfect matching and showed that under certain conditions a connected bipartite graph with a unique perfect matching has an inverse. This result extended some known results, providing us with a larger class of graphs possessing inverses. This results can also be seen to complement the results by Tifenbach and Kirkland. Below we mention that result recalling the necessary definitions while supplying an illustration.

DEFINITION 1.7. [4] Let G be a connected graph with a unique perfect matching $\mathcal{M}$ and $[u, v] \notin \mathcal{M}$. An *extension* at $[u, v]$ is an nn-alternating u-v-path other than $[u, v]$. An extension at $[u, v]$ is called *even type* (resp., *odd type*) if the number of nonmatching edges on that extension is even (resp., odd). For example, in the graph G shown in Figure 2, the path $[i', x_1, x_1', x_2, x_2, i_1]$ is an odd type extension at $[i', i_1]$ and the path $[i_2', u_1, u_1', u_2, u_2', u_3, u_3', i_3]$ is an even type extension.

REMARK 1.8. Let $G \in \mathcal{H}$ and Q_e be an extension at e in G. Then Q_e can never be an extension at some other nonmatching edge in G as G is simple.

The following definition differentiates between the nonmatching edges of a graph in $\mathcal{H}$.

DEFINITION 1.9. [4] The nonmatching edge $[u, v]$ is said to be an *odd type* edge, if either there are no extensions at $[u, v]$ or each extension at $[u, v]$ is odd type. We say $[u, v]$ is an *even type* edge, if each extension at $[u, v]$ is even type. We say $[u, v]$ is *mixed type*, if it has an even type extension and an odd type extension. Let $\mathcal{E}$ be the set of all even type edges of G.

EXAMPLE 1.10. [4] In the graph G shown in Figure 2, the nn-alternating paths $[i_2', u_1, u_1', u_2, u_2', u_3, u_3', i_3]$ and $[i_2', u_1, u_1', u_2, u_2', v_1, v_1', v_2, v_2', u_3, u_3', i_3]$ are two even type extensions at $[i_2', i_3]$. These are the only extensions at $[i_2', i_3]$. Hence, $[i_2', i_3]$ is an even type edge. The edge $[i', i_1]$ is mixed type. Every other nonmatching edge is of odd type.
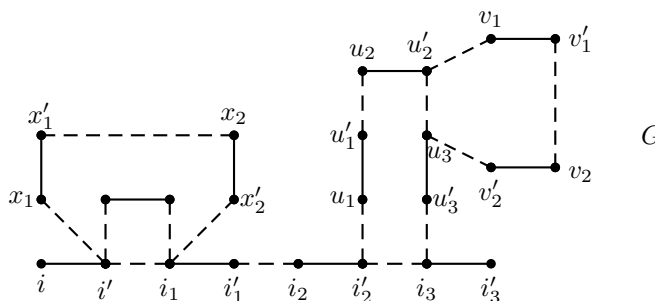
FIGURE 2. *Here, the solid edges are the matching edges.*

REMARK 1.11. Let $G \in \mathcal{H}$. Assume that $\mathcal{E}$, $\mathcal{O}$ and $\mathcal{F}$ are the set of even type, odd type and mixed type edges in G, respectively. Then the graph $G - \mathcal{E} - \mathcal{O} - \mathcal{F}$ is connected.

DEFINITION 1.12. [4] Let $\mathcal{H}_{nm}$ be the class of all graphs $G \in \mathcal{H}$ such that G has no mixed type edges. Here 'nm' is an abbreviation of 'no mixed type edges'.

DEFINITION 1.13. [4] By $\mathcal{H}_{nmc}$, we denote the class of graphs G in $\mathcal{H}$ such that

i) $G \in \mathcal{H}_{nm}$,
ii) G satisfies the condition 'C',

C: *The extensions at two distinct even type edges never have an odd type edge in common.* Here 'nmc' is an abbreviation of 'no mixed type edges and a condition'. Thus,

$$\mathcal{H}_{nmc} = \{G \in \mathcal{H} \mid G \text{ has no mixed type edges and } G \text{ satisfies condition C}\}.$$

DEFINITION 1.14. Let $G \in \mathcal{H}$. By $(G - \mathcal{E})/\mathcal{M}$ denote the graph which is obtained from G by deleting all the even type edges and then contracting each matching edge to a single vertex.

THEOREM 1.15. [4] *Let $G \in \mathcal{H}_{nmc}$. Then G^+ exists if and only if $(G - \mathcal{E})/\mathcal{M}$ is bipartite.*

REMARK 1.16. There are invertible and non-invertible bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nmc}$.

A) Consider the graph G shown in Figure 1. The graph $G \in \mathcal{H} \setminus \mathcal{H}_{nmc}$, as G has mixed type edge $[1', 4]$. We have already seen that this graph is invertible.
B) Consider the graph H shown in Figure 1. The graph $H \in \mathcal{H} \setminus \mathcal{H}_{nmc}$, as H does not satisfy condition 'C'(that is, two distinct even type extensions $[1', 2, 2', 3]$ at $[1', 3]$ and $[2', 3, 3', 4]$ at $[2', 4]$ in H have an odd type edge $[2', 3]$ in common). We have already seen that H is not invertible.

Remarks 1.6 and 1.16 tell us that we can not identify the invertible bicyclic graphs in $\mathcal{H}$ using Theorems 1.5 and 1.15. This brings the following natural question. *Characterize the bicyclic graphs in $\mathcal{H}$ which possess inverses.* In this article, we present a complete characterization of the bicyclic graphs with unique perfect matchings which possess inverses. Let $G \in \mathcal{H}$ be a bicyclic graph. Then G is either in $\mathcal{H}_{nm}$ or in $\mathcal{H} \setminus \mathcal{H}_{nm}$. In Section 2, we characterize the bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$ which possess inverses. In Section 3, we characterize the bicyclic graphs in $\mathcal{H}_{nm}$ which possess inverses.

We briefly mention some of the literature in this area. Characterizing the self-inverse graphs $G \in \mathcal{H}$ with $G/\mathcal{M}$ is bipartite was done in [6]. In [5], the authors supplied a constructive characterization of a class of graphs H such that H is an inverse graph of some graph G in $\mathcal{H}$ with $G/\mathcal{M}$ is bipartite. In the same

article, the authors extended the notion of inverse graph to weighted graphs. In [8], the author supplied a necessary and sufficient condition for a graph G in $\mathcal{H}$ to be self-inverse.

Before going to start our discussions, we first supply a set of examples of bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$.

EXAMPLE 1.17. Examples of bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$. In the following table, we list the graphs used in Figure 3.

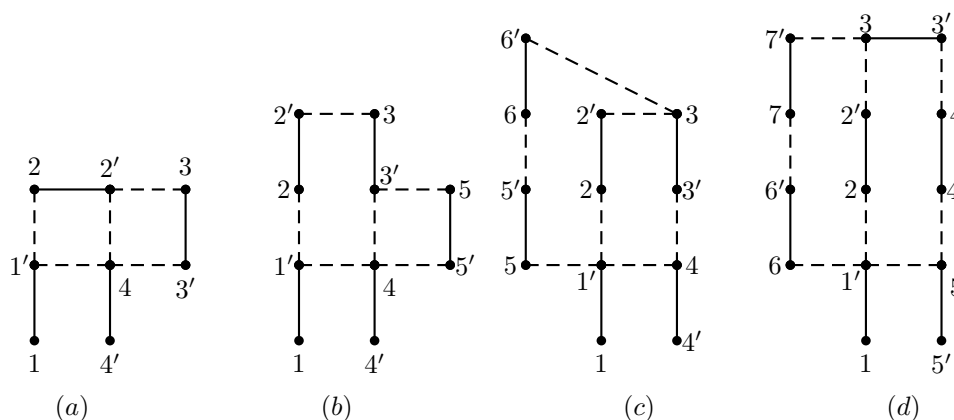| Graphs G	Mixed type edges	even type extensions	odd type extensions
Figure 3(a)	$[1', 4]$	$[1', 2, 2', 4]$	$[1', 2, 2', 3, 3', 4]$
Figure 3(b)	$[1', 4]$	$[1', 2, 2', 3, 3', 5, 5', 4]$	$[1', 2, 2', 3, 3', 4]$
Figure 3(c)	$[1', 4]$	$[1', 5, 5', 6, 6', 3, 3', 4]$	$[1', 2, 2', 3, 3', 4]$
Figure 3(d)	$[1', 5]$	$[1', 2, 2', 3, 3', 4, 4', 5]$	$[1', 6, 6', 7, 7', 3, 3', 4, 4', 5]$

FIGURE 3. *Here, the solid edges are the matching edges.*

2. Inverses of bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$. In this section, we characterize the bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$ which possess inverses. Each bicyclic graph of order n has $n + 1$ edges. Hence, each bicyclic graph has *at most three cycles* and there are *at most four paths* from one vertex to another vertex in G. We first supply some structural descriptions of bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$ which are necessary to state and prove our main result of this section. We begin our discussions by supplying the following result.

LEMMA 2.1. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph. Assume that $m = [u, v]$ is a mixed type edge in G. Then G has at least two distinct cycles containing the edge m. Furthermore, G has exactly three cycles.*

Proof. Let $Q_1(u, v)$ and $Q_2(u, v)$ be even and odd type extensions at $m = [u, v]$, respectively. Consider $\Gamma_1 = [Q_1(u, v), [v, u]]$, $\Gamma_2 = [Q_2(u, v), [v, u]]$. It is clear that Γ_i is a cycle in G for $i = 1, 2$. Notice that $\Gamma_1 \neq \Gamma_2$ as $Q_1(u, v) \neq Q_2(u, v)$. Hence, G has at least two distinct cycles in G containing the edge $m = [u, v]$.

Since $Q_1(u, v)$ and $Q_2(u, v)$ are two paths from u to v, we have a cycle Γ created by these two paths. The edge $m \notin \Gamma$ because $m \notin Q_1(u, v) \cup Q_2(u, v)$. Hence, Γ, Γ_1 and Γ_2 are three distinct cycles in G. Therefore, G has three cycles.   □

The following observation tells us about the number of mixed type edges for a bicyclic graph in $\mathcal{H} \setminus \mathcal{H}_{nm}$.

LEMMA 2.2. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph. Then G has exactly one mixed type edge.*

Proof. Suppose that G has two mixed type edges m_1 and m_2. By using Lemma 2.1, G has at least two distinct cycles containing the edge m_i for $i = 1, 2$. Hence, G has at least four distinct cycles, a contradiction to the fact that G has three cycles. □

REMARK 2.3. Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph. Then G has exactly one mixed type edge. Henceforth, we use $m = [u, v]$ to denote the mixed type edge in G.

To proceed further we need the following known result.

LEMMA 2.4. [4] *Let $G \in \mathcal{H}$ and let $[u, v]$ be an odd type or even type edge in G. Let $Q(u, v)$ be an extension at $[u, v]$. Then each nonmatching edge on $Q(u, v)$ is odd type.*

The following lemma tells about the number of even type edges for a bicyclic graph in $\mathcal{H} \setminus \mathcal{H}_{nm}$.

LEMMA 2.5. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph of order n. Then G has at most one even type edge. Furthermore, if G has an even type edge e, then e has exactly one even type extension.*

Proof. Suppose that G has two even type edges, say e and f. Let m be the mixed type edge in G. We notice that $G - \{e, f, m\}$ is connected. Therefore, $G - \{e, f, m\}$ has at least $n - 1$ edges. Hence, G has at least $n + 2$ edges, a contradiction to the fact that G is bicyclic.

We assume that G has an even type edge $e = [x, y]$. We show that e has exactly one even type extension. Suppose that e has two even type extensions. Let $Q_1(x, y)$ and $Q_2(x, y)$ be two even type extensions at e. By using Lemma 2.4, each nonmatching edge on $Q_i(x, y)$ is odd type for $i = 1, 2$. Then $m \notin Q_i$ for $i = 1, 2$. Let $Q_3(u, v)$ and $Q_4(u, v)$ be two extensions at $m = [u, v]$. Then the cycles $[Q_1(x, y), [y, x]]$, $[Q_2(x, y), [y, x]]$, $[Q_3[u, v], [v, u]]$ and $[Q_4(u, v), [v, u]]$ are four distinct cycles in G, a contradiction that G has exactly three cycles. Hence, e has exactly one even type extension. □

LEMMA 2.6. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph. Assume that G has an even type edge $e = [x, y]$. Then e must be present on some extension at $m = [u, v]$.*

Proof. Suppose that e is not present on any extension at m. Consider the graph $G - e$. Notice that $(G - e) \in \mathcal{H} \setminus \mathcal{H}_{nm}$. By using Lemma 2.1, $G - e$ has three cycles. Therefore, G has at least four cycles, a contradiction to the fact that G has exactly three cycles. □

To proceed further we need the following lemma.

LEMMA 2.7. [4] *Let $G \in \mathcal{H}$ and $P(i, j)$ be an mm-alternating i-j-path. Let $[u, v]$ be a nonmatching edge on $P(i, j)$ and $Q(u, v)$ be an extension at $[u, v]$. Then $Q(u, v)$ contains no vertex of $P(i, j)$ other than u and v. That is, $V(P(i, j)) \cap V(Q(u, v)) = \{u, v\}$.*

The following lemma tells us about the number of extensions for a bicyclic graph in $\mathcal{H} \setminus \mathcal{H}_{nm}$.

LEMMA 2.8. *Let G be a bicyclic graph in $\mathcal{H}$. Then the graph G has at most two extensions.*

Proof. Suppose that G has three extensions in G, say, Q_1, Q_2 and Q_3. There are three cases.

Case I. Assume that Q_1, Q_2 and Q_3 are extensions at $e = [i, j]$. Then there are four distinct cycles in G which are $[Q_1(i, j), [j, i]]$, $[Q_2(i, j), [j, i]]$, $[Q_3(i, j), [j, i]]$ and a cycle created by the paths Q_1 and Q_2, a contradiction to the fact that G has at most three cycles.

Case II. Assume that Q_1 and Q_2 are extensions at $e_1 = [i_1, j_1]$ and Q_3 is an extension at $e_2 = [i_2, j_2]$,

where $e_1 \neq e_2$. Then there are four distinct cycles in G which are $[Q_1(i_1, j_1), [j_1, i_1]]$, $[Q_2(i_1, j_1), [j_1, i_1]]$, $[Q_3(i_2, j_2), [j_2, i_2]]$ and a cycle created by the paths Q_1 and Q_2, a contradiction to the fact that G has at most three cycles.

Case III. Assume that Q_1, Q_2 and Q_3 are extensions at $e_1 = [i_1, j_1]$ $e_2 = [i_2, j_2]$ and $e_3 = [i_3, j_3]$, respectively, where $e_1 \neq e_2 \neq e_3$. The graph $G - \{e_1, e_2, e_3\}$ is a connected graph with at least $n - 1$ edges. Then G has at least $n + 2$ edges, a contradiction to the fact that G is bicyclic. Hence, G has at most two extensions.   □

EXAMPLE 2.9. Here, we supply a bicyclic graph in $\mathcal{H} \setminus \mathcal{H}_{nm}$ with an even type edge. Consider the graph G shown in Figure 4. The edge $[1', 5]$ is the mixed type edge with the extensions $[1', 2, 2', 3, 3', 4, 4', 5]$ and $[1', 2, 2', 6, 6', 7, 7', 8, 8', 3, 3', 4, 4', 5]$ in G. The edge $[3', 4]$ is the even type edge with the extension $[3', 6, 6', 7, 7', 8, 8', 4]$ and this even type edge is also an edge of the extension $[1', 2, 2', 3, 3', 4, 4', 5]$ at the mixed type edge $[1', 5]$.
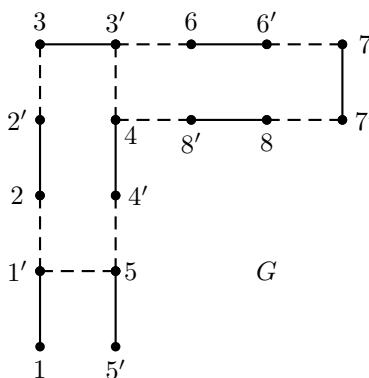

FIGURE 4. *A graph in $\mathcal{H} \setminus \mathcal{H}_{nm}$ with even type edge. Here, the solid edges are the matching edges.*

REMARK 2.10. Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ with an even type edge $[x, y]$. By using Lemma 2.6, $[x, y]$ must be present on some extension at $m = [u, v]$. Let $Q_1(u, v)$ be such extension at m. Let $Q_2(x, y)$ be the even type extension at $[x, y]$. Then the following are true.

• Using Lemma 2.8, there are exactly two extensions at $m = [u, v]$. By using Lemma 2.7, $V(Q_1(u, v)) \cap V(Q_2(x, y)) = \{x, y\}$. Then the path $[Q_1(u, x), Q_2(x, y), Q_1(y, v), [v, u]]$ is the other extension at m.

• By using Lemma 2.1, the graph G has exactly three cycles which are $[[v, u], Q_1(u, x), Q_2(x, y), Q_1(y, v)]$, $[Q_1(u, v), [v, u]]$ and $[Q_2(x, y), [y, x]]$. Hence, each cycle in G contains either the even type edge or the mixed type edge.

• There is no mm-alternating path from one vertex to another vertex in G containing both the even type edge and the mixed type edge, simultaneously. Suppose that there is such path, say, $P(i, j)$. We consider the even type extension Q at m which contains the even type edge $[x, y]$. Then $\{x, y, u, v\} \subseteq V(P(i, j)) \cap V(Q(u, v))$ which is not possible by Lemma 2.7.

The following observation tells us about the number of mm-alternating paths from one vertex to another vertex not containing the even type edge and the mixed type edge.

LEMMA 2.11. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph and i, j be two vertices. Assume that there is an mm-alternating i-j-path in G. Then there is exactly one mm-alternating i-j-path in G not containing the even type edge and the mixed type edge.*

Proof. Let $P(i, j)$ be an mm-alternating i-j-path in G. If $P(i, j)$ does not contain the even type and the mixed type edges, then there is nothing to show. We assume that either the even type edge or the mixed type edge must be present on $P(i, j)$. By using Remark 2.10, both the even type edge and the mixed type edge cannot be present on $P(i, j)$, simultaneously. First we assume that $P(i, j)$ contains the even type edge, say, $e = [x, y]$. Let $Q(x, y)$ be the even type extension at e. Consider the path $[P(i, x), Q(x, y), P(y, j)]$. This is an mm-alternating i-j-path not containing the even type edge and the mixed type edge. Similar arguments work if $P(i, j)$ contains the mixed type edge.

We now prove the uniqueness. Suppose that there are two mm-alternating i-j-paths not containing the even type edge and the mixed type edge. Let $P_1(i, j)$ and $P_2(i, j)$ be such paths. Then these two paths create a cycle not containing the even type edge and the mixed type edge. But either the even type edge or the mixed type edge must be present on each cycle in G by Remark 2.10. Hence, there is exactly one mm-alternating i-j-path in G not containing the even type edge and the mixed type edge. □

The following observation tells us about the number of mm-alternating paths from one vertex to another vertex containing the even (resp., mixed) type edge.

LEMMA 2.12. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph. Then there is at most one mm-alternating path from one vertex to another vertex in G containing the even (resp., mixed) type edge.*

Proof. Suppose that there are two vertices i, j in G such that there are two mm-alternating i-j-paths containing the even (resp., mixed) type edge $e = [x, y]$. Let $P_1(i, j)$ and $P_2(i, j)$ be two such paths. By using Remark 2.10, both the even type edge and the mixed type edge cannot be present on $P(i, j)$, simultaneously. Then these two paths create a cycle not containing the even type and the mixed type edges. But either the even type edge or the mixed type edge must be present on each cycle in G by Remark 2.10. Hence, there is at most one mm-alternating i-j-path containing the even (resp., mixed) type edge. □

The following observation supplies a bound on the number of mm-alternating paths from one vertex to another vertex.

LEMMA 2.13. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be a bicyclic graph. Then there are at most three mm-alternating paths from one vertex to another vertex in G.*

Proof. Suppose that there are two vertices i, j in G such that the number of mm-alternating i-j-paths is more than three. By using Lemma 2.12, there is at most one mm-alternating i-j-path containing an even (resp., mixed) type edge. Then there are at least two mm-alternating i-j-paths not containing the even type edge and the mixed type edge which is not possible by Lemma 2.11. Hence, there are at most three mm-alternating paths in G from one vertex to another vertex. □

The following description of the inverse of the adjacency matrix of a connected bipartite graph with a unique perfect matching is a restatement of results contained in [1, 2]. We follow the convention that a sum over an empty set is zero.

LEMMA 2.14. [1, 2] *Let $G \in \mathcal{H}$. Let $B = [b_{ij}]$, where*

$$b_{ij} = \sum_{P(i,j) \in \mathcal{P}(i,j)} (-1)^{(\|P(i,j)\| - 1)/2},$$

where $\mathcal{P}(i,j)$ is the set of mm-alternating i-j-paths in G and $\|P(i,j)\|$ denotes the number of edges in $P(i,j)$. Then $B = A(G)^{-1}$.

REMARK 2.15. Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ and i, j be two vertices. Then exactly one of the following happens.

- There are no mm-alternating i-j-paths in G.

- There is exactly one mm-alternating i-j-path in G.

- There are exactly two mm-alternating i-j-paths in G, say, P_1 and P_2. By using Lemma 2.11, one of these two paths does not contain the even type and the mixed type edges, and the other one contains either the even type edge or the mixed type edge. Without loss of generality we assume that P_1 is the path not containing the even type and the mixed type edges. Then the path P_2 contains either the even type edge or the mixed type edge. But the path P_2 cannot contain the mixed type edge, otherwise there are three mm-alternating i-j-paths.

The path P_1 contains an even (resp., odd) number of nonmatching edges if and only if P_2 contains an odd (resp., even) number of nonmatching edges. By using Lemma 2.14, $A_{i,j}^{-1} = 0$.

- There are exactly three mm-alternating i-j paths, say, P_1, P_2 and P_3. By using Lemma 2.11, there is exactly one path not containing the even type and the mixed type edges. By using Lemma 2.12 and Remark 2.10, there is at most one mm-alternating i-j-path containing the even (resp., mixed) type edge but not containing the mixed (resp., even) type edge. Since G has exactly three mm-alternating i-j-paths, there are exactly two mm-alternating i-j-paths such that one path contains the even type edge and the other one contains the mixed type edge. Without loss of generality we assume that P_1 does not contain the even type and the mixed type edges and P_2 contains the even type edge e and P_3 contains the mixed type edge m. Since G is bicyclic, the path P_1 must contain the odd type extension at m.

The path P_1 contains an even (resp., odd) number of nonmatching edges if and only if P_2 contains an odd (resp., even) number of nonmatching edges if and only if P_3 contains an even (resp., odd) number of nonmatching edges. By using Lemma 2.14, $A_{i,j}^{-1} = 1$ if P_1 contains an even number of nonmatching edges, otherwise $A_{i,j}^{-1} = -1$.

To proceed further we need the following lemmas.

LEMMA 2.16. [1] Let $G \in \mathcal{H}$. Consider $A(G)^{-1}$ and construct a weighted graph $\hat{G}$ from G as follows: for each pair of vertices i, j take i adjacent to j in $\hat{G}$ whenever $\sum_{P \in \mathcal{P}_{ij}} (-1)^{\frac{\|P\|-1}{2}} \neq 0$, and let the weight of that edge be 1 or -1 according as $\sum_{P \in \mathcal{P}_{ij}} (-1)^{\frac{\|P\|-1}{2}}$ is positive or negative. Then $A(G)^{-1}$ is diagonally similar to a non-negative matrix if and only if the product of the edge weights on any cycle in $\hat{G}$ is 1.

LEMMA 2.17. Let $G \in \mathcal{H}$ and $[p,q]$ be an odd type edge. Then $A(G)_{p',q'}^{-1} < 0$.

Proof. Since $[p,q]$ is an odd type edge in G, all the extensions at $[p,q]$ are odd type. Let $Q_i(p,q)$ be the odd type extension at $[p,q]$ for $i = 1, \ldots, k$. Then the paths $[p', p, q, q'], [p', Q_i(p,q), q']$ are mm-alternating p'-q'-paths for $i = 1, \ldots, k$. These are the only mm-alternating p'-q'-paths in G. By using Lemma 2.14 and Remark 2.15, $A(G)_{p',q'}^{-1} = -(k+1)$. Hence, $A(G)_{p',q'}^{-1} < 0$. □

The following is our main result of this section. This result supplies a characterization of the bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$ which possess inverses.

THEOREM 2.18. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ be bicyclic. Assume that $m = [u, v]$ is the mixed type edge in G. Then G^+ exists if and only if G has an even type edge which must be present on the even type extension at m.*

Proof. First we assume that G^+ exists. Suppose that G has no even type edges. Since $[u, v]$ is the mixed type edge in G, by using Lemma 2.8, there is exactly one even type extension and exactly one odd type extension at $[u, v]$. Let $Q_1(u, v)$ and $Q_2(\mathrm{u,v})$ be the even type and odd type extensions at $[u, v]$, respectively. There are exactly three mm-alternating u'-v' paths in G which are $[u', u, v, v']$, $[u', Q_1(u, v), v']$ and $[u', Q_2(u, v), v']$. Let $Q_1(u, v) = [u, u_1, u'_1, \ldots, u_{2k-1}, u'_{2k-1}, v]$. Since G has no even type edge, all the nonmatching edges on $Q_1(u, v)$ are odd type. By using Lemmas 2.14 and 2.17, we see that

i) $A^{-1}_{u_i, u'_i} = 1$ for all $i = 1, \ldots, 2k - 1$;

ii) $A^{-1}_{u', u'_1}, A^{-1}_{v', u_{2k-1}}, A^{-1}_{u_i, u'_{i+1}} < 0$ for all $i = 1, \ldots, 2k - 2$, as each nonmatching edge on $Q_1(u, v)$ is odd type;

iii) $A^{-1}_{u', v'} = -1$.

We see that the cycle $[u', u'_1, u_1, u'_2, u_2, \ldots, u'_{2k-1}, u_{2k-1}, v', u']$ is available in $\hat{G}$ with the product of the edge weights is equal to -1. Using Lemma 2.16, G^+ cannot exist. Therefore, G has an even type edge and by using Lemmas 2.5 and 2.6 it is unique and it must be present on some extension at $m = [u, v]$.

We now show that the even type edge must be present on the even type extension at $m = [u, v]$. Let $[x, y]$ be the even type edge. Suppose that $[x, y]$ is present on the odd type extension at $[u, v]$. Let $Q_1(u, v)$ be the odd type extension at $[u, v]$. Let $Q_2(x, y)$ be the even type extension at $[x, y]$. By using Lemma 2.7, $V(Q_2(x, y)) \cap V_1(Q(u, v)) = \{x, y\}$. Then the path $[Q_1(u, x), Q_2(x, y), Q_1(y, v)]$ is an even type extension at $[u, v]$. Since $Q_1(u, v)$ does not contain the edge $[u, v]$ and $Q_2(x, y)$ does not contain the edge $[x, y]$, all the nonmatching edges on $[Q_1(u, x), Q_2(x, y), Q_1(y, v)]$ are odd type. By using previous arguments, we deduce that G^+ cannot exist. Hence, G has exactly one even type edge which must be present on the even type extension at $[u, v]$.

We now prove the converse. Let $Q_1(u, v)$ be the even type extension at $[u, v]$ such that the extension $Q_1(u, v)$ contains the even type edge $[x, y]$. Let $Q_2(x, y)$ be the even type extension at $[x, y]$. Then $G - \{[x, y], [u, v]\}$ is a tree.

As $G - \{[x, y], [u, v]\}$ is a tree, take the vertex 1, define $s_1 = 1$. Now to define s_i, take the path from 1 to i in $G - \{[x, y], [u, v]\}$. If it has an odd many nonmatching edges define $s_i = -1$, otherwise define $s_i = 1$. The matrix S is well defined. Notice that $s_i s_j = 1$ if the path from i to j in $G - \{[x, y], [u, v]\}$ contains an even number of nonmatching edges, otherwise $s_i s_j = -1$.

Let i and j be two any vertices in G. If there is no mm-alternating i-j-path in G, then $s_i A^{-1}_{i,j} s_j = 0$. So we assume that there are mm-alternating i-j-paths in G. By using Lemma 2.13, there are at most three mm-alternating i-j-paths in G. There are three cases.

Case I. There is exactly one mm-alternating i-j path, say $P(i, j)$ in G. All the nonmatching edges on $P(i, j)$ are odd type, otherwise there are at least two mm-alternating i-j-paths in G. Then $P(i, j)$ is also an mm-alternating path in $G - \{[x, y], [u, v]\}$. Then $s_i A^{-1}_{i,j} s_j = s_i s_j A^{-1}_{i,j} = (-1)^{(\|P(i,j)\|-1)/2}(-1)^{(\|P(i,j)\|-1)/2} = 1 > 0$.

Case II. There are exactly two mm-alternating i-j paths. By using Remark 2.15, $A^{-1}_{i,j} = 0$. Hence, $s_i A^{-1}_{i,j} s_j = 0$.

Case III. There are exactly three mm-alternating i-j paths, say P_1, P_2 and P_3. There are two cases.

Case III(A). The path P_1 contains an even number of nonmatching edges. By using Remark 2.15, we have $A(G)_{i,j}^{-1} = 1$. Since the path P_1 does not contain the even type and the mixed type edges, the path P_1 is also an mm-alternating i-j-path in $G - \{[x, y], [u, v]\}$. By definition of S, we have $s_i s_j = 1$. Hence, $s_i A(G)_{i,j}^{-1} s_j = 1$.

Case III(B). The path P_1 contains an odd number of nonmatching edges. By using Remark 2.15, we have $A(G)_{i,j}^{-1} = -1$. Since the path P_1 does not contain the even type and the mixed type edges, the path P_1 is also an mm-alternating i-j-path in $G - \{[x, y], [u, v]\}$. By definition of S, we have $s_i s_j = -1$. Hence, $s_i A(G)_{i,j}^{-1} s_j = 1$. Thus, G^+ exists.   □

REMARK 2.19. We now explain why the graph G shown in Figure 1 is invertible. We already noticed that $G \in \mathcal{H} \setminus \mathcal{H}_{nm}$ and G has exactly one even type edge $[2', 4]$ which is present on the even type extension $[1', 2, 2', 4, 4']$ at the mixed type edge $[1', 4]$. Hence, by Theorem 2.18, G is invertible.

3. Inverses of bicyclic graphs in $\mathcal{H}_{nm}$. In the previous section, we characterized the bicyclic graphs in $\mathcal{H} \setminus \mathcal{H}_{nm}$ which possess inverses. We now proceed to characterize the bicyclic graphs in $\mathcal{H}_{nm}$ which possess inverses. If $G \in \mathcal{H}_{nm}$, then either $G \in \mathcal{H}_{nmc}$ or G does not satisfy the condition 'C' in Definition 1.13.

If $G \in \mathcal{H}_{nmc}$, then by using Theorem 1.15, we can say G^+ exists or not. So to characterize invertible bicyclic graphs in $\mathcal{H}_{nm}$, it is sufficient to have a characterization of invertible bicyclic graphs in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$.

REMARK 3.1. Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph. Then it has exactly two even type edges and each even type edge has exactly one even type extension. Let $e = [x_1, y_1]$ and $f = [x_2, y_2]$ be the even type edges in G with the even type extensions Q_e and Q_f, respectively. We have $E(V(Q_e)) \cap E(V(Q_f)) \neq \phi$.

• There is no mm-alternating path which contains both the even type edges e and f. If possible, let $P(i, j)$ be an mm-alternating path containing both the even type edges. Consider the path $P_1(i, j) = [P(i, x_1), Q_e(x_1, y_1), P(y_1, j)]$. Then P_1 is an mm-alternating path containing the even type extension Q_e and the even type edge $f = [x_2, y_2]$. By using Lemma 2.7, $V(P_1(i, j)) \cap V(Q_f) = \{x_2, y_2\}$. Then $E(V(Q_e)) \cap E(V(Q_f)) = \phi$ which is not possible.

• The graph G has exactly three cycles which are $[Q_e(x_1, y_1), [y_1, x_1]]$, $[Q_f(x_2, y_2), [y_2, x_2]]$ and $(Q_e \cup Q_f) \setminus E(V(Q_e)) \cap E(V(Q_f))$. Hence, each cycle in G contains an even type edge.

• The graph $G - \{e, f\}$ is a tree.

The following observation tells us about the number of mm-alternating paths from one vertex to another vertex not containing even type edges for a graph in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$.

LEMMA 3.2. *Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph and i, j be two vertices. Assume that there is an mm-alternating i-j-path in G. Then there is exactly one mm-alternating i-j-path in G not containing an even type edge.*

Proof. Let $P(i, j)$ be an mm-alternating i-j-path. If $P(i, j)$ does not contain an even type edge, then there is nothing to show. We assume that $P(i, j)$ contains an even type edge. By using Remark 3.1, $P(i, j)$ contains exactly one even type edge. Let $e = [x_1, y_1]$ be the even type edge with the even type extension Q_e. Consider the path $[P(i, x_1), Q_e(x_1, y_1), P(y_1, j)]$. This is an mm-alternating i-j-path not containing an even type edge.

We now prove the uniqueness. Suppose that there are two mm-alternating i-j-paths not containing an even type edge. Let $P_1(i, j)$ and $P_2(i, j)$ be such paths. Then these two paths create a cycle not containing

an even type edge. But each cycle in G must contain an even type edge by Remark 3.1. Hence, there is exactly one mm-alternating i-j-path in G not containing an even type edge. $\square$

LEMMA 3.3. *Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph and i, j be two vertices. Assume that there is an mm-alternating i-j-path. Then no two mm-alternating i-j-paths contain the same even type edge.*

Proof. Suppose that two mm-alternating i-j-paths contain the same even type edge $e = [x, y]$. Let $P_1(i, j)$ and $P_2(i, j)$ be two such paths. By using Remark 3.1, both the even type edges cannot be present on P_i's, simultaneously for $i = 1, 2$. Then these two paths create a cycle not containing an even type edge. But each cycle in G must contain an even type edge by Remark 3.1. Hence, no two mm-alternating i-j-paths containing the same even type edge. $\square$

The following observation supplies a bound on the number of mm-alternating paths from one vertex to another vertex for a graph in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$.

LEMMA 3.4. *Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph. Then there are at most three mm-alternating paths in G from one vertex to another vertex.*

Proof. Suppose that there are two vertices i and j in G such that the number of mm-alternating i-j-paths is more than three. By using Lemma 3.3, there are exactly two mm-alternating i-j-paths such that each path containing an even type edge. Then there are at least two mm-alternating i-j-paths not containing an even type edge which is not possible by Lemma 3.2. Hence, there are at most three mm-alternating paths in G from one vertex to another vertex. $\square$

The following is a necessary condition for a graph in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ to have exactly three mm-alternating paths from one vertex to another vertex.

LEMMA 3.5. *Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph and i, j be two vertices. Assume that there are three mm-alternating i-j-paths in G. Then there is exactly one mm-alternating i-j-path containing both the even type extensions.*

Proof. Let P_1, P_2 and P_3 be the mm-alternating i-j-paths in G. By using Lemmas 3.2 and 3.3, each even type edge must be present on exactly one mm-alternating path. Without loss of generality we assume that P_1 contains the even type edge $e = [x_1, y_1]$ and P_2 contains the even type edge $f = [x_2, y_2]$. Let $P_1'(i, j)$ and $P_2'(i, j)$ be two mm-alternating paths containing the even type extensions Q_e and Q_f, respectively. Then $f \notin E(P_1'(i, j))$, otherwise by using Lemma 2.7, $V(P_1'(i, j)) \cap V(Q_f) = \{x_2, y_2\} \Rightarrow E(Q_e) \cap E(Q_f) = \phi$ which is not possible as $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$. Similarly $e \notin E(P_1'(i, j))$. Then by using Lemma 3.2, $P_1'(i, j) = P_2'(i, j) = P_3$. $\square$

REMARK 3.6. Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph. Let x and y be any two vertices in G. Then exactly one of the following holds.

- There are no mm-alternating x-y-paths in G.

- There is exactly one mm-alternating x-y-path in G.

- There are exactly two mm-alternating x-y-paths in G, say, P_1 and P_2. By using Lemma 3.2, one of these two paths does not contain even type edges and the other path contains exactly one even type edge. Without loss of generality we assume that P_1 contains exactly one even type edge, say, e. Since there are exactly two mm-alternating x-y-paths in G, the path P_2 must contain the even type extension at e.

The path P_1 contains an even (resp., odd) number of nonmatching edges if and only if P_2 contains an odd (resp., even) number of nonmatching edges. By using Lemma 2.14, $A_{i,j}^{-1} = 0$.

• There are exactly three mm-alternating x-y paths, say, P_1, P_2 and P_3. By using Lemmas 3.2 and 3.5 and Remark 3.1, one of these three paths contains both the even type extensions other two paths contain exactly one even type edge. Without loss of generality we assume that P_1 contains both the even type extensions. By using Lemma 3.3, the paths P_2 and P_3 do not contain the same even type edge. We assume that P_2 contains the even type edge e and P_3 contains the even type edge f.

An even type extension contains an even number of nonmatching edges; then P_1 contains an even (resp., odd) number of nonmatching edges if and only if P_2 and P_3 contain an odd (resp., even) number of nonmatching edges. By using Lemma 2.14, $A_{i,j}^{-1} = -1$ if P_1 contains an even number of nonmatching edges, otherwise $A_{i,j}^{-1} = 1$.

The following is our main result of this section. This result supplies a characterization of the bicyclic graphs in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$.

THEOREM 3.7. *Let $G \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ be a bicyclic graph. Then G^+ exists if and only if G has no mm-alternating path which contains both the even type extensions.*

Proof. First we assume that G^+ exists. We have to show that G has no mm-alternating path which contains both the even type extensions. Suppose that there is an mm-alternating path $P(u_1, u_m') = [u_1, u_1', \ldots, u_m, u_m']$ in G such that $Q_e = P(u_{k_1}, u_{k_2}')$ and $Q_f = P(u_{l_1}', u_{l_2}')$ are two even type extensions at $[u_{k_1}', u_{k_2}']$ and $[u_{l_1}', u_{l_2}']$, respectively, where $1 \leq k_1 < l_1 < k_2 < l_2 \leq m$. By using Lemma 2.14 and Remark 3.6, we see

i) $A(G)_{u_i, u_i'}^{-1} = 1$, for $i = 1, \ldots, m$,
ii) $A(G)_{u_i, u_{i+1}'}^{-1} = -1$ for $i = 1, \ldots, m-1$,
iii) $A(G)_{u_1, u_m'}^{-1} = -1$ if $P(u_1, u_m')$ contains an even number of nonmatching edges, otherwise $A(G)_{u_1, u_m'}^{-1} = 1$.

First assume that $P(u_1, u_m')$ contains an even number of nonmatching edges. The total number of non-matching edges on $P(u_1, u_m')$ is $\frac{\|P(u_1, u_m')\| - 1}{2} = \frac{2m - 1 - 1}{2} = m - 1$ which is even. Then we see that the cycle $[u_1, u_2', u_2, \ldots, u_m', u_1]$ is available in $\hat{G}$ (for $\hat{G}$, see Lemma 2.16). This cycle contains m number of nonmatching edges and each nonmatching edge has weight -1. Then the product of the edge weights on $[u_1, u_2', u_2, \ldots, u_m', u_1]$ is equal to -1, as m is odd. Using Lemma 2.16, G^+ cannot exist. Similar arguments work if $P(u_1, u_m')$ contains an odd number of nonmatching edges. Therefore, G has no mm-alternating path which contains both the even type extensions.

We now assume that G has no mm-alternating path which contains both the even type extensions. By using Lemma 3.5, there are at most two mm-alternating paths from one vertex to another vertex. Let e and f be the even type edges in G and Q_e and Q_f be the even type extensions at e and f, respectively. Since G is bicyclic, the graph $G - \{e, f\}$ is a tree.

As $G - \{e, f\}$ is a tree, take the vertex 1, define $s_1 = 1$. Now to define s_i, take the path from 1 to i in $G - \{e, f\}$. If it has odd many nonmatching edges define $s_i = -1$, otherwise define $s_i = 1$. The matrix S is well defined. Notice that $s_i s_j = 1$ if the path from i to j in $G - \{e, f\}$ contains an even number of nonmatching edges, otherwise $s_i s_j = -1$.

If there is no mm-alternating i-j-path in G, then $s_i A_{i,j}^{-1} s_j = 0$. So we assume that there are mm-alternating i-j-paths in G. By using Lemma 3.5, there are two mm-alternating i-j-paths in G. There are two cases.

Case I. There is exactly one mm-alternating i-j-path, say $P(i,j)$ in G. All the nonmatching edges are odd type, otherwise there is another mm-alternating i-j-path. The path $P(i,j)$ is also a path in $G - \{e, f\}$. Then

$$s_i A_{i,j}^{-1} s_j = s_i s_j A_{i,j}^{-1} = (-1)^{(\|P(i,j)\|-1)/2}(-1)^{(\|P(i,j)\|-1)/2} = 1 > 0.$$

Case II. There are exactly two mm-alternating i-j paths. By using Remark 3.6, $A_{i,j}^{-1} = 0$. Then $s_i A_{i,j}^{-1} s_j = 0$. Hence, G^+ exists. $\qquad\square$

REMARK 3.8. We now explain why the graph H shown in Figure 1 is not invertible. We noticed that $H \in \mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$ and the mm-alternating path $[1, 1', 2, 2', 3, 3', 4, 4']$ contains both the even type extensions $[1', 2, 2', 3]$ and $[2', 3, 3', 4]$. Hence, by Theorem 3.7, G is not invertible.

The following is our main result in this article. This result supplies a characterization of bicyclic graphs in $\mathcal{H}$ which posses inverses.

THEOREM 3.9. *Let $G \in \mathcal{H}$ be bicyclic. Then G^+ exists if and only if*

1. *there is no mm-alternating path in G which contains two even type extensions at two distinct even type edges such that both the extensions have an odd type edge in common,*
2. *the graph $(G - \mathcal{E})/\mathcal{M}$ is bipartite.*

Proof. First we assume that G^+ exists. There are two cases.

Case I. The graph G is in $\mathcal{H} \setminus \mathcal{H}_{nm}$. Then by Theorem 2.18, G has exactly one even type edge which must be present on the even type extension at $m = [u, v]$ (mixed type edge). Hence, there is no mm-alternating path in G which contains two even type extensions at two distinct even type edges such that both the extensions have an odd type edge in common.

Let $Q_1(u, v)$ be the even type extension at m which contains the even type edge. Let $[x, y]$ be the even type edge and $Q_2(x, y)$ be the even type extension at $[x, y]$. Then by using Remark 2.10, the path $[Q_1(u, x), Q_2(x, y), Q_1(y, v)]$ is the odd type extension at $m = [u, v]$. Then the cycle $[Q_1(u, x), Q_2(x, y), Q_1(y, v), [v, u]]$ is the only cycle in the graph $(G - \mathcal{E})$. This cycle contains an even number of matching edges. Hence, the graph $(G - \mathcal{E})/\mathcal{M}$ is bipartite.

Case II. The graph G is in $\mathcal{H}_{nm}$. Then by using Theorems 1.15 and 2.18, there is no mm-alternating path in G which contains two even type extensions at two distinct even type edges such that both the extensions have an odd type edge in common in G and the graph $(G - \mathcal{E})/\mathcal{M}$ is bipartite.

We now prove the converse. First we assume that G is in $\mathcal{H} \setminus \mathcal{H}_{nm}$. If G has no even type edges, then $(G - \mathcal{E})/\mathcal{M} = G/\mathcal{M}$ is not bipartite. Hence, G has an even type edge. By using Lemmas 2.5 and 2.6, G has exactly one even type edge say $e = [x, y]$ which must be present on some extension at $m = [u, v]$ (mixed type edge). Suppose that e is present on an odd type extension at $m = [u, v]$. Let $Q_1(u, v)$ be such an odd type extension and $Q_2(x, y)$ be the even type extension at $e = [x, y]$. Then we have a cycle $[Q_1(u, x), Q_2(x, y), Q_1(y, v), [v, u]]$ in G which contains an odd number of matching edges and which does not contain the edge e. Hence, $(G - e)/\mathcal{M}$ is not bipartite, a contradiction. Thus, e must be present on the

even type extension at m. By using Theorem 2.18, G^+ exists. If $G \in \mathcal{H}_{nm}$, then by using Theorems 1.15 and 3.7, G^+ exists. ☐

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