EVENTUAL CONE INVARIANCE[*]

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Abstract. Eventually nonnegative matrices are square matrices whose powers become and remain (entrywise) nonnegative. Using classical Perron-Frobenius theory for cone preserving maps, this notion is generalized to matrices whose powers eventually leave a proper cone $K \subset \mathbb{R}^n$ invariant, that is, $A^m K \subseteq K$ for all sufficiently large m. Also studied are the related notions of eventual cone invariance by the matrix exponential, as well as other generalizations of M-matrix and dynamical system notions.

Key words. Eventually nonnegative matrix, Exponentially nonnegative matrix, Perron-Frobenius, Proper cone.

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1. Introduction. Perron-Frobenius theory commonly refers to the study of square (entrywise) non-negative matrices A. The name reflects the basic properties of such matrices, which are the tenets of the celebrated Perron-Frobenius theorem; that is, in its simplest form, the spectral radius is necessarily an eigenvalue of A corresponding to a nonnegative eigenvector. More generally, however, Perron-Frobenius theory encompasses the study of matrices A that leave a proper cone $K \subset \mathbb{R}^n$ invariant, namely $AK \subseteq K$. This indeed generalizes nonnegative matrices, which leave the nonnegative orthant, $\mathbb{R}^n_+$, invariant. One can include in this area the study of functionals and operators in finite or infinite dimensions, in contexts where the main tenets of the Perron-Frobenius theory apply. One such instance is the study of eventually nonnegative matrices A, namely, matrices for which A^m is nonnegative for all sufficiently large m.

The first explicit interest in eventually nonnegative matrices appears in Friedland [6] in the context of the inverse eigenvalue problem for nonnegative matrices. The direct association of matrix eventual nonnegativity and positivity to Perron-Frobenius theory appears in Handelman [7]. Subsequently, this concept is further studied in [3], [4], [9], [12], [13], [16], [17], [18], [24], [25], [26].

The purpose of this paper is to pursue and record a comprehensive generalization of eventual nonnegativity from $K = \mathbb{R}^n_+$ to general proper cones $K \subset \mathbb{R}^n$. This generalization materializes herein in a manner that parallels existing theory and proof techniques for eventually nonnegative matrices found mainly in [16], [17], [19]. There are, however, some clear and some subtle differences, which are due to $\mathbb{R}^n_+$ being a self-dual, polyhedral cone, while these attributes are not necessarily ascribed to a general proper cone K. In addition, our efforts bring to light some interesting facts: Firstly, it is well-known that essential nonnegativity (i.e., $A + aI$ being nonnegative for some $a \geq 0$) and exponential nonnegativity (i.e., e^{tA} being nonnegative for all $t \geq 0$) are two equivalent notions, due to the fact that $\mathbb{R}^n_+$ is a polyhedral cone. This equivalence fails for non-polyhedral cones. We will see, however, that the equivalence of two related notions, namely, eventual positivity and eventual exponential positivity holds relative to any proper cone K, whether K is polyhedral or not; see Remark 9 and Example 10. The latter equivalence was indeed shown for $K = \mathbb{R}^n_+$ in [17], yet its general validity was previously masked by the polyhedrality of $\mathbb{R}^n_+$. Secondly, we discover that although eventual nonnegativity implies eventual exponential nonnegativity under an assumption on the index of the

eigenvalue 0, this implication does not generalize to non-polyhedral cones; see Example 14.

This paper is organized as follows. Section 2 contains all notation and the definitions of the concepts used. Section 3 has some basic results from the theory of invariant cones. Sections 4 and 5 contain the main results on eventually (exponentially) K-positive and K-nonnegative matrices, respectively. In Section 6, we study initial points $x_0 = x(0)$ giving rise to solutions of $\dfrac{dx(t)}{dt} = Ax(t)$ that reach and stay in a proper cone K for all but a finite amount of time. Finally, in Section 7, we consider a generalization of M-matrices by studying matrices of the form $A = sI - B$, where B is eventually K-nonnegative and s is greater than or equal to the spectral radius of B.

2. General notation, definitions and preliminaries. Given an $n \times n$ matrix A, the spectrum of A is denoted by $\sigma(A)$ and its spectral radius by $\rho(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\}$. An eigenvalue λ of A is said to be *dominant* if $|\lambda| = \rho(A)$. The *spectral abscissa* of A is defined and denoted by $\lambda(A) = \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$. By $\operatorname{index}_0(A)$ we denote the degree of 0 as a root of the minimal polynomial of A. Consequently, when we say $\operatorname{index}_0(A) \leq 1$, we mean that either A is invertible or that the size of the largest nilpotent Jordan block in the Jordan canonical form of A is 1×1.

The following geometric concepts will be used in the sequel.

The *dual* of a set $S \subseteq \mathbb{R}^n$ is $S^* = \{z \in \mathbb{R}^n : z^T y \geq 0 \text{ for all } y \in S\}$.

A nonempty convex set $K \subseteq \mathbb{R}^n$ is said to be a *cone* if $\alpha K \subseteq K$ for all $\alpha \geq 0$. A cone K is called *proper* if it is (i) *closed* (in the Euclidean space $\mathbb{R}^n$), (ii) *pointed* (i.e., $K \cap (-K) = \{0\}$), and (iii) *solid* (i.e., the topological interior of K, $\operatorname{int} K$, is nonempty).

The *nonnegative orthant*, that is the nonnegative vectors in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. A *polyhedral cone* $K \subseteq \mathbb{R}^n$ is a cone consisting of all nonnegative linear combinations of a finite set of vectors in $\mathbb{R}^n$, which are called the *generators* of K. Thus, K is polyhedral if and only if $K = X\mathbb{R}^m_+$ for some $n \times m$ matrix X; when $m = n$ and X is invertible, $K = X\mathbb{R}^n_+$ is called a *simplicial* cone in $\mathbb{R}^n_+$. Note that simplicial cones in $\mathbb{R}^n$ are proper cones.

For any set $S \subseteq \mathbb{R}^n$, S^* is a proper cone. A cone K is polyhedral if and only if K^* is polyhedral. If for a cone K we have $K = K^*$, we call K *self-dual*. The nonnegative orthant, $\mathbb{R}^n_+$, is a self-dual cone.

DEFINITION 1. Given a cone K in $\mathbb{R}^n$, a matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called:

- *K-nonnegative* (resp., *K-positive*) if $AK \subseteq K$ $(A(K \setminus \{0\}) \subseteq \operatorname{int} K)$;
- *K-primitive* if it is K-nonnegative and there exists a natural number m such that A^m is K-positive;
- *essentially K-nonnegative* (resp., *essentially K-positive*) if there exists an $\alpha \geq 0$ such that $A + \alpha I$ is K-nonnegative (K-positive);
- *exponentially K-nonnegative* (resp., *exponentially K-positive*) if for every $t \geq 0$, $e^{tA}K \subseteq K$ (resp., $e^{tA}(K \setminus \{0\}) \subseteq \operatorname{int} K$);
- *eventually K-nonnegative* (resp., *eventually K-positive*) if there exists a positive integer k_0 such that $A^k K \subseteq K$ (resp., $A^k K \setminus \{0\} \subseteq \operatorname{int} K$) for all $k \geq k_0$. We denote the smallest such positive integer by $k_0 = k_0(A)$ and refer to it as the *power index* of A;
- *eventually exponentially K-nonnegative* (resp., *eventually exponentially K-positive*) if there is a $t_0 \in [0, \infty)$ such that, for all $t \geq t_0$, $e^{tA}K \subseteq K$ (resp., $e^{tA}(K \setminus \{0\}) \subseteq \operatorname{int} K$). We denote the smallest such nonnegative number by $t_0 = t_0(A)$ and refer to it as the *exponential index* of A.

LEMMA 2. *Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:*

(i) *A is eventually exponentially K-nonnegative.*

(ii) *There exists an $a \in \mathbb{R}$ such that $A + aI$ is eventually exponentially K-nonnegative.*

(iii) *For all $a \in \mathbb{R}$, $A + aI$ is eventually exponentially K-nonnegative.*

Proof. The equivalences follow readily from the fact that

$$e^{t(A+aI)} \;=\; e^{atI}\, e^{tA} \;=\; e^{at} e^{tA}$$

since aI and A commute.   □

We conclude with some notions crucial to the analysis in Sections 4 and 5.

DEFINITION 3. Given a proper cone K in $\mathbb{R}^n$, we say that $A \in \mathbb{R}^{n \times n}$ has

- the *K-Perron-Frobenius property* if $\rho(A) > 0$, $\rho(A) \in \sigma(A)$ and there exists an eigenvector of A in K corresponding to $\rho(A)$;
- the *strong K-Perron-Frobenius property* if, in addition to having the K-Perron-Frobenius property, $\rho(A)$ is a simple eigenvalue such that

$$\rho(A) > |\lambda| \quad \text{for all } \lambda \in \sigma(A), \ \lambda \neq \rho(A),$$

as well as there is an eigenvector of A in int K corresponding to $\rho(A)$.

3. Results from the theory of invariant cones. Let K be a proper cone in $\mathbb{R}^n$. By the Perron-Frobenius Theorem, every non-nilpotent K-nonnegative matrix A has the K-Perron-Frobenius property, and every K-primitive matrix A has the strong K-Perron-Frobenius property; see [1].

Given a proper cone K in $\mathbb{R}^n$, the set of all matrices in $\mathbb{R}^{n \times n}$ that are K-nonnegative is itself a proper cone in $\mathbb{R}^{n \times n}$, denoted by $\pi(K)$. The relative interior of $\pi(K)$ in $\mathbb{R}^{n \times n}$, int $\pi(K)$ consists of all K-positive matrices in $\mathbb{R}^{n \times n}$.

In the following lemma, we review a well-known relation between the notions of exponential K-nonnegativity and essential K-nonnegativity; see [21], [22] and [1, Chapter 6, Theorem 3.12].

LEMMA 4. *Let K be a cone in $\mathbb{R}^n$. If A is essentially K-nonnegative, then $A \in \mathbb{R}^{n \times n}$ is exponentially K-nonnegative. The converse is true only when K is a polyhedral cone.*

Proof. The equivalence of essential and exponential K-nonnegativity when K is polyhedral is found in [21]. The fact that exponential K-nonnegativity does not imply essential K-nonnegativity when K is non-polyhedral is supported by the following counterexample.   □

EXAMPLE 5. Consider the non-polyhedral cone (see [2, p. 79])

$$K = \left\{ x = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 \leq x_3^2, \ x_2 \geq 0, \ x_3 \geq 0 \right\}$$

and the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

It can be verified that for all $t \geq 0$, $e^{tA} K \subseteq K$. Notice, however, that for $x = [-1 \ 0 \ 1]^T \in K$ and for all $a \geq 0$, $(A + aI)x \notin K$. That is, A is exponentially K-nonnegative but not essentially K-nonnegative.

LEMMA 6. *Let $A \in \mathbb{R}^{n \times n}$ so that $\lambda \in \mathbb{C}$ is a simple eigenvalue of A and $|\lambda| > |\mu|$ for all eigenvalues $\mu \neq \lambda$ of A. Let u, v be right and left eigenvectors of λ, respectively, such that $u^T v = 1$. Then*

$$\lim_{k \to \infty} \left(\frac{A}{\rho(A)} \right)^k = uv^T.$$

Proof. By assumption, u and v span the eigenspaces of A and A^T corresponding to $\lambda \neq 0$, respectively, and $\rho(A) = |\lambda| > |\mu|$ for every eigenvalue $\mu \neq \lambda$ of A. The result follows from [8, Lemma 8.2.7]. ☐

4. Eventually (exponentially) K-positive matrices. In this section, we characterize eventual (exponential) K-positivity.

THEOREM 7. *Let K be a proper cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. The following are equivalent:*

(i) *A has the strong K-Perron-Frobenius property and A^T has the strong K^*-Perron-Frobenius property.*

(ii) *A is eventually K-positive.*

(iii) *A^T is eventually K^*-positive.*

Proof. First, it is convenient to observe that (ii) and (iii) are equivalent.

(ii) $\iff$ (iii). Recall that a matrix is K-positive if and only if its transpose is K^*-positive [1, Chapter 2, (2.23)]. Thus, A^k is K-positive for all sufficiently large k if and only if $(A^T)^k$ is K^*-positive for all sufficiently large k.

(i) $\implies$ (ii). Suppose (i) holds and let $Au = \rho(A)u$, $v^T A = \rho(A)v^T$ with $u \in \mathrm{int}\, K$ and $v \in \mathrm{int}\, K^*$ such that $u^T v = 1$. By Lemma 6, we can set

$$B = \lim_{k \to \infty} \left(\frac{A}{\rho(A)} \right)^k = uv^T.$$

For every $w \in K \setminus \{0\}$, we have that $Bw = (v^T w) u \in \mathrm{int}\, K$ because $v^T w > 0$. Thus, B is K-positive, i.e., $B \in \mathrm{int}\, \pi(K)$. Therefore, on account of $\pi(K)$ being a proper cone in $\mathbb{R}^{n \times n}$ [21, Lemma 5] and the convergence of positively scaled powers of A to B, there exists a nonnegative integer k_0 such that A^k is K-positive for all $k \geq k_0$. That is, A is eventually K-positive.

(ii) $\implies$ (i) Assume there exists a nonnegative integer k_0 such that A^k is K-positive for all $k \geq k_0$. Thus, for all $k \geq k_0$, $\rho(A^k)$ is a simple positive eigenvalue of A^k greater than the magnitude of any other eigenvalue of A^k, having a corresponding eigenvector in $\mathrm{int}\, K$ [1, Chapter 2, Theorem 3.26]. It follows that A^k, and thus, A possess the strong K-Perron-Frobenius property. Since (ii) implies (iii), the above argument applied to A^T also shows that A^T has the strong K^*-Perron-Frobenius property. ☐

THEOREM 8. *Let K be a proper cone in $\mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{n \times n}$ the following properties are equivalent:*

(i) *There exists an $a \geq 0$ such that $A + aI$ has the K-strong Perron-Frobenius property and $A^T + aI$ have the K^*-strong Perron-Frobenius property.*

(ii) *$A + aI$ is eventually K-positive for some $a \geq 0$.*

(iii) *$A^T + aI$ is eventually K^*-positive for some $a \geq 0$.*

(iv) *A is eventually exponentially K-positive.*

(v) A^T *is eventually exponentially K^*-positive.*

Proof. The equivalences of (i)–(iii) are the content of Theorem 7 applied to $A + aI$. We will argue the equivalence of (ii) and (iv), with the equivalence of (iii) and (v) being analogous:

Let $a \geq 0$ such that $A + aI$ is eventually K-positive, and let k_0 be a positive integer such that $(A + aI)^k(K \setminus \{0\}) \subseteq \operatorname{int} K$ for all $k \geq k_0$. As K is solid, there exists large enough $t_0 > 0$ so that the first $k_0 - 1$ terms of the series

$$e^{t(A+aI)} = \sum_{m=0}^{\infty} \frac{t^m(A + aI)^m}{m!}$$

are dominated by the remaining terms, rendering $e^{t(A+aI)}$ K-positive for all $t \geq t_0$. It follows that $e^{tA} = e^{-ta}e^{t(A+aI)}$ is K-positive for all $t \geq t_0$. That is, A is eventually exponentially K-positive. Conversely, suppose A is eventually exponentially K-positive. As $(e^A)^k = e^{kA}$, it follows that e^A is eventually K-positive. Thus, by Theorem 7, e^A has the K-strong Perron-Frobenius property. Recall that $\sigma(e^A) = \{e^\lambda : \lambda \in \sigma(A)\}$ and so $\rho(e^A) = e^\lambda$ for some $\lambda \in \sigma(A)$. Then for each $\mu \in \sigma(A)$ with $\mu \neq \lambda$ we have

$$e^\lambda > |e^\mu| = |e^{\operatorname{Re}\mu + i\operatorname{Im}\mu}| = e^{\operatorname{Re}\mu}.$$

Hence, λ is the spectral abscissa of A, namely, $\lambda > \operatorname{Re}\mu$ for all $\mu \in \sigma(A)$ with $\mu \neq \lambda$. In turn, this means that there exists large enough $a > 0$ such that

$$\lambda + a > |\mu + a| \quad \text{for all } \mu \in \sigma(A), \ \mu \neq \lambda.$$

As $A + aI$ shares eigenspaces with e^A, it follows that $A + aI$ has the strong K-Perron-Frobenius property. By Theorem 7, $A + aI$ is eventually K-positive. $\qquad\Box$

REMARK 9. It is worth emphasizing that the role of polyhedrality differs in the context of eventual (exponential) nonnegativity from the role presented in Lemma 4. In Example 5, we presented a non-polyhedral cone K and a matrix A such that A is exponentially K-nonnegative but not essentially K-nonnegative. Nevertheless, perhaps contrary to one's intuition, Theorem 8 shows that for every proper cone (polyhedral or not), eventual exponential K-positivity implies (in fact, is equivalent) to eventual K-positivity of $A + \alpha I$ for some $\alpha \geq 0$. This finding is illustrated by examining below the matrix and the cone of Example 5.

EXAMPLE 10. Recall from Example 5 the non-polyhedral cone

$$K = \left\{x = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 \leq x_3^2, \ x_2 \geq 0, \ x_3 \geq 0\right\}$$

and the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

We find that $\rho(A + I) = 4.1304$ with corresponding right and left eigenvectors given, respectively, by

$$x = \begin{bmatrix} -0.17494 \\ 0.48446 \\ 0.85715 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} -0.27516 \\ 0.58620 \\ 0.76200 \end{bmatrix}.$$

The dual of K can be computed to be

$$K^* = K \cup \left\{x = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3 \ : \ x_3 \geq |x_1|, \ x_2 \geq 0\right\}.$$

The interiors of K and K^* are the corresponding subsets for which the defining inequalities are strict. Notice then that $x \in \operatorname{int} K$ and $y \in \operatorname{int} K^*$. Thus, $A + I$ and its transpose satisfy the strong K-Perron-Frobenius and the strong K^*-Perron-Frobenius property, respectively. It follows by Theorem 7 that $A + I$ is eventually K-positive, and thus, by Theorem 8, it is eventually exponentially K-positive. Recall that in Example 5, it was shown that A is not essentially K-positive.

5. Eventually (exponentially) K-nonnegative matrices. We now turn our attention to necessary conditions for eventual (exponential) cone nonnegativity.

THEOREM 11. *Let K be a proper cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ be an eventually K-nonnegative matrix which is not nilpotent. Then A has the K-Perron-Frobenius property and A^T has the K^*-Perron-Frobenius property.*

Proof. Since A^k is K-nonnegative for some sufficiently large k, $\rho(A^k)$ is an eigenvalue of A^k and has a corresponding eigenvector in K [1, Chapter 2, Theorem 3.2]. It follows that A possesses the Perron-Frobenius property. Recall now that A is K-positive if and only if A^T is K^*-positive from which the second part of the conclusion follows. $\square$

THEOREM 12. *Let K be a proper cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ be an eventually exponentially K-nonnegative matrix. Then the following hold:*

(i) e^A *has the K-Perron-Frobenius property and e^{A^T} has the K^*-Perron-Frobenius property.*

(ii) *If $\rho(e^A)$ is a simple eigenvalue of e^A and $\rho(e^A) = e^{\rho(A)}$, then there exists an $a_0 \geq 0$ such that $\lim_{k \to \infty} ((A + aI)/(\rho(A + aI)))^k = xy^T$ for all $a > a_0$, where $x \in K$ and $y \in K^*$ are, respectively, right and left eigenvectors of A corresponding to $\rho(A)$, satisfying $x^T y = 1$. The limit matrix xy^T belongs to $\pi(K)$.*

Proof. (i) Let A be eventually exponentially K-nonnegative. As $(e^A)^k = e^{kA}$, it follows that e^A is eventually K-nonnegative. Thus, by Theorem 11 and since e^A and e^{A^T} are not nilpotent, e^A has the K-Perron-Frobenius property and e^{A^T} has the K^*-Perron-Frobenius property.

(ii) From (i) we specifically have that $\rho(e^A) \in \sigma(e^A)$. Let $x \in K$, $y \in K^*$ be right and left eigenvectors of e^A, respectively, corresponding to $\rho(e^A)$ and normalized so that $x^T y = 1$. As in the proof of Theorem 8, $\rho(e^A) = e^\lambda$ for some $\lambda \in \sigma(A)$ with $\lambda > \operatorname{Re} \mu$, $\forall \mu \in \sigma(A) \setminus \{\lambda\}$. This means that there exists an $a_0 \geq 0$, such that for all $a > a_0$,

$$\rho(A + aI) = \lambda + a > |\mu + a|, \quad \text{for all } \mu \in \sigma(A), \ \mu \neq \lambda.$$

As $A + aI$ and e^A share eigenvectors, we obtain that for all $a > a_0$, $A + aI$ and $A^T + aI$ both have the Perron-Frobenius property relative to cones K and K^*, respectively, with $\lambda + a$ being simple and their only dominant eigenvalue. Applying Lemma 6 to $A + aI$, we thus obtain

$$\lim_{k \to \infty} \frac{(A + aI)^k}{(\rho(A + aI))^k} = xy^T. \tag{5.1}$$

Note that for every $w \in K$, as $y \in K^*$ we have $y^T w \geq 0$; that is $xy^T w \in K$. $\square$

REMARK 13. When $K = \mathbb{R}^n_+$, it was shown in [17, Theorem 3.7]) that if A is eventually nonnegative with $\operatorname{index}_0(A) \leq 1$, then A is eventually exponentially nonnegative. This result generalizes to simplicial cones $K = X\mathbb{R}^n_+$, where X invertible, simply by working with XAX^{-1}. However, it is not in general true that eventual K-nonnegativity and an index assumption imply eventual exponential K-nonnegativity. This is shown by the following counterexample.

EXAMPLE 14. Consider the non-polyhedral, proper (ice-cream) cone (see [10])

$$K = \left\{ s\,[x_1\ x_2\ 1]^T \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1,\ s \geq 0 \right\},$$

as well as the matrix A and its powers given by

$$A = \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^m = \begin{bmatrix} a^m & ma^{m-1}b & 0 \\ 0 & a^m & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a = \frac{1}{2}$, $b = -5$. For any nonzero $x = s[x_1\ x_2\ 1]^T \in K$, consider $\frac{1}{s} A^m x = [y_1\ y_2\ 1]^T$. We have that

$$y_1^2 + y_2^2 = a^{2m}x_1^2 + (a^{2m} + m^2 a^{2m-2}b^2)x_2^2 + 2ma^{2m-1}bx_1 x_2. \tag{5.2}$$

We look at two cases: Firstly, if $x_1 x_2 > 0$ and since $b = -5 < 0$, then

$$a^{2m}x_1^2 + (a^{2m} + m^2 a^{2m-2}b^2)x_2^2 + 2ma^{2m-1}bx_1 x_2 \leq a^{2m}x_1^2 + (a^{2m} + m^2 a^{2m-2}b^2)x_2^2. \tag{5.3}$$

Observe that $\lim_{m\to\infty} a^{2m} = 0$ and $\lim_{m\to\infty} m^2 a^{2m-2} = 0$ since $a = \frac{1}{2}$. Therefore, from (5.2) and (5.3), we get for sufficiently large m that

$$y_1^2 + y_2^2 \leq a^{2m}x_1^2 + (a^{2m} + m^2 a^{2m-2}b^2)x_2^2 \leq x_1^2 + x_2^2 \leq 1.$$

Hence, $\frac{1}{s} A^m x \in K$, and thus, $A^m x \in K$ for all sufficiently large m.

Secondly, if $x_1 x_2 < 0$, let us without loss of generality assume $|x_1| \geq |x_2|$. Then, for all sufficiently large m,

$$\begin{aligned} y_1^2 + y_2^2 &= a^{2m}x_1^2 + (a^{2m} + m^2 a^{2m-2}b^2)x_2^2 + 2ma^{2m-1}bx_1 x_2 \\ &\leq (a^{2m} - 2ma^{2m-1}b)x_1^2 + (a^{2m} + m^2 a^{2m-2}b^2)x_2^2 \\ &\leq x_1^2 + x_2^2 \leq 1. \end{aligned}$$

Thus, $A^m x \in K$ for all sufficiently large m. That is, A is eventually K-nonnegative.

Next we will argue that A is not eventually exponentially K-nonnegative by considering

$$e^{tA} = \begin{bmatrix} e^{at} & te^{a(t-1)}c & 0 \\ 0 & e^{ta} & 0 \\ 0 & 0 & e^t \end{bmatrix},$$

where $a = \frac{1}{2}$ and where $c = b\sum_{m=1}^{\infty} \frac{m}{m!2^{m-1}}$. Thus,

$$0 < |c| = |b| \sum_{m=1}^{\infty} \frac{m}{m!2^{m-1}} \leq |b| \sum_{m=1}^{\infty} \frac{1}{2^{m-1}} = 2|b|.$$

Let now $x = [0\ 1\ 1]^T \in K$ so that $e^{tA}x = \begin{bmatrix} cte^{a(t-1)} \\ e^{at} \\ e^t \end{bmatrix}$. It follows that $(cte^{a(t-1)})^2 + (e^{at})^2 > 1$, since

$c^2 > 0$ and $a = \frac{1}{2}$. That is, $e^t\left(c^2 t^2 e^{-1} + 1\right) > 1$. We can thus conclude that $e^{tA}x \notin K$ for any $t > 0$, i.e., A is not eventually exponentially K-nonnegative.

6. Points of K-potential. In this section, $K \subset \mathbb{R}^n$ is a proper cone and $A \in \mathbb{R}^{n \times n}$ denotes an eventually exponentially K-nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$. We will study points of K-potential, that is, the set

$$X_A(K) = \left\{ x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\,(\forall t \geq \hat{t})\,[e^{tA}x_0 \in K] \right\}. \tag{6.1}$$

That is, $X_A(K)$ comprises all initial points giving rise to solutions (trajectories) of $\dfrac{dx}{dt} = Ax$ that reach K at some finite time and stay in K for all finite time thereafter.

Given the eventually exponentially K-nonnegative matrix $A \in \mathbb{R}^{n \times n}$ with exponential index $t_0 = t_0(A) \geq 0$, we define the cone

$$K_0 = e^{t_0 A}\,K = \{x_0 \in \mathbb{R}^n \mid (\exists\,y \in K)\,[x_0 = e^{t_0 A}y]\}$$

and consider the sets

$$Y_A(K_0) = \{x_0 \in \mathbb{R}^n \mid (\exists\,\hat{t} = \hat{t}(x_0) \geq 0)\,[e^{\hat{t}A}x_0 \in K_0]\} \tag{6.2}$$

and

$$X_A(K_0) = \left\{ x_0 \in \mathbb{R}^n \mid (\exists\,\hat{t} = \hat{t}(x_0) \geq 0)\,(\forall t \geq \hat{t})\,[e^{tA}x_0 \in K_0] \right\}. \tag{6.3}$$

LEMMA 15. *Let K_0, $Y_A(K_0)$ as defined above. Then $K_0 \subseteq K \subseteq Y_A(K_0)$.*

Proof. We have that $K_0 \subseteq K$ since $e^{t_0 A}K \subseteq K$. If $x_0 \in K$, then for $\hat{t} = 2t_0$, $e^{\hat{t}A}x_0 = e^{t_0 A}\,(e^{t_0 A}x_0) \in K_0$. Hence, $K \subseteq Y_A(K_0)$. $\quad\square$

Note that the sets $Y_A(K_0)$, $X_A(K_0)$ and $X_A(K)$ are convex cones. They are not necessarily closed sets, however. For example, when $K = \mathbb{R}^2_+$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

it can be shown that $X_A(\mathbb{R}^2_+)$ consists of the whole upper plane excluding the negative x-axis.

The set $Y_A(K_0)$ comprises initial points for which the trajectories enter K_0 at some time. The set $X_A(K_0)$ comprises initial points for which the trajectories enter K at some time and remain in K_0 for all time thereafter. The set of points of K-potential, $X_A(K)$ comprises initial points for which the trajectories at some time reach K and remain in K for all time thereafter. Next we shall argue that $Y_A(K_0)$, $X_A(K_0)$ and $X_A(K)$ coincide and interpret this result subsequently.

PROPOSITION 16. *Let $K \subset \mathbb{R}^n$ be a proper cone and let $A \in \mathbb{R}^{n \times n}$ be an eventually exponentially K-nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$. Let $K_0 = e^{t_0 A}\,K$. Then*

$$Y_A(K_0) = X_A(K) = X_A(K_0).$$

Proof. We begin by proving the first equality. If $x_0 \in Y_A(K_0)$, then there exists a $\hat{t} \geq 0$ and $y \in K$ such that $e^{\hat{t}A}x_0 = e^{t_0 A}y$. Thus, $x_0 = e^{(t_0 - \hat{t})A}y$ and so $e^{tA}x_0 = e^{(t + t_0 - \hat{t})A}y \in K$ if $t + t_0 - \hat{t} \geq t_0$, i.e., for all $t \geq \hat{t}$. It follows that $x_0 \in X_A(K)$, i.e., $Y_A(K) \subseteq X_A(K)$. For the opposite containment, let $x_0 \in X_A(K)$; that is, there exists a $\hat{t} \geq 0$ such that $e^{tA}x_0 \in K$ for all $t \geq \hat{t}$. Let $\tilde{t} = \hat{t} + t_0$. Then $e^{\tilde{t}A}x_0 = e^{t_0 A}(e^{\hat{t}A}x_0) \in K_0$, proving that $X_A(K) \subseteq Y_A(K_0)$ and thus equality holds.

For the second equality, we clearly have $X_A(K_0) \subseteq X_A(K)$ since $K_0 \subseteq K$. To show the opposite containment, let $x_0 \in X_A(K)$. Then there exists a $\hat{t} \geq 0$ such that $e^{t_0 A}e^{sA}x_0 \in K_0$ for all $s \geq \hat{t}$. That is, $e^{tA}x_0 \in K_0$ for all $t \geq t_0 + \hat{t}$, and thus, $x_0 \in X_A(K_0)$. $\quad\square$

REMARK 17. Referring to Proposition 16, we make the following observations:

(i) If $t_0 = 0$ (i.e., if $e^{tA}K \subseteq K$ for all $t \geq 0$), then $K_0 = K$. In this case, $X_A(K)$ coincides with the *reachability cone* of K for the essentially K-nonnegative matrix; this cone is studied in detail in [14] and [15], especially when $K = \mathbb{R}^n_+$.

(ii) The equality $X_A(K) = X_A(K_0)$, in conjunction with Lemma 15, can be interpreted as saying that the cone $K_0 = e^{t_0 A}K$ serves as an "attractor" set for trajectories emanating at points of K-potential; in other words, trajectories emanating in $X_A(K)$ always reach and remain in $K_0 \subseteq K$ after a finite time.

(iii) Our observations so far imply that the trajectory emanating from a point of K-potential will indeed enter cone K; however, it may subsequently exit K_0 while it remains in K, and it will eventually re-enter K_0 and remain in K_0 for all finite time thereafter. This situation is illustrated by [17, Example 4.4] for $K = \mathbb{R}^n_+$. As a consequence, a natural question arises: When is it possible that all trajectories emanating in $X_A(K)$ reach and never exit K_0? This is equivalent to asking whether or not $e^{tA}K_0 \subseteq K_0$ for all $t \geq 0$, which is resolved in the next corollary.

COROLLARY 18. *Let $K \subseteq \mathbb{R}^n$ be a proper cone and let $A \in \mathbb{R}^{n \times n}$ be an eventually exponentially K-nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$. Let $K_0 = e^{t_0 A}K$. Then $e^{tA}K_0 \subseteq K_0$ for all $t \geq 0$ if and only if $t_0 = 0$.*

Proof. If $t_0 = 0$, then $K_0 = K$ and $e^{tA}K \subseteq K$ for all $t \geq 0$. For the converse, suppose $e^{tA}K_0 \subseteq K_0$ for all $t \geq 0$. We must show that $t_0 = 0$. Let $y \in K$ and consider $x_0 = e^{t_0 A}y \in K_0$. As $e^{tA}x_0 \in K_0$ for all $t \geq 0$, there must exist $z \in K$ such that

$$e^{(t+t_0)A}y = e^{t_0 A}z \quad \text{for all } t \geq 0.$$

But this means $e^{tA}y = z \in K$ for all $t \geq 0$. Since y was taken arbitrary in K, we have $e^{tA}K \subseteq K$ for all $t \geq 0$; that is, $t_0 = 0$. □

REMARK 19. Referring to the assumptions and notation of Corollary 18, when K is a *polyhedral* proper cone, we can invoke Lemma 4 and Corollary 18 to assert that $e^{tA}K_0 \subseteq K_0$ for all $t \geq 0$ if and only if A is essentially K-nonnegative.

7. Generalizing M-matrices based on eventual K-nonnegativity. This section generalizes some of the results in [19] and concerns $M_{\vee,K}$ - matrices, namely, matrices of the form $A = sI - B \in \mathbb{R}^{n \times n}$, where K is a proper cone in $\mathbb{R}^n$, B is an eventually K-nonnegative matrix and $s \geq \rho(B) \geq 0$. In the remainder, every $M_{\vee,K}$ - matrix A is assumed to be in the above form for some proper cone K.

We begin with some basic properties of $M_{\vee,K}$ - matrices, which are immediate consequences of the fact that the eventually nonnegative (resp. positive) matrix B satisfies Theorem 11 (resp. Theorem 7).

THEOREM 20. *Let $A = sI - B \in \mathbb{R}^{n \times n}$ be an $M_{\vee,K}$ - matrix. Then*

(i) *$s - \rho(B) \in \sigma(A)$;*

(ii) *$\operatorname{Re}\lambda \geq 0$ for all $\lambda \in \sigma(A)$;*

(iii) *$\det A \geq 0$ and $\det A = 0$ if and only if $s = \rho(B)$;*

(iv) *if, in particular, $\rho(B) > 0$, then there exists an eigenvector $x \in K$ of A and an eigenvector $y \in K^*$ of A^T corresponding to $\lambda(A) = s - \rho(B)$;*

(v) *if, in particular, B is eventually K-positive and $s > \rho(B)$, then in* (iv),
$x \in \operatorname{int} K$, $y \in \operatorname{int} K$ *and in* (ii) $\operatorname{Re} \lambda > 0$ *for all* $\lambda \in \sigma(A)$.

In the following result, different representations of an $M_{\vee,K}$ - matrix are considered (analogous to different representations of an M-matrix).

THEOREM 21. *Let $A \in \mathbb{R}^{n \times n}$ be an $M_{\vee,K}$ - matrix. Then in any representation $A = tI - \hat{B}$ with $\hat{B}$ being eventually K-nonnegative, it follows that $t \geq \rho(\hat{B})$. If, in addition, A is nonsingular, then $t > \rho(\hat{B})$.*

Proof. Since A is an $M_{\vee,K}$ - matrix, $A = sI - B \in \mathbb{R}^{n \times n}$ for some eventually K-nonnegative B and some $s \geq \rho(B) \geq 0$. Let $A = tI - \hat{B}$, where $\hat{B}$ is eventually K-nonnegative. If B is nilpotent, then $0 = \rho(B) \in \sigma(B)$, and by Theorem 11, this containment also holds if B is not nilpotent. A similar containment holds for $\hat{B}$. If $t \geq s$, then $\rho(\hat{B}) = \rho((t-s)I + B) = \rho(B) + t - s$. Hence, $t = s - \rho(B) + \rho(\hat{B}) \geq \rho(\hat{B})$. Similarly, if $t \leq s$, then $\rho(B) = \rho(\hat{B}) + s - t$. Hence, $t = s - \rho(B) + \rho(\hat{B}) \geq \rho(\hat{B})$. If A is nonsingular, it follows by Theorem 20 (iii) that $s > \rho(B)$ and so $t > \rho(\hat{B})$. □

As with M-matrices (see [1, Chapter 6, Lemma 4.1]), we can now show that the class of $M_{\vee,K}$ - matrices is the closure of the class of nonsingular $M_{\vee,K}$ - matrices.

PROPOSITION 22. *Let $A = sI - B \in \mathbb{R}^{n \times n}$, where B is eventually K-nonnegative. Then A is an $M_{\vee,K}$ - matrix if and only if $A + \epsilon I$ is a nonsingular $M_{\vee,K}$ - matrix for each $\epsilon > 0$.*

Proof. If $A + \epsilon I = (s + \epsilon)I - B$ is a nonsingular $M_{\vee,K}$ - matrix for each $\epsilon > 0$, then by Theorem 21, $s + \epsilon > \rho(B)$ for each $\epsilon > 0$. Letting $\epsilon \to 0+$ gives $s \geq \rho(B)$, i.e., A is an $M_{\vee,K}$ - matrix. Conversely, let $A = sI - B$ be an $M_{\vee,K}$ - matrix, where B is eventually K-nonnegative and $s \geq \rho(B) \geq 0$. Thus, for every $\epsilon > 0$, $A + \epsilon I = (s + \epsilon)I - B$ with B eventually K-nonnegative and $s + \epsilon > s \geq \rho(B)$. That is, $A + \epsilon I$ is a nonsingular $M_{\vee,K}$ - matrix. □

Given an M-matrix A, clearly $-A$ is essentially nonnegative, i.e., $-A + \alpha I \geq 0$ for all sufficiently large $\alpha \geq 0$. Thus, $e^{-tA} = e^{-t\alpha} e^{-t(A - \alpha I)} \geq 0$ for all $t \geq 0$; that is $-A$ is exponentially nonnegative. In the following theorem, this property is extended to a special subclass of $M_{\vee,K}$ - matrices with exponential nonnegativity replaced by eventual exponential positivity.

THEOREM 23. *Let $A = sI - B \in \mathbb{R}^{n \times n}$ be an $M_{\vee,K}$ - matrix with B being eventually K-nonnegative (and thus, $s \geq \rho(B) > 0$). Then $-A$ is eventually exponentially K-positive.*

Proof. Let $A = sI - B$, where $B = sI - A$ is eventually K-nonnegative with power index k_0. As B^m is K-nonnegative for all $m \geq k_0$, there exists sufficiently large $t_0 > 0$ so that for all $t \geq t_0$, the sum of the first $k_0 - 1$ terms of the series $e^{tB} = \sum_{m=0}^{\infty} \dfrac{t^m B^m}{m!}$ is dominated by the term $\dfrac{t^{k_0} B^{k_0}}{k_0!}$, and thus, since $\pi(K)$ is also a proper cone, e^{tB} is K-positive for all $t \geq t_0$. It follows that $e^{-tA} = e^{-ts} e^{tB}$ is positive for all $t \geq t_0$. That is, $-A$ is eventually exponentially K-positive as claimed. □

There are several properties of a Z-matrix A (namely, a matrix with non-positive off-diagonal entries) that are equivalent to A being an M-matrix. These properties are documented in the often cited Theorems 2.3 and 4.6 in [1]: positive stability, semipositivity, inverse nonnegativity and monotonicity among others. In the cone-theoretic generalizations of M-matrices (see [22]), these properties are generalized and shown to play an analogous characterizing role. In the following theorems we examine the form and role these properties take in the context of $M_{\vee,K}$ - matrices.

THEOREM 24. *Let $A = sI - B \in \mathbb{R}^{n \times n}$, where B is eventually K-nonnegative and has power index $k_0 \geq 0$. Let $\hat{K}$ be the cone defined as $\hat{K} = B^{k_0} K$. Consider the following conditions:*

(i) *A is an invertible $M_{\vee, \hat{K}}$ - matrix.*

(ii) *$s > \rho(B)$ (positive stability of A).*

(iii) *A^{-1} exists and $A^{-1}\hat{K} \subseteq K$ (inverse K-nonnegativity).*

(iv) *$Ax \in \hat{K} \implies x \in K$ (K-monotonicity).*

Then (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv). *If, in addition, B is not nilpotent, then all conditions* (i)-(iv) *are equivalent.*

Proof. (i)$\Longrightarrow$(ii). This implication follows by Theorem 21 and invertibility of A.

(ii)$\Longrightarrow$(i). It follows from definition of an $M_{\vee,K}$ - matrix and Theorem 20 (i).

(iii)$\Longrightarrow$(iv). Assume (iii) holds and consider $y = Ax \in \hat{K}$. As A^{-1} exists, $x = A^{-1}y \in K$.

(iv)$\Longrightarrow$(iii). Assume (iv) holds. First notice that A must be invertible because if $Au = 0 \in \hat{K}$, then $u \in K$; also $A(-u) = 0 \in \hat{K}$ and so $u \in -K$, that is, as K is pointed, $u = 0$. Consider now $y = A^{-1}B^{k_0}x$, where $x \in K$. Then $Ay = B^{k_0}x \in \hat{K}$ and so $y \in K$.

(ii)$\Longrightarrow$(iii). If (ii) holds, then $\rho(B/s) < 1$ and so

$$A^{-1} = \frac{1}{s}(I - B/s)^{-1} = \frac{1}{s}\sum_{q=0}^{\infty}\frac{B^q}{s^q}.$$

Consequently, for all $x \in K$,

$$A^{-1}B^{k_0}x = \frac{1}{s}\sum_{q=0}^{\infty}\frac{B^{q+k_0}}{s^q}x \in K.$$

Now suppose that B is not nilpotent, that is, $\rho(B) > 0$. To prove that (i)-(iv) are equivalent it is sufficient to show that (iii)$\Longrightarrow$(ii).

(iii)$\Longrightarrow$(ii). By Theorem 11, B has the K-Perron-Frobenius property, i.e., there exists a nonzero $x \in K$ so that $Bx = \rho(B)x$. Assume (iii) holds and consider $\mu = s - \rho(B) \in \sigma(A) \cap \mathbb{R}$. As $B^{k_0}x = \rho(B)^{k_0}x$ and since $\rho(B) > 0$, it follows that $x \in \hat{K}$, and thus, $A^{-1}x \in K$. But $Ax = \mu x$ and so $x = \mu A^{-1}x$. It follows that $\mu > 0$. □

REMARK 25. **(a)** The implication (iii)$\Longrightarrow$(ii) or (i) in Theorem 24 is not in general true if B is nilpotent. For example, consider $K = \mathbb{R}_+^2$ and the eventually K-nonnegative matrix $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \overset{\mathrm{v}}{\geq} 0$, which has power index $k_0 = 2$. Thus, $\hat{K} = B^2\mathbb{R}_+^2 = \{0\}$. For any $s < 0$, $A = sI - B$ is invertible and $A^{-1}\hat{K} = \{0\} \subset \mathbb{R}_+^2$; however, A is not an $M_{\vee,\hat{K}}$ - matrix because its eigenvalues are negative.

(b) It is well known that when an M-matrix is invertible, its inverse is nonnegative. As mentioned earlier, in [9, Theorem 8] it is shown that the inverse of a pseudo M-matrix is eventually positive. In [11, Theorem 4.2] it is shown that if B is an irreducible eventually nonnegative matrix with $index_0(B) \leq 1$, then there exists a $t > \rho(B)$ such that for all $s \in (\rho(B), t)$, $(sI - B)^{-1} > 0$. The situation with the inverse of an $M_{\vee,K}$ - matrix A is different. Notice that condition (iii) of Theorem 24 is equivalent to $A^{-1}B^{k_0} \in \pi(K)$. In

general, if A is an invertible $M_{\vee,K}$ - matrix, A^{-1} is neither K-nonnegative nor eventually K-nonnegative; for example, if $A = I - B$ with B as in Remark 25 (a), then the $(1,2)$ entry of $(A^{-1})^k$ is negative for all $k \geq 1$.

In the next theorem we present some properties of singular $M_{\vee,K}$ - matrices analogous to the properties of singular, irreducible M-matrices found in [1, Chapter 6, Theorem 4.16].

THEOREM 26. *Let $A = sI - B \in \mathbb{R}^{n \times n}$ be a singular $M_{\vee,K}$ - matrix, where B is eventually K-positive. Then the following hold.*

(i) *A has rank $n - 1$.*

(ii) *There exists a vector $x \in \operatorname{int} K$ such that $Ax = 0$.*

(iii) *If for some vector u, $Au \in K$, then $Au = 0$* (almost monotonicity).

Proof. As A is singular, by Theorem 20 (iii) it follows that $s = \rho(B)$.

(i) By Theorem 7, B has the strong K-Perron-Frobenius property and so $\rho(B)$ is a simple eigenvalue of B. Thus, $0 = s - \rho(B)$ is a simple eigenvalue of A.

(ii) As B has the strong K-Perron-Frobenius property, there exists an $x \in \operatorname{int} K$ such that $Bx = \rho(B)x$, i.e., $Ax = \rho(B)x - Bx = 0$.

(iii) By Theorem 7, B^T also has the strong K^*-Perron-Frobenius property and so there exists a $z \in \operatorname{int} K^*$ such that $z^T B = \rho(B)z^T$. Let u be such that $Au \in K$. If $Au \neq 0$, then $z^T Au > 0$. However,

$$z^T Au = \rho(B)z^T u - z^T Bu = \rho(B)z^T u - \rho(B)z^T u = 0,$$

a contradiction, showing that $Au = 0$.   □

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