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SOLUTIONS OF THE SYSTEM OF OPERATOR EQUATIONS $BXA = B = AXB$ VIA $*$-ORDER*

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Abstract. In this paper, some necessary and sufficient conditions are established for the existence of solutions to the system of operator equations $BXA = B = AXB$ in the setting of bounded linear operators on a Hilbert space, where the unknown operator X is called the inverse of A along B. After that, under some mild conditions, it is proved that an operator X is a solution of $BXA = B = AXB$ if and only if $B \overset{*}{\leq} AXA$, where the $*$-order $C \overset{*}{\leq} D$ means $CC^* = DC^*, C^*C = C^*D$. Moreover, the general solution of the equation above is obtained. Finally, some characterizations of $C \overset{*}{\leq} D$ via other operator equations, are presented.

Key words. $*$-Order, Moore–Penrose inverse, Matrix equation, Operator equation.

AMS subject classifications. 15A24, 15B48, 47A62, 46L05.

1. Introduction and preliminaries. Throughout the paper, $\mathscr{H}$ and $\mathscr{K}$ are complex Hilbert spaces. We denote the space of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ by $\mathbb{B}(\mathscr{H}, \mathscr{K})$, and write $\mathbb{B}(\mathscr{H})$ when $\mathscr{H} = \mathscr{K}$. Recall that an operator $A \in \mathbb{B}(\mathscr{H})$ is positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and then we write $A \geq 0$. We shall write $A > 0$ if A is positive and invertible. An operator $A \in \mathbb{B}(\mathscr{H})$ is a generalized projection if $A^2 = A^*$. Let $\mathscr{S}(\mathscr{H}), \mathscr{Q}(\mathscr{H}), \mathscr{OP}(\mathscr{H}), \mathscr{GP}(\mathscr{H})$ be the set of all self-adjoint operators on $\mathscr{H}$, the set of all idempotents, the set of orthogonal projections and the set of all generalized projections on $\mathscr{H}$, respectively.

For $A \in \mathbb{B}(\mathscr{H}, \mathscr{K})$, let $\mathscr{R}(A)$ and $\mathscr{N}(A)$ be the range and the null space of A, respectively. The projection corresponding to a closed subspace $\mathscr{M}$ of $\mathscr{H}$ is denoted by $P_{\mathscr{M}}$. The symbol A^- stands for an arbitrary generalized inner inverse of A, that is, an operator A^- satisfying $AA^-A = A$. The Moore–Penrose inverse of a closed range operator A is the unique operator $A^\dagger \in \mathbb{B}(\mathscr{H})$ satisfying the following equations:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

Then, $A^*AA^\dagger = A^* = A^\dagger AA^*$, and we have the following properties:

$$\begin{aligned} \mathscr{R}(A^\dagger) = \mathscr{R}(A^*) = \mathscr{R}(A^\dagger A) = \mathscr{R}(A^*A), \quad \mathscr{N}(A^\dagger) = \mathscr{N}(A^*) = \mathscr{N}(AA^\dagger), \\ \mathscr{R}(A) = \mathscr{R}(AA^\dagger) = \mathscr{R}(AA^*), \quad P_{\mathscr{R}(A)} = AA^\dagger \quad \text{and} \quad P_{\mathscr{R}(A^*)} = A^\dagger A. \end{aligned} \tag{1.1}$$

For $A, B \in \mathscr{S}(\mathscr{H})$, $A \leq B$ means $B - A \geq 0$. The order $\leq$ is said to be the Löwner order on $\mathscr{S}(\mathscr{H})$. If there exists $C \in \mathscr{S}(\mathscr{H})$ such that $AC = 0$ and $A + C = B$, then we write $A \preceq B$. The order $\preceq$ is said to be the logic order on $\mathscr{S}(\mathscr{H})$.

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For $A, B \in \mathbb{B}(\mathscr{H})$, let $A \overset{*}{\leq} B$ mean

$$AA^* = BA^*, \quad A^*A = A^*B. \tag{1.2}$$

It is known that, for $A, B \in \mathscr{S}(\mathscr{H})$, $A \preceq B$ if and only if $A \overset{*}{\leq} B$; see [6]. We denote by $A \overset{*}{\wedge} B$ the infimum (or the greatest lower bound) of A and B over the $*$-order and $A \overset{*}{\vee} B$ the supremum (or the least upper bound) of A and B over the $*$-order, if they exist; cf. [12].

It is known that if $A \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ has closed range, then by considering

$$\mathscr{H} = \mathscr{R}(A^*) \oplus \mathscr{N}(A) \quad \text{and} \quad \mathscr{K} = \mathscr{R}(A) \oplus \mathscr{N}(A^*),$$

we can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix}, \tag{1.3}$$

where $A_1 : \mathscr{R}(A^*) \to \mathscr{R}(A)$ is invertible; see [7, Lemma 2.1]. Therefore, the Moore–Penrose generalized inverse of A can be represented as

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix}. \tag{1.4}$$

Many results have been obtained on the solvability of equations for matrices and operators on Hilbert spaces and Hilbert C^*- modules. In 1976, Mitra [11] considered the matrix equations $AX = B, AXB = C$ and the system of linear equations $AX = C, XB = D$. He got the necessary and sufficient conditions for existence and expressions of general Hermitian solutions. In 1966, the celebrated Douglas Lemma was established in [8]. It gives some conditions for the existence of a solution to the equation $AX = B$ for operators on a Hilbert space. Using the generalized inverses of operators, in 2007, Dajić and Koliha [4] got the existence of the common Hermitian and positive solutions to the system $AX = C, XB = D$ for operators acting on a Hilbert space. In 2008, Xu [17] extended these results to the adjointable operators. Several general operator equations and systems in some general settings such as Hilbert C^*-modules have been studied by some mathematicians; see, e.g., [9, 10, 13, 16].

The matrix equation $AXB = C$ is consistent if and only if $AA^-CB^-B = C$ for some A^-, B^-, and the general solution is $X = A^-CB^- + Y - A^-AYBB^-$, where Y is an arbitrary matrix; see [11]. In 2010, Gonzalez [1] got some necessary and sufficient conditions for existence of a solution to the equation $AXB = C$ for operators on a Hilbert space.

Let A, B or C have closed range. Then, the operator equation $AXB = C$ is solvable if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$; see [1, Theorem 3.1]. Therefore, if A or C has closed range, then the equation $AXC = C$ is solvable if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$, and $CXA = C$ is solvable if and only if $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$. Deng [5] investigated the equation $CAX = C = XAC$, which is essentially different from ours. In this paper, we first characterize the existence of solutions of the system of operator equations $BXA = B = AXB$ by means of $*-$ order. After that, we generalize the solutions to the system of operator equations $BXA = B = AXB$ in a new fashion.

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2. The existence of solutions of the system $BXA = B = AXB$. We start our work with the celebrated Douglas lemma.

LEMMA 2.1 (Douglas Lemma, [8]). *Let $A, C \in \mathbb{B}(\mathscr{H})$. Then, the following statements are equivalent:*

(a) $\mathscr{R}(C) \subseteq \mathscr{R}(A)$.
(b) There exists $X \in \mathbb{B}(\mathscr{H})$ *such that* $AX = C$.
(c) There exists a positive number λ *such that* $CC^* \leq \lambda^2 AA^*$.

If one of these conditions holds, then there exists a unique solution $\widetilde{X} \in \mathbb{B}(\mathscr{H})$ *of the equation* $AX = C$ *such that* $\mathscr{R}(\widetilde{X}) \subseteq \overline{\mathscr{R}(A^*)}$ *and* $\mathscr{N}(\widetilde{X}) = \mathscr{N}(C)$.

LEMMA 2.2. *Let* $A, B \in \mathbb{B}(\mathscr{H})$. *If* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$, *then* $B = B_1 \bigoplus 0$, *where* $B_1 \in \mathbb{B}(\overline{\mathscr{R}(A^*)}, \overline{\mathscr{R}(A)})$.

Proof. Let A, B be operators from the decomposition $\mathscr{H} = \overline{\mathscr{R}(A^*)} \bigoplus \mathscr{N}(A)$ into the decomposition $\mathscr{H} = \overline{\mathscr{R}(A)} \bigoplus \mathscr{N}(A^*)$. If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, then, by Lemma 2.1, there exists $C \in \mathbb{B}(\mathscr{H})$ such that $B = AC$ and $\mathscr{N}(C) = \mathscr{N}(B)$. Since $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$, so $\mathscr{R}(C^*) \subseteq \overline{\mathscr{R}(C^*)} = \overline{\mathscr{R}(B^*)} \subseteq \overline{\mathscr{R}(A^*)} = \mathscr{N}(P_{\mathscr{N}(A)})$. Hence, $P_{\mathscr{N}(A)}C^* = 0$ and so $CP_{\mathscr{N}(A)} = 0$. It follows from $\mathscr{N}(C) = \mathscr{N}(B)$ that $BP_{\mathscr{N}(A)} = 0$.

If $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$, then a similar reasoning shows that $P_{\mathscr{N}(A^*)}B = 0$. Therefore, $P_{\overline{\mathscr{R}(A)}}BP_{\mathscr{N}(A)} = P_{\mathscr{N}(A^*)}BP_{\overline{\mathscr{R}(A^*)}} = P_{\mathscr{N}(A^*)}BP_{\mathscr{N}(A)} = 0$. Hence, $B = B_1 \bigoplus 0$, where $B_1 = P_{\overline{\mathscr{R}(A)}}BP_{\overline{\mathscr{R}(A^*)}}$. ∎

THEOREM 2.3. *Let* $A \in \mathbb{B}(\mathscr{H})$ *and* $B \in \mathscr{S}(\mathscr{H})$. *If* A *has closed range, then the following statements are equivalent:*

(1) The system of operator equations $BXA = B = AXB$ *is solvable.*
(2) $AA^\dagger BA^\dagger A = B$.
(3) $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(B) \subseteq \mathscr{R}(A^*)$.

Proof. $((1) \Longrightarrow (2))$: Using (1.1) and $B = BXA$, we get that $\mathscr{R}(B) \subseteq \mathscr{R}(A^*) = \mathscr{R}(A^\dagger A)$. Hence, by Lemma 2.1, there exists $C^* \in \mathbb{B}(\mathscr{H})$ such that $B = A^\dagger AC^*$. Hence, $B = CA^\dagger A$. Applying (1.1) and $AXB = B$, we derive that $\mathscr{R}(B) \subseteq \mathscr{R}(A) = \mathscr{R}(AA^\dagger)$. Thus, by Lemma 2.1, there exists $\widetilde{C} \in \mathbb{B}(\mathscr{H})$ such that $B = AA^\dagger\widetilde{C}$. It follows that

$$AA^\dagger BA^\dagger A = AA^\dagger(AA^\dagger\widetilde{C})A^\dagger A = AA^\dagger\widetilde{C}A^\dagger A = BA^\dagger A = (CA^\dagger A)A^\dagger A = CA^\dagger A = B.$$

$((2) \Longrightarrow (3))$: Let $AA^\dagger BA^\dagger A = B$. Then, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. It follows from $B = B^* = (AA^\dagger BA^\dagger A)^* = A^\dagger ABAA^\dagger$ and (1.1) that $\mathscr{R}(B) \subseteq \mathscr{R}(A^\dagger) = \mathscr{R}(A^*)$.

$((3) \Longrightarrow (1))$: Let $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(B) \subseteq \mathscr{R}(A^*)$. Upon applying Lemma 2.2, $B = B_1 \bigoplus 0$, where $B_1 = P_{\overline{\mathscr{R}(A)}}BP_{\overline{\mathscr{R}(A^*)}}$. Since A has closed rang, so by using (1.3) and (1.4) we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $AA^\dagger B = B$ and $BA^\dagger A = B$. Thus $X = A^\dagger$ is a solution of the system $BXA = B = AXB$. ∎

PROPOSITION 2.4. *Let* $A, B, X \in \mathbb{B}(\mathscr{H})$. *Then,*

$$\mathscr{R}(A) \subseteq \mathscr{R}(B), \quad \mathscr{N}(B) \subseteq \mathscr{N}(A) \quad \textit{and} \quad BXA = B = AXB$$

if and only if

$$\mathscr{N}(B) = \mathscr{N}(A), \quad \mathscr{R}(B) = \mathscr{R}(A) \quad \textit{and} \quad AXA = A.$$

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Proof. $(\Longrightarrow)$: Suppose that $\mathscr{R}(A) \subseteq \mathscr{R}(B), \mathscr{N}(B) \subseteq \mathscr{N}(A)$ and $BXA = B = AXB$. It follows from $BXA = B$ and $\mathscr{N}(B) \subseteq \mathscr{N}(A)$ that $\mathscr{N}(A) \subseteq \mathscr{N}(B) \subseteq \mathscr{N}(A)$. Hence, $\mathscr{N}(A) = \mathscr{N}(B)$. It follows from $AXB = B$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ that $\mathscr{R}(A) \subseteq \mathscr{R}(B) \subseteq \mathscr{R}(A)$. Therefore, $\mathscr{R}(A) = \mathscr{R}(B)$. Moreover, $(I - AX)B = 0$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$. Hence, we derive that $(I - AX)A = 0$. So, $AXA = A$.

$(\Longleftarrow)$: Suppose that $\mathscr{N}(B) = \mathscr{N}(A), \mathscr{R}(B) = \mathscr{R}(A)$ and $AXA = A$. Hence,

$$(I - AX)A = 0 \Longrightarrow \mathscr{R}(A) \subseteq \mathscr{N}(I - AX) \Longrightarrow \mathscr{R}(B) \subseteq \mathscr{N}(I - AX) \Longrightarrow B = AXB,$$
$$A(I - XA) = 0 \Longrightarrow \mathscr{R}(I - XA) \subseteq \mathscr{N}(A) \Longrightarrow \mathscr{R}(I - XA) \subseteq \mathscr{N}(B) \Longrightarrow B = BXA. \qquad \square$$

3. System of operator equations $BXA = B = AXB$ **via $*$-order.** We know that $(\mathbb{B}(\mathscr{H}), \overset{*}{\leq})$ is a partially ordered set; see [2]. Let $G_1, G_2 \in \mathbb{B}(\mathscr{H})$ be invertible and $G_1 \overset{*}{\leq} A, G_2 \overset{*}{\leq} A$. Then, $G_1G_1^* = AG_1^*$ and $G_2G_2^* = AG_2^*$. Hence, we obtain $G_1 = G_2 = A$. This fact leads us to consider the characterizations of $A \overset{*}{\leq} B$. Now we state the necessary and sufficient conditions in which the common $*-$ lower or $*-$ upper bounds of A and B exist.

We need the following essential lemma.

Lemma 3.1. [18, Lemma 2.1]. *Let $A, B \in \mathbb{B}(\mathscr{H})$ and $\overline{\mathscr{M}}$ denote the closure of a space $\mathscr{M}$. Then,*

(a) $AA^* = BA^* \Longleftrightarrow A = BP_{\overline{\mathscr{R}(A^*)}} \Longleftrightarrow A = BQ$ for some $Q \in \mathscr{OP}(\mathscr{H})$;
(b) $A^*A = A^*B \Longleftrightarrow A = P_{\overline{\mathscr{R}(A)}}B \Longleftrightarrow A = PB$ for some $P \in \mathscr{OP}(\mathscr{H})$;
(c) $A \overset{*}{\leq} B \Longleftrightarrow B = A + P_{\mathscr{N}(A^*)}BP_{\mathscr{N}(A)}$;
(d) $A \overset{*}{\leq} B \Longleftrightarrow A = P_{\overline{\mathscr{R}(A)}}B = BP_{\overline{\mathscr{R}(A^*)}} = P_{\overline{\mathscr{R}(A)}}BP_{\overline{\mathscr{R}(A^*)}}$;
(e) $A \overset{*}{\leq} B \Longleftrightarrow A = A_1 \bigoplus 0, B = A_1 \bigoplus B_1$;

where $A_1 \in \mathbb{B}(\overline{\mathscr{R}(A^*)}, \overline{\mathscr{R}(A)}), B_1 \in \mathbb{B}(\mathscr{N}(A), \mathscr{N}(A^*))$ and $A \bigoplus B$ means the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

The following Lemma is a version of Lemma 2.1 when the operator A has closed range.

Lemma 3.2. [4, Theorem 3.1]. *Let $A \in \mathbb{B}(\mathscr{H})$ have closed range. Then, the equation $AX = C$ has a solution $X \in \mathbb{B}(\mathscr{H})$ if and only if $AA^\dagger C = C$, and this if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$. In this case, the general solution is $X = A^\dagger C + (I - A^\dagger A)T$, where $T \in \mathbb{B}(\mathscr{H})$ is arbitrary.*

Proposition 3.3. *Let $A, B \in \mathbb{B}(\mathscr{H})$. Then*

(a) If A has closed range and $B \overset{*}{\leq} A$, then $X = A^\dagger$ is a solution of the system $BXA = B = AXB$.
(b) If B has closed range and $B \overset{*}{\leq} A$, then $X = B^\dagger$ is a solution of the system $BXA = B = AXB$.

Proof. (a) Let A be a closed range operator and $B \overset{*}{\leq} A$. It follows from Lemma 3.1 (d) that $B = AP_{\overline{\mathscr{R}(B^*)}}$ and $B = P_{\overline{\mathscr{R}(B)}}A$. Hence, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$. It follows from $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and Lemma 3.2 that $AA^\dagger B = B$. It follows from $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$ and Lemma 3.2 that $BA^\dagger A = \big((A^\dagger A)^*B^*\big)^* = \Big(A^*A^{\dagger *}B^*\Big)^* = B$. Hence, $X = A^\dagger$ is a solution of the system of operator equations $BXA = B = AXB$.

(b) Let B be a closed range operator and $B \overset{*}{\leq} A$. It follows from Lemma 3.1 that $B = AP_{\mathscr{R}(B^*)}$ and $B = P_{\mathscr{R}(B)}A$. Applying (1.1), we conclude that $AB^\dagger B = B$ and $BB^\dagger A = B$. Hence, $X = B^\dagger$ is a solution of the system $BXA = B = AXB$. $\square$

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Proposition 3.4. *Let $A, B, X \in \mathbb{B}(\mathscr{H})$. If $A \overset{*}{\leq} B$ and $BXA = B = AXB$, then $\mathscr{N}(B) = \mathscr{N}(A)$, $\mathscr{R}(B) = \mathscr{R}(A)$ and $AXA = A$.*

Proof. Let $A \overset{*}{\leq} B$ and $BXA = B = AXB$. Applying Lemma 3.1 (d), we have $A = P_{\overline{\mathscr{R}(A)}}B = BP_{\overline{\mathscr{R}(A^*)}}$. Hence, $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{N}(B) \subseteq \mathscr{N}(A)$. Using Proposition 2.4,

$$\mathscr{N}(B) = \mathscr{N}(A), \quad \mathscr{R}(B) = \mathscr{R}(A) \quad \text{and} \quad AXA = A. \qquad \square$$

Remark 3.5. Note that the converse of Proposition 3.4 is not true, in general. Set $A^\dagger, A^*, A$ instead of A, B, X. If $A \in \mathbb{B}(\mathscr{H})$ has closed range, then, by (1.1), we have $\mathscr{R}(A^*) = \mathscr{R}(A^\dagger), \mathscr{N}(A^*) = \mathscr{N}(A^\dagger)$ and $A^\dagger AA^\dagger = A^\dagger$ but not $A^\dagger \overset{*}{\leq} A^*$. Indeed, if $A^\dagger \overset{*}{\leq} A^*$, then by utilizing Lemma 3.1 (d), we have $A^\dagger = P_{\mathscr{R}(A^\dagger)}A^*$. It follows from $\mathscr{R}(A^\dagger) = \mathscr{R}(A^*)$ that $A^\dagger = P_{\mathscr{R}(A^*)}A^* = A^*$.

Theorem 3.6. *Let $A, B \in \mathbb{B}(\mathscr{H})$ and $B \overset{*}{\leq} A$. Then, the following statements are equivalent:*

(a) There exists a solution $X \in \mathbb{B}(\mathscr{H})$ of the system $BXA = B = AXB$.
(b) $B \overset{*}{\leq} AXA$.

Proof. ((a) $\Longrightarrow$ (b)) : Let $X \in \mathbb{B}(\mathscr{H})$ is a solution of the system $BXA = B = AXB$. Hence, $B - BXA = 0$ and $B - AXB = 0$. It follows from the assumption $B \overset{*}{\leq} A$ and Lemma 3.1 (d) that $B = P_{\overline{\mathscr{R}(B)}}A$ and $B = AP_{\overline{\mathscr{R}(B^*)}}$. Hence,

$$P_{\overline{\mathscr{R}(B)}}(B - AXA) = B - P_{\overline{\mathscr{R}(B)}}AXA = B - BXA = 0$$

and

$$(B - AXA)P_{\overline{\mathscr{R}(B^*)}} = B - AXAP_{\overline{\mathscr{R}(B^*)}} = B - AXB = 0.$$

Therefore, $B \overset{*}{\leq} AXA$.

((b) $\Longrightarrow$ (a)) : Suppose that $B \overset{*}{\leq} AXA$. Applying Lemma 3.1 (d), we infer that $P_{\overline{\mathscr{R}(B)}}(B - AXA) = 0$ and $(B - AXA)P_{\overline{\mathscr{R}(B^*)}} = 0$. It follows from the assumption $B \overset{*}{\leq} A$ and Lemma 3.1 (d) that $B = P_{\overline{\mathscr{R}(B)}}A$ and $B = AP_{\overline{\mathscr{R}(B^*)}}$, whence

$$B - BXA = B - P_{\overline{\mathscr{R}(B)}}AXA = P_{\overline{\mathscr{R}(B)}}(B - AXA) = 0$$

and

$$B - AXB = B - AXAP_{\overline{\mathscr{R}(B^*)}} = (B - AXA)P_{\overline{\mathscr{R}(B^*)}} = 0.$$

Therefore, X is a solution of the system $BXA = B = AXB$. $\square$

Let $A, B \in \mathbb{B}(\mathscr{H})$ have closed ranges. It follows from Proposition 3.3 that $A^\dagger$ and $B^\dagger$ are solutions of the system $BXA = B = AXB$. Therefore, we are interested in the study of the following system of operator equations:

$$BXA = B = AXB, \tag{3.5}$$

$$BAX = B = XAB. \tag{3.6}$$

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Let $A, B \in \mathbb{B}(\mathscr{H})$. An operator $C \in \mathbb{B}(\mathscr{H})$ is said to be an inverse of A along B if it fulfills one of the equations (3.5) or (3.6). If $A \in \mathbb{B}(\mathscr{H})$ is invertible, then $X = A^{-1}$ is a solution of the system $XA = I = AX$. Hence, A^{-1} is an inverse of A along I, where I is the identity of $\mathbb{B}(\mathscr{H})$.

Let $A \in \mathbb{B}(\mathscr{H})$ have closed range. Using (1.1), we have $AA^\dagger A = A = AA^\dagger A$. Hence, $A^\dagger$ satisfies Eq. (3.5). Therefore, $A^\dagger$ is the inverse of A along A.

It follows from (1.1) that $A^*AA^\dagger = A^* = A^\dagger AA^*$. Hence, $A^\dagger$ satisfies Eq. (3.6). Therefore, A is the inverse of A along A^*.

Lemma 3.7. [11, Theorem 2.1]. *Let $C \in \mathbb{B}(\mathscr{H})$ and $A, B \in \mathbb{B}(\mathscr{H})$ have closed ranges. Then, the equation $AXB = C$ has a solution $X \in \mathbb{B}(\mathscr{H})$ if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A), \mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$, and this if and only if $AA^\dagger CB^\dagger B = C$. In this case, $X = A^\dagger CB^\dagger + U - A^\dagger AUBB^\dagger$, where $U \in \mathbb{B}(\mathscr{H})$ is arbitrary.*

In the next result, we provide a general solution of the system $BXA = B = AXB$.

Theorem 3.8. *Let $A, B \in \mathbb{B}(\mathscr{H})$ have closed ranges and $B \overset{*}{\leq} A$. Then, the general solution of the system of operator equations $BXA = B = AXB$ is*

$$\begin{aligned} X = {} & A^\dagger BB^\dagger + A^\dagger \left[B(I - AA^\dagger) + (A - B)S\right](A - B)^\dagger + T - A^\dagger AT(A - B)^\dagger(A - B) \\ & -A^\dagger B(I - AA^\dagger)(A - B)^\dagger BB^\dagger - A^\dagger(A - B)S(A - B)^\dagger BB^\dagger \\ & -A^\dagger ATBB^\dagger + A^\dagger AT(A - B)^\dagger(A - B)BB^\dagger, \end{aligned}$$

where $S, T \in \mathbb{B}(\mathscr{H})$.

Proof. Let A, B have closed ranges. It follows from the assumption $B \overset{*}{\leq} A$ and Lemma 3.1 (d) that $B = AP_{\mathscr{R}(B^*)}$. Hence, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. Using Lemma 3.2, we have $AA^\dagger B = B$. It follows from $AA^\dagger BB^\dagger B = B$ and Lemma 3.7 that the equation $AXB = B$ is solvable. In this case, the general solution is

$$X = A^\dagger BB^\dagger + W - A^\dagger AWBB^\dagger, \tag{3.7}$$

where $W \in \mathbb{B}(\mathscr{H})$ is arbitrary. If X satisfies the equation $BXA = B$, then

$$B(A^\dagger BB^\dagger + W - A^\dagger AWBB^\dagger)A = B.$$

It follows from the assumption $B \overset{*}{\leq} A$ and Lemma 3.1 (d) that $B = P_{\mathscr{R}(B)}A$. Applying (1.1), $BB^\dagger A = B$. Hence,

$$BA^\dagger B + BWA - BA^\dagger AWB = B.$$

Therefore, $B(A^\dagger B + WA - A^\dagger AWB) = B$. So, $A^\dagger B + WA - A^\dagger AWB$ is a solution of the equation $BX = B$. Utilizing Lemma 3.2 again, we have

$$A^\dagger B + WA - A^\dagger AWB = B^\dagger B + (I - B^\dagger B)S, \tag{3.8}$$

where $S \in \mathbb{B}(\mathscr{H})$ is arbitrary. Multiply the left hand side of Eq. (3.8) by A, to get

$$AA^\dagger B + AWA - AA^\dagger AWB = AB^\dagger B + A(I - B^\dagger B)S.$$

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It follows from the assumption $B \overset{*}{\leq} A$ and Lemma 3.1 (d) that $B = AP_{\mathscr{R}(B^*)}$. Applying (1.1), $AB^\dagger B = B$. We derive that

$$AA^\dagger B + AWA - AWB = B + (A-B)S.$$

Now, we get $AW(A-B) = B(I - AA^\dagger) + (A-B)S$. So, W is a solution of the equation $AX(A-B) = B(I-AA^\dagger) + (A-B)S$. Using Lemma 3.7, we get that

$$W = A^\dagger \left[B(I-AA^\dagger) + (A-B)S\right](A-B)^\dagger + T - A^\dagger AT(A-B)^\dagger(A-B),$$

where $T \in \mathbb{B}(\mathscr{H})$ is arbitrary. By putting W in Eq. (3.7), we reach

$$\begin{aligned}
X &= A^\dagger BB^\dagger + A^\dagger \left[B(I-AA^\dagger) + (A-B)S\right](A-B)^\dagger + T - A^\dagger AT(A-B)^\dagger(A-B)\\
&\quad - A^\dagger A(A^\dagger \left[B(I-AA^\dagger) + (A-B)S\right](A-B)^\dagger\\
&\quad + T - A^\dagger AT(A-B)^\dagger(A-B)BB^\dagger\\
&= A^\dagger BB^\dagger + A^\dagger \left[B(I-AA^\dagger) + (A-B)S\right](A-B)^\dagger + T - A^\dagger AT(A-B)^\dagger(A-B)\\
&\quad - A^\dagger AA^\dagger B(I-AA^\dagger)(A-B)^\dagger BB^\dagger - A^\dagger AA^\dagger(A-B)S(A-B)^\dagger BB^\dagger\\
&\quad - A^\dagger ATBB^\dagger + A^\dagger AT(A-B)^\dagger(A-B)BB^\dagger \qquad \text{(by (1.1))}\\
&= A^\dagger BB^\dagger + A^\dagger \left[B(I-AA^\dagger) + (A-B)S\right](A-B)^\dagger + T - A^\dagger AT(A-B)^\dagger(A-B)\\
&\quad - A^\dagger B(I-AA^\dagger)(A-B)^\dagger BB^\dagger - A^\dagger(A-B)S(A-B)^\dagger BB^\dagger\\
&\quad - A^\dagger ATBB^\dagger + A^\dagger AT(A-B)^\dagger(A-B)BB^\dagger. \qquad \square
\end{aligned}$$

THEOREM 3.9. *Let $A, B \in \mathbb{B}(\mathscr{H})$ where A has closed range. If the system $BXA = B = AXB$ is solvable, then the system $XB = A^\dagger B, BX = BA^\dagger$ is solvable. Conversely, If $B \overset{*}{\leq} A$ and the system $XB = A^\dagger B, BX = BA^\dagger$ is solvable, then the system $BXA = B = AXB$ is solvable.*

Proof. $(\Longrightarrow)$: Let $\widetilde{X}$ be a solution of the system $BXA = B = AXB$. It follows from $B = A\widetilde{X}B$ that $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. Using Lemma 3.2, $AA^\dagger B = B$. It follows from (1.1) that

$$P_{\overline{\mathscr{R}(A^*)}}\widetilde{X}AA^\dagger B = (A^\dagger A)\widetilde{X}(AA^\dagger)B = (A^\dagger A)\widetilde{X}(AA^\dagger B) = A^\dagger(A\widetilde{X}B) = A^\dagger B.$$

So, $P_{\overline{\mathscr{R}(A^*)}}\widetilde{X}AA^\dagger$ is a solution of the equation $XB = A^\dagger B$. Since $B^* = (B\widetilde{X}A)^* = A^*\widetilde{X}^*B^*$, we have $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$. Applying Lemma 2.1, there exists $Y \in \mathbb{B}(\mathscr{H})$ such that $B = YA$. Hence,

$$\begin{aligned}
BP_{\overline{\mathscr{R}(A^*)}}\widetilde{X}AA^\dagger &= B(A^\dagger A)\widetilde{X}(AA^\dagger) = Y(AA^\dagger A)\widetilde{X}(AA^\dagger)\\
&= (YA\widetilde{X}A)A^\dagger = (B\widetilde{X}A)A^\dagger = BA^\dagger.
\end{aligned}$$

Therefore, $P_{\overline{\mathscr{R}(A^*)}}\widetilde{X}AA^\dagger$ is a solution of the equation $B = BA^\dagger$. Thus $P_{\overline{\mathscr{R}(A^*)}}\widetilde{X}AA^\dagger$ is a solution of the system $XB = A^\dagger B, BX = BA^\dagger$.

$(\Longleftarrow)$: Suppose that $\widetilde{X}$ is a solution of the system $XB = A^\dagger B, BX = BA^\dagger$. It follows from the assumption $B \overset{*}{\leq} A$ that $B = AP_{\overline{\mathscr{R}(B^*)}}$ and $B = P_{\overline{\mathscr{R}(B)}}A$. Hence, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$. It follows from $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ to Lemma 3.2 that $AA^\dagger B = B$. Hence, $A\widetilde{X}B = A(A^\dagger B) = AA^\dagger B = B$. It follows from $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$ and Lemma 2.1 that there exists $Z^* \in \mathbb{B}(\mathscr{H})$ such that $B = ZA$. Hence,

$$B\widetilde{X}A = (BA^\dagger)A = BA^\dagger A = ZAA^\dagger A = ZA = B.$$

Therefore, $\widetilde{X}$ is a solution of the system $BXA = B = AXB$. $\square$

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LEMMA 3.10. [4, Theorem 4.2]. *Let $A, B, C, D \in \mathbb{B}(\mathscr{H})$ and $A, B, M = B^*(I - A^\dagger A)$ have closed ranges. Then, the system $AX = C,\ XB = D$ has a hermitian solution $X \in \mathbb{B}(\mathscr{H})$ if and only if*

$$AA^\dagger C = C, \quad DB^\dagger B = D, \quad AD = CB$$

and AC^* and B^*D are hermitian. In this case, the general hermitian solution is

$$\begin{aligned} X = {} & A^\dagger C + (I - A^\dagger A)M^\dagger s(T) \\ & + (I - A^\dagger A)(I - M^\dagger M)\left[A^\dagger C + (I - A^\dagger A)M^\dagger s(T)\right]^* \\ & + (I - A^\dagger A)(I - M^\dagger M)W(I - M^\dagger M)^*(I - A^\dagger A)^*, \end{aligned}$$

where $W \in \mathbb{B}(\mathscr{H})$ is hermitian and $s(T) = D^* - B^*A^\dagger C$ is the so-called Schur complement of the block matrix $T = \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix}$.

THEOREM 3.11. *Suppose that $A, B \in \mathbb{B}(\mathscr{H})$ have closed ranges. If $B \overset{*}{\leq} A$ and $B^*A^\dagger B, BA^{\dagger *}B^*$ are hermitian, then the system $BXA = B = AXB$ has a hermitian solution.*

Proof. Replace A, B, C, D in Lemma 3.10 by $B, B, BA^\dagger, A^\dagger B$ to get

$$AA^\dagger C = BB^\dagger(BA^\dagger) = BA^\dagger = C, \quad DB^\dagger B = (A^\dagger B)B^\dagger B = A^\dagger B = D$$

and

$$AD = B(A^\dagger B) = (BA^\dagger)B = CB, \quad AC^* = B(BA^\dagger)^* = BA^{\dagger *}B^*, \quad B^*D = B^*A^\dagger B.$$

Using Lemma 3.10, the system $XB = A^\dagger B, BX = BA^\dagger$ has a hermitian solution, say, $\widetilde{X}$. It follows from the assumption $B \overset{*}{\leq} A$ that $B = AP_{\overline{\mathscr{R}(B^*)}}$ and $B = P_{\overline{\mathscr{R}(B)}}A$. Hence, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$. It follows from $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and Lemma 3.2 that $AA^\dagger B = B$. Hence, $A\widetilde{X}B = A(A^\dagger B) = AA^\dagger B = B$. It follows from $\mathscr{R}(B^*) \subseteq \mathscr{R}(A^*)$ and Lemma 2.1 that there exists $Z \in \mathbb{B}(\mathscr{H})$ such that $B = ZA$. Hence,

$$B\widetilde{X}A = (BA^\dagger)A = BA^\dagger A = ZAA^\dagger A = ZA = B.$$

Therefore, $\widetilde{X}$ is a hermitian solution of the system $BXA = B = AXB$. ∎

4. $*$-Order via other operator equations. Generally speaking, the inequality $PB \overset{*}{\leq} B$ dose not hold for any $P \in \mathscr{P}(\mathscr{H})$ even if $\mathscr{R}(P) \subseteq \overline{\mathscr{R}(B)}$. In [2, Lemma 2.6], some conditions are mentioned which give a one-sided description of the relation $A \overset{*}{\leq} B$ regarding (1.2).

The next result is known.

PROPOSITION 4.1. [2, Proposition 2.6]. *Let $B \in \mathbb{B}(\mathscr{H})$.*

(a) If $P \in \mathscr{O}\mathscr{P}(\mathscr{H})$ and $\mathscr{R}(P) \subseteq \overline{\mathscr{R}(B)}$, then $PB \overset{*}{\leq} B$ if and only if $PBB^* = BB^*P$.
(b) If $Q \in \mathscr{O}\mathscr{P}(\mathscr{H})$ and $\mathscr{R}(Q) \subseteq \overline{\mathscr{R}(B^*)}$, then $BQ \overset{*}{\leq} B$ if and only if $QB^*B = B^*BQ$.

In the following, we state a generalization of Proposition 4.1.

PROPOSITION 4.2. *Let $B \in \mathbb{B}(\mathscr{H})$. If there exist $P, Q \in \mathscr{O}\mathscr{P}(\mathscr{H})$ such that $\mathscr{R}(P) \subseteq \overline{\mathscr{R}(B)}$and $\mathscr{R}(Q) \subseteq \overline{\mathscr{R}(B^*)}$, then $PBQ \overset{*}{\leq} B$ if and only if $PBQB^* = BQB^*P$ and $QB^*PB = B^*PBQ$.*

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Proof. $(\Longrightarrow):$ Let $PBQ \overset{*}{\leq} B$. Applying (1.2), we get that

$$PBQB^* = (PBQ)B^* = B(PBQ)^* = BQB^*P$$

and

$$B^*PBQ = B^*(PBQ) = (PBQ)^*B = QB^*PB.$$

$(\Longleftarrow):$ Let $PBQB^* = BQB^*P$ and $QB^*PB = B^*PBQ$. Applying (1.2), we obtain that

$$(PBQ)(PBQ)^* = PBQB^*P = (BQB^*P)P = BQB^*P = B(PBQ)^*$$

and

$$(PBQ)^*(PBQ) = QB^*PBQ = Q(QB^*PB) = QB^*PB = (PBQ)^*B. \qquad \square$$

The next known theorem gives a characterization of the order $\overset{*}{\leq}$.

Theorem 4.3. [6, Theorem 2.3]. *Let $A \in \mathbb{B}(\mathscr{H})$ and $C \in \mathscr{Q}(\mathscr{H})$. Then, $C \overset{*}{\leq} A$ if and only if there exists $X \in \mathbb{B}(\mathscr{H})$ such that $A = C + (I - C^*)X(I - C^*)$.*

In the following, we establish an analogue of Theorem 4.3 for generalized projections on a Hilbert space. Recall that an operator $A \in \mathbb{B}(\mathscr{H})$ is a generalized projection if $A^2 = A^*$.

Lemma 4.4. [14, Theorem A.2]. *Let $A \in \mathbb{B}(\mathscr{H})$ be a generalized projection.Then, A is a closed range operator and A^3 is an orthogonal projection on $\mathscr{R}(A)$. Moreover, $\mathscr{H}$ has decomposition*

$$\mathscr{H} = \mathscr{R}(A) \bigoplus \mathscr{N}(A)$$

and A has the following matrix representation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A) \end{bmatrix},$$

where the restriction $A_1 = A|_{\mathscr{R}(A)}$ is unitary on $\mathscr{R}(A)$.

Theorem 4.5. *Let $A \in \mathbb{B}(\mathscr{H})$ and $B \in \mathscr{GP}(\mathscr{H})$. Then, $B \overset{*}{\leq} A$ if and only if there exists $X \in \mathbb{B}(\mathscr{H})$ such that $A = B + (I - BB^*)X(I - B^*B)$.*

Proof. $(\Longrightarrow):$ Let $B \in \mathscr{GP}(\mathscr{H})$ and $B \overset{*}{\leq} A$. Employing Lemma 4.4, we infer that B has closed range and $B^3 = P_{\mathscr{R}(B)}$. It follows from (1.1) that

$$\mathscr{R}(B^*) = \mathscr{R}(B^*B) = \mathscr{R}(B^3) = \mathscr{R}(BB^*) = \mathscr{R}(B).$$

Hence, $P_{\mathscr{R}(B)} = P_{\mathscr{R}(B^*)} = BB^* = B^*B$. Therefore, $P_{\mathscr{N}(B)} = P_{\mathscr{N}(B^*)} = I - BB^* = I - B^*B$. Applying Lemma 3.1 (c), we get $A = B + P_{\mathscr{N}(B^*)}AP_{\mathscr{N}(B)}$. Hence, $A = B + (I - BB^*)A(I - B^*B)$.

$(\Longleftarrow):$ Let $X \in \mathbb{B}(\mathscr{H})$ be a solution of the equation $A = B + (I - BB^*)X(I - B^*B)$. Since B is a generalized projection, so $B^*BB^* = B^*$. Hence,

$$B^*A = B^*B + B^*(I - BB^*)X(I - B^*B) = B^*B$$

and

$$AB^* = BB^* + (I - BB^*)X(I - B^*B)B^* = BB^*.$$

Therefore, $B \overset{*}{\leq} A$ by (1.2). $\square$

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In the next result, we show that if A is a generalized projection and $B \overset{*}{\leq} A \overset{*}{\wedge} A^*$, then AA^* can be written as the sum of two idempotents.

THEOREM 4.6. *Let $A \in \mathscr{GP}(\mathscr{H})$ and $B \in \mathbb{B}(\mathscr{H})$. If $B \overset{*}{\leq} A \overset{*}{\wedge} A^*$, then B is an idempotent and there exists an idempotent X such that $AA^* = B + X$ and $B^*X = XB^* = 0$.*

Proof. Let $B \overset{*}{\leq} A \overset{*}{\wedge} A^*$. It follows from the assumption $A^2 = A^*$ and Lemma 3.1 (d) that

$$B^2 = (P_{\overline{\mathscr{R}(B)}}A^*)(A^*P_{\overline{\mathscr{R}(B^*)}}) = P_{\overline{\mathscr{R}(B)}}{A^*}^2P_{\overline{\mathscr{R}(B^*)}} = P_{\overline{\mathscr{R}(B)}}AP_{\overline{\mathscr{R}(B^*)}} = BP_{\overline{\mathscr{R}(B^*)}} = B.$$

Using Lemma 3.1, we get that

$$AB = A(AP_{\overline{\mathscr{R}(B^*)}}) = A^2P_{\overline{\mathscr{R}(B^*)}} = A^*P_{\overline{\mathscr{R}(B^*)}} = B,$$

$$BA = (P_{\overline{\mathscr{R}(B)}}A)A = P_{\overline{\mathscr{R}(B)}}A^2 = P_{\overline{\mathscr{R}(B)}}A^* = B,$$

$$A^*B = A^*(A^*P_{\overline{\mathscr{R}(B^*)}}) = {A^*}^2P_{\overline{\mathscr{R}(B^*)}} = AP_{\overline{\mathscr{R}(B^*)}} = B$$

and

$$BA^* = (P_{\overline{\mathscr{R}(B)}}A^*)A^* = P_{\overline{\mathscr{R}(B)}}{A^*}^2 = P_{\overline{\mathscr{R}(B)}}A = B.$$

Let $X = AA^* - B$. It follows from the assumption $B \overset{*}{\leq} A \overset{*}{\wedge} A^*$ that

$$\begin{aligned} X^2 = (AA^* - B)^2 =& (AA^*)^2 + B^2 - AA^*B - BAA^* \\ =& AA^* + B - AB - BA^* \\ =& AA^* + B - B - B = AA^* - B = X. \end{aligned}$$

Hence, X is an idempotent. Applying (1.2), we have

$$B^*X = B^*(AA^* - B) = B^*AA^* - B^*B = B^*A^*A - B^*B = B^*A - B^*B = 0$$

and

$$XB^* = (AA^* - B)B^* = AA^*B^* - BB^* = AB^* - BB^* = 0. \qquad \square$$

LEMMA 4.7. *Let $A \in \mathscr{Q}(\mathscr{H})$ and $B \in \mathbb{B}(\mathscr{H})$. Then, $B \overset{*}{\leq} A$ if and only if B is an idempotent and there exists an idempotent X such that $A = B + X$ and $B^*X = XB^* = 0$.*

Proof. $(\Longrightarrow)$: Let $B \overset{*}{\leq} A$. It follows from the assumption $A^2 = A$ and Lemma 3.1 (d) that

$$B^2 = (P_{\overline{\mathscr{R}(B)}}A)(AP_{\overline{\mathscr{R}(B^*)}}) = P_{\overline{\mathscr{R}(B)}}A^2P_{\overline{\mathscr{R}(B^*)}} = (P_{\overline{\mathscr{R}(B)}}A)P_{\overline{\mathscr{R}(B^*)}} = BP_{\overline{\mathscr{R}(B^*)}} = B.$$

Utilizing Lemma 3.1 (d), we obtain that

$$AB = A(AP_{\overline{\mathscr{R}(B^*)}}) = A^2P_{\overline{\mathscr{R}(B^*)}} = AP_{\overline{\mathscr{R}(B^*)}} = B$$

and

$$BA = (P_{\overline{\mathscr{R}(B)}}A)A = P_{\overline{\mathscr{R}(B)}}A^2 = P_{\overline{\mathscr{R}(B)}}A = B.$$

Hence, $X = A - B$ is an idempotent and $B^*X = B^*(A - B) = 0$ and $XB^* = (A - B)B^* = 0$.

$(\Longleftarrow)$: Let $A = B + X$ and $B^*X = XB^* = 0$ for some idempotent X. Then, $B^*(A - B) = B^*X = 0$ and $(A - B)B^* = XB^* = 0$. Therefore, $B \overset{*}{\leq} A$ by (1.2). $\square$

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Corollary 4.8. *Let $A \in \mathscr{GP}(\mathscr{H})$ and $B \in \mathbb{B}(\mathscr{H})$. Then, $B \overset{*}{\leq} AA^*$ if and only if B is an idempotent and there exists an idempotent X such that $AA^* = B + X$ and $B^*X = XB^* = 0$.*

Proof. Let $A \in \mathscr{GP}(\mathscr{H})$. Then, $(AA^*)^2 = AA^*AA^* = AA^*$. Hence, AA^* is an idempotent. Now apply Lemma 4.7. ∎

We end our work with the following result.

Proposition 4.9. *Let $A \in \mathbb{B}(\mathscr{H})$ and $C \in \mathscr{GP}(\mathscr{H})$. Then, $B \in \mathbb{B}(\mathscr{H})$ is common $*-$ lower bound of A and CC^* if and only if B is an idempotent and there exist $X, Y \in \mathbb{B}(\mathscr{H})$ such that*

$$A = B + (I - B^*)X(I - B^*) \quad \text{and} \quad CC^* = B + Y,$$

where $B^*Y = YB^* = 0$.

Proof. $(\Longrightarrow):$ If B be a common $*-$ lower bound of A and CC^*, then $B \overset{*}{\leq} A$ and $B \overset{*}{\leq} CC^*$. It follows from the assumption $B \overset{*}{\leq} CC^*$ and Lemma 4.7 that B is an idempotent and there exists an idempotent $Y \in \mathbb{B}(\mathscr{H})$ such that $CC^* = B + R$, where $B^*R = RB^* = 0$. Since B is an idempotent and $B \overset{*}{\leq} A$, by Theorem 4.3, there exists $S \in \mathbb{B}(\mathscr{H})$ such that $A = B + (I - B^*)S(I - B^*)$.

$(\Longleftarrow):$ If there exists an idempotent Y such that $CC^* = B + Y$ with $B^*Y = 0$ and $YB^* = 0$, then $B \overset{*}{\leq} CC^*$. The assumption $A = B + (I - B^*)S(I - B^*)$ and the fact that B is an idempotent yield $B^*(A - B) = 0$ and $(A - B)B^* = 0$. Hence, $B \overset{*}{\leq} A$ and B is a common $*-$ lower bound of A and CC^*. ∎

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