



## PROJECTIVE PARTITIONS OF VECTOR SPACES\*

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**Abstract.** Given infinite-dimensional real vector spaces  $V, W$  with  $|W| \leq |V|$ , it is shown that there exists a collection of subspaces of  $V$  that are isomorphic to  $W$ , mutually intersect only at 0, and altogether cover  $V$ .

**Key words.** Vector space partitions, Sphere partitions, Projective spaces.

**AMS subject classifications.** 15A03, 54H05.

**1. Introduction.** A topological space  $X$  is said to partition another topological space  $Y$  if there exists a collection of topological embeddings of  $X$  into  $Y$  so that the images of the embeddings partition  $Y$  i.e., they are mutually disjoint and altogether cover  $Y$ . Topological partitions were studied by Bankston and McGovern in [4], where they showed that every nonempty subset of the  $n$ -dimensional sphere  $S^n$  partitions the  $(2n + 1)$ -dimensional Euclidean space  $\mathbb{R}^{2n+1}$  *isometrically*, which means that each embedding in the partition is an isometric embedding. If one is not restricted to isometric embeddings, one can show that any nonempty subset of  $S^n$  partitions  $\mathbb{R}^{n+2}$  [3]. On the other hand,  $S^n$  does not partition  $\mathbb{R}^{n+1}$  [10].

In this paper, we study *projective* partitions of real vector spaces. For a vector space  $V$ , let  $\mathbb{P}(V)$  be the projective space modeled by  $V$  comprised of 1-dimensional subspaces of  $V$ .

DEFINITION 1.1. Let  $V$  and  $W$  be real vector spaces. We say that  $W$  projectively partitions  $V$  if  $\mathbb{P}(W)$  partitions  $\mathbb{P}(V)$  isomorphically, or equivalently, if there exists a collection  $\Lambda$  of subspaces of  $V$  such that (i) each element of  $\Lambda$  is isomorphic to  $W$ , (ii) if  $U_1, U_2$  are distinct elements of  $\Lambda$ , then  $U_1 \cap U_2 = \{\mathbf{0}\}$ , and (iii)  $V = \bigcup_{U \in \Lambda} U$ .

General subspace coverings of vector spaces have been studied extensively. For example, Khare [11] obtained an expression for  $\nu(\mathbb{F}, V, k)$ , the least number of proper subspaces of codimension  $k$  of a vector space  $V$  over a field  $\mathbb{F}$  needed to cover  $V$ ; see also [6]. Subspace partitions in the context of finite-dimensional vector spaces over a finite field are also discussed extensively; for example, see [5, 7, 9] and the survey article [8]. Projective partitions are distinct from the subspace partitions in these combinatorial studies, where the subspaces can have potentially different dimensions.

In Section 2, we consider the finite dimensional case and show that  $\mathbb{R}^n$  partitions  $\mathbb{R}^m$  if and only if  $m \geq 2n$  (Theorem 2.2). As a corollary, it follows that the sphere  $S^{n-1}$  partitions  $S^{m-1}$  isometrically if and only if  $m \geq 2n$ . In section 2, we also discuss the problem of partitioning Euclidean spaces without using the Axiom of Choice (which is needed in the proof of Theorem 2.2) to arrive at the familiar Hopf fibrations.

In Section 3, we consider the infinite-dimensional case and show that in general if  $W$  and  $V$  are real vector spaces where their cardinalities satisfy the inequality  $|W| \leq |V|$ , then  $W$  projectively partitions  $V$ .

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**2. The finite dimensional case.** For a set  $A$ , let  $|A|$  denote the cardinality of  $A$ . To consider the finite dimensional case, we first need a lemma.

LEMMA 2.1. *Let  $U$  be a real vector space and  $\{U_\alpha : \alpha \in J\}$  be a collection of  $k$ -dimensional proper subspaces of  $U$ , where  $k$  is a fixed positive integer. If  $|J| < |U|$ , then  $U \neq \bigcup_{\alpha \in J} U_\alpha$ .*

*Proof.* Let  $V$  be any  $(k + 1)$ -dimensional subspace of  $U$  (since each  $U_\alpha$  is  $k$ -dimensional and a proper subspace of  $U$ , such a  $V$  exists). Since  $U = \bigcup_{\alpha \in J} U_\alpha$ , we have  $V = \bigcup_{\alpha \in J} (V \cap U_\alpha)$ , where  $V \cap U_\alpha$  is a proper subspace of  $V$  for each  $\alpha \in J$ . By the main result of [6], we conclude that  $|J| \geq |\mathbb{R}|$ . Then, we have the contradiction

$$\left| \bigcup_{\alpha \in J} U_\alpha \right| = |J| \cdot |\mathbb{R}| = |J| < |U|,$$

and the lemma follows.  $\square$

THEOREM 2.2. *Let  $R$  and  $S$  be real vector spaces of respective dimensions  $m$  and  $n$ , where  $m \in \mathbb{N} \cup \{\infty\}$  and  $n \in \mathbb{N}$ . Then  $S$  projectively partitions  $R$  if and only if  $m \geq 2n$ .*

*Proof.* Without loss of generality, we let  $S = \mathbb{R}^n$ . Suppose that  $\mathbb{R}^n$  projectively partitions  $R$  i.e., there exists a covering of  $R$  by  $n$ -dimensional subspaces that mutually intersect only at the origin. Let  $U_1$  and  $U_2$  be two distinct members of the covering. Let  $B_i$  be a basis of  $U_i$ ,  $i = 1, 2$ . Then  $B_1 \cup B_2$  is a linearly independent set of  $2n$  vectors in  $R$ . It follows that  $m \geq 2n$ .

For the converse, let  $\omega$  denote the least uncountable ordinal that has the same cardinality as  $R$ ; in particular, by the well-ordering principle, one can identify  $R \setminus \{\mathbf{0}\}$  with the set of ordinals less than  $\omega$  i.e., we can let  $R \setminus \{\mathbf{0}\} = \{P_\alpha : \alpha < \omega\}$ . We now construct a collection of  $n$ -dimensional subspaces that mutually intersect only at the origin and altogether cover  $R$  by transfinite induction. Let  $V_0$  be any  $n$ -dimensional subspace of  $R$  that contains  $P_0$ . For a given  $\alpha < \omega$ , suppose by induction we have defined a set  $\Lambda_\alpha$  of  $n$ -dimensional subspaces of  $R$  that mutually intersect only at the origin and altogether contain  $\{P_\beta : \beta \leq \alpha\}$ . If  $P_{\alpha+1}$  is included in the union of members of  $\Lambda_\alpha$ , then let  $\Lambda_{\alpha+1} = \Lambda_\alpha$ . Thus, suppose  $P_{\alpha+1}$  is not included in the union of members of  $\Lambda_\alpha$ . We construct an  $n$ -dimensional subspace  $V_{\alpha+1} \subseteq R$  that contains  $P_{\alpha+1}$  and intersects each subspace in  $\Lambda_\alpha$  only at the origin. Let  $u_1 = P_{\alpha+1}$  and suppose  $u_i$  is defined for  $1 \leq i \leq k$ , where  $1 \leq k < n$ . Let  $W_k$  be the subspace spanned by  $u_1, \dots, u_k$ . By Lemma 2.1, the set

$$U_k = \bigcup_{V \in \Lambda_\alpha} V \oplus W_k$$

does not coincide with  $R$ , since each  $V \oplus W_k$  is a proper  $(n + k)$ -dimensional subspace of  $R$  (note that  $n + k < 2n \leq m$ ). It follows that there exists a vector  $u_{k+1} \in \mathbb{R}^m \setminus U_k$ . Clearly, the subspace spanned by  $u_1, \dots, u_{k+1}$ , intersects each subspace in  $\Lambda_\alpha$  only at the origin, and the inductive step in defining  $u_i$ ,  $i = 1, \dots, n$ , is completed. Now, let  $V_{\alpha+1}$  be the subspace spanned by  $u_1, \dots, u_n$ . Then  $V_{\alpha+1}$  is an  $n$ -dimensional subspace containing  $P_{\alpha+1}$  that intersects each subspace in  $\Lambda_\alpha$  only at the origin. We let  $\Lambda_{\alpha+1} = \Lambda_\alpha \cup \{V_{\alpha+1}\}$ . This completes the inductive step of the transfinite induction. Finally, the set  $\bigcup_{\alpha < \omega} \Lambda_\alpha$  gives the desired partition of  $R$  by  $\mathbb{R}^n$ .  $\square$

For  $m, n \in \mathbb{N}$ , projective partitions of  $\mathbb{R}^m$  by  $\mathbb{R}^n$  are in one-to-one correspondence with isometric partitions of  $S^{m-1}$  by  $S^{n-1}$ . Therefore, we have the following corollary of Theorem 2.2.

COROLLARY 2.3. *For positive integers  $m$  and  $n$ ,  $S^{n-1}$  partitions  $S^{m-1}$  isometrically if and only if  $m \geq 2n$ .*

In the proof of Theorem 2.2, we used the Axiom of Choice in its equivalent formulation as the existence of a well-ordering for any set. When  $n = 1, 2, 4, 8$ , and  $m$  is a multiple of  $n$ , one can construct the partition explicitly without the Axiom of Choice. First, we need a lemma.

LEMMA 2.4. *Suppose that there exist real  $n \times n$  matrices  $A_1, \dots, A_n$  such that for every  $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  the matrix  $x_1 A_1 + \dots + x_n A_n$  is invertible. Then  $\mathbb{R}^n$  projectively partitions  $\mathbb{R}^{kn}$  for all  $k \geq 1$  (not assuming the Axiom of Choice).*

*Proof.* The proof is by induction on  $k \geq 1$ . The claim is trivial when  $k = 1$ . Suppose the claim is true for  $k$ , and let  $\Lambda_k$  be a collection of  $n$ -dimensional subspaces of  $\{\mathbf{0}\} \times \mathbb{R}^{kn}$  that mutually intersect only at the origin and altogether cover  $\{\mathbf{0}\} \times \mathbb{R}^{kn} \subseteq \mathbb{R}^{(k+1)n}$ . For each  $u = (u_1, \dots, u_k) \in (\mathbb{R}^n)^k$ , let

$$S_u = \text{lin} \{(e_i, A_i u_1, A_i u_2, \dots, A_i u_k) : 1 \leq i \leq n\},$$

where  $\text{lin}(S)$  means the linear span of  $S$ , and  $e_i$  denotes the  $n$ -tuple in  $\mathbb{R}^n$  with 1 in the  $i$ th place and zero everywhere else,  $1 \leq i \leq n$ . Then the collection  $\Lambda_{k+1} = \{S_u : u \in \mathbb{R}^{kn}\} \cup \Lambda_k$  of  $n$ -dimensional subspaces is a vector space partition of  $\mathbb{R}^{(k+1)n}$ . To see this, let  $(x, y_1, \dots, y_k) \in \mathbb{R}^{(k+1)n} \setminus \{\mathbf{0}\}$ ,  $x, y_1, \dots, y_k \in \mathbb{R}^n$ . Suppose first that  $x = (x_1, \dots, x_n) \neq \mathbf{0}$ . Since  $x_1 A_1 + \dots + x_n A_n$  is invertible, the equations

$$(x_1 A_1 + \dots + x_n A_n) u_i = y_i, \quad 1 \leq i \leq k,$$

have unique solutions  $u_1, \dots, u_k$ , and so

$$(x, y_1, \dots, y_k) = \sum_{i=1}^n x_i (e_i, A_i u_1, \dots, A_i u_k).$$

If  $x = \mathbf{0}$ , then  $(x, y_1, \dots, y_n) \in \{\mathbf{0}\} \times \mathbb{R}^{kn}$  and so it is included in a unique member of  $\Lambda_k$ . This completes the induction step and the lemma follows.  $\square$

COROLLARY 2.5. *Let  $n \in \{1, 2, 4, 8\}$ . Then  $\mathbb{R}^n$  projectively partitions  $\mathbb{R}^{kn}$  for all  $k \geq 1$  (not assuming the Axiom of Choice).*

*Proof.* The maximum number of real  $n \times n$  matrices  $A_1, \dots, A_l$  such that the matrices  $x_1 A_1 + \dots + x_l A_l$  are invertible for all nonzero vectors  $x = (x_1, \dots, x_l)$ , is given by the Radon-Hurwitz function [1, 2]:

$$\rho(n) = 2^c + 8d,$$

where  $n = (2a + 1)2^{c+4d}$  and  $0 \leq c \leq 3$ . It can easily be seen that the only values of  $n$  for which  $\rho(n) = n$  are  $n = 1, 2, 4, 8$ . The claim then follows from Lemma 2.4.  $\square$

It follows from Corollary 2.5 that if  $n \in \{1, 2, 4, 8\}$  and  $n \mid m$ , then there exists a collection of isometric copies of  $\mathbb{S}^{n-1}$  that partition  $\mathbb{S}^{m-1}$ . Many of these cases follow also from the Hopf fibrations:

$$\begin{aligned} S^1 &\hookrightarrow S^{2k-1} \hookrightarrow \mathbb{C}\mathbb{P}^k, \\ S^3 &\hookrightarrow S^{4k-1} \hookrightarrow \mathbb{H}\mathbb{P}^k, \\ S^7 &\hookrightarrow S^{8k-1} \hookrightarrow \mathbb{O}\mathbb{P}^k \quad (k < 3), \end{aligned}$$

where  $\mathbb{C}\mathbb{P}^k, \mathbb{H}\mathbb{P}^k$ , and  $\mathbb{O}\mathbb{P}^k$  are the  $k$ -dimensional complex, quaternionic, and octonionic projective spaces respectively.

**3. The infinite dimensional case.** Next, we consider the infinite-dimensional case. By the results of [13, 14], given an infinite-dimensional separable real Hilbert space, there exist orthogonal linear operators  $U_i : H \rightarrow H$ ,  $i \geq 1$ , such that

$$U_i^2 = -I \quad \text{and} \quad U_i U_j = -U_j U_i,$$

for all distinct  $i, j \geq 1$ . In particular, if  $\{x_i\}_{i=1}^\infty$  is a sequence of real numbers such that  $0 < \sum_{i=1}^\infty x_i^2 < \infty$ , then  $U = \sum_{i=1}^\infty x_i U_i$  is invertible since

$$U^2 = \sum_{i=1}^\infty x_i^2 U_i^2 + \sum_{i \neq j} x_i x_j (U_i U_j + U_j U_i) = - \left( \sum_{i=1}^\infty x_i^2 \right) I.$$

One can then modify the proof of Lemma 2.4 to show that every separable infinite dimensional Hilbert space can be partitioned by countable-dimensional subspaces. However, we use the theorem below to show the existence of projective partitions in all infinite dimensions.

**THEOREM 3.1.** *Let  $V$  be an infinite-dimensional real vector space. Then there exists a set  $\Lambda$  of linear maps  $V \rightarrow V$  such that  $|\Lambda| = |V|$  and*

$$\sum_{i=1}^n x_i T_i$$

*is an isomorphism of  $V$  for all  $n \geq 1$ , all vectors  $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ , and all distinct elements  $T_1, \dots, T_n \in \Lambda$ .*

*Proof.* Let  $Q$  be a positive definite quadratic form on  $V$ , and let  $W = Cl(V, Q)$ , the Clifford algebra generated by  $V$  and  $Q$ . In other words,  $W$  is the free algebra generated by  $V$  subject to the conditions  $v^2 = Q(v)1_W$ , where  $1_W$  is the multiplicative identity. Since in each representation of an element of  $W$  there are only a finite number of elements of  $V$ , the sets  $V$  and  $W$  have the same cardinality, hence they are isomorphic. Thus, it is sufficient to prove the claim of the theorem for  $W$  instead of  $V$ . Let  $B$  be a Hamel basis of  $V$ . We let  $\Lambda = \{T_v : v \in B\}$ , where for each  $v \in B$ , we define  $T_v : W \rightarrow W$  by setting  $T_v(w) = vw$  (the algebra multiplication in  $W$ ). Each  $T_v$  is a linear map. Moreover, if  $(x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ , then

$$T = \sum_{i=1}^n x_i T_{v_i}$$

is invertible, where  $v_i \in B$  are distinct,  $1 \leq i \leq n$ . To see this, note that the equation  $Tw = u$  is equivalent to

$$\left( \sum_{i=1}^n x_i v_i \right) w = u,$$

which has a unique solution, since  $v = \sum_{i=1}^n x_i v_i$  is invertible in  $W$  with inverse  $v^{-1} = Q(v)^{-1}v$ .  $\square$

Now, we are ready to prove the main result of this article.

**THEOREM 3.2.** *Let  $V$  be an infinite-dimensional real vector space and  $W$  be a real vector space such that  $|W| \leq |V|$ . Then  $W$  projectively partitions  $V$ .*

*Proof.* The case where  $W$  is finite-dimensional follows from Theorem 2.2. Thus, suppose  $W$  is infinite-dimensional. By Theorem 3.1, there exists a collection  $\{T_i : i \in J\}$  of linear maps such that  $|J| = |W|$  and every nontrivial linear combination of the maps in the collection is invertible. Let  $B$  be a Hamel basis of  $V$ . We write  $B = \bigcup_{\alpha \in J} B_\alpha$ , where each  $B_\alpha = \{e_i^\alpha : i \in I\}$  has the same cardinality as  $W$  and the sets  $B_\alpha$ ,

$\alpha \in J$ , are mutually disjoint. Let  $V_\alpha$  be the subspace spanned by  $B_\alpha$ . Then  $V = \bigoplus_{\alpha \in J} V_\alpha$ , and each  $V_\alpha$  is a subspace of  $V$  isomorphic to  $W$ . For each  $\alpha \in J$ , let  $P_\alpha : V \rightarrow V_\alpha$  be the projection onto  $V_\alpha$ ; moreover, choose an isomorphism  $\theta_\alpha : V_\alpha \rightarrow W$ .

Choose an order  $<$  on  $J$  such that  $(J, <)$  is a well-ordered set. For each  $\alpha \in J$ , let  $U_\alpha = \bigoplus_{\beta > \alpha} V_\beta$ . For  $\alpha \in J$  and  $z \in U_\alpha$ , let  $X_{\alpha,z}$  be the subspace spanned by the vectors

$$e_i^\alpha + \sum_{\beta > \alpha} \theta_\beta^{-1} \circ T_i \circ \theta_\beta(P_\beta(z)), \quad i \in I. \tag{3.1}$$

For each  $z \in V$ , there are only finitely many  $\beta \in J$  such that  $P_\beta(z) \neq 0$ , and so the summation in equation (3.1) is finite. We show that the collection  $\{X_{\alpha,z} : \alpha \in J, z \in U_\alpha\}$  is a collection of linear subspaces isomorphic to  $W$  that altogether cover  $V$  and mutually intersect only at the origin. The vectors in (3.1) are linearly independent (since for example their projections on  $V_\alpha$  are linearly independent). Therefore, each  $X_{\alpha,z}$  has a Hamel basis whose cardinality is  $|I| = |W|$ , and so it is isomorphic to  $W$ .

Next, we show that for every nonzero  $v \in V$ , there exists  $\alpha \in J$  and  $z \in U_\alpha$  such that  $v \in X_{\alpha,z}$ . Let  $\alpha$  be the least element of  $J$  such that  $P_\alpha(v) \neq 0$  and write

$$v = P_\alpha(v) + \sum_{\beta > \alpha} P_\beta(v) = \sum_{i \in I} c_i e_i^\alpha + \sum_{\beta > \alpha} P_\beta(v),$$

where all but a positive number of  $c_i$ 's are zero. Therefore, for each  $\beta > \alpha$ , the map  $S_\beta = \sum_{i \in I} c_i \theta_\beta^{-1} T_i \theta_\beta$  on  $V_\beta$  is invertible. It follows that there exists a unique vector  $z_\beta \in V_\beta$  such that  $S_\beta(z_\beta) = P_\beta(v)$ . Let  $z = \sum_{\beta > \alpha} z_\beta$  (since there are only finitely many  $\beta > \alpha$  with  $P_\beta(v) \neq 0$ , this is a finite sum). It follows that

$$\begin{aligned} v &= \sum_{i \in I} c_i e_i^\alpha + \sum_{\beta > \alpha} S_\beta(z_\beta) \\ &= \sum_{i \in I} c_i e_i^\alpha + \sum_{\beta > \alpha} \sum_{i \in I} c_i \theta_\beta^{-1} T_i \theta_\beta(z_\beta) \\ &= \sum_{i \in I} c_i e_i^\alpha + \sum_{i \in I} c_i \sum_{\beta > \alpha} \theta_\beta^{-1} T_i \theta_\beta(z_\beta) \\ &= \sum_{i \in I} c_i \left( e_i^\alpha + \sum_{\beta > \alpha} \theta_\beta^{-1} T_i \theta_\beta(P_\beta(z)) \right), \end{aligned}$$

which belongs to  $X_{\alpha,z}$ .

It is left to show that  $X_{\alpha,z} \cap X_{\alpha',z'} = \{0\}$  if  $\alpha \neq \alpha'$  or  $\alpha = \alpha'$  but  $z \neq z'$ . First suppose that  $\alpha \neq \alpha'$ , and without loss of generality, let  $\alpha < \alpha'$ . If  $v \in X_{\alpha,z} \cap X_{\alpha',z'}$  is nonzero, it follows from the definition (3.1) that there exist real numbers  $c_i, d_i, i \in I$ , such that

$$\sum_{i \in I} c_i \left( e_i^\alpha + \sum_{\beta > \alpha} \theta_\beta^{-1} T_i \theta_\beta(P_\beta(z)) \right) = \sum_{i \in I} d_i \left( e_i^{\alpha'} + \sum_{\beta > \alpha'} \theta_\beta^{-1} T_i \theta_\beta(P_\beta(z')) \right). \tag{3.2}$$

By taking  $P_\alpha$  of both sides of (3.2), we have  $\sum_{i \in I} c_i e_i^\alpha = 0$ , which implies that  $c_i = 0$  for all  $i \in I$ , and so  $v = 0$ . Next, suppose that  $\alpha = \alpha'$ . By taking  $P_\beta$  of both sides of (3.2) for each  $\beta > \alpha$ , we have  $\theta_\beta^{-1} T_i \theta_\beta(P_\beta(z)) = \theta_\beta^{-1} T_i \theta_\beta(P_\beta(z'))$ , and so  $P_\beta(z) = P_\beta(z')$  for all  $\beta > \alpha$ , which implies that  $z = z'$ . This completes the proof of Theorem 3.2.  $\square$

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