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# SINGULAR VALUE AND NORM INEQUALITIES ASSOCIATED WITH $2 \times 2$ POSITIVE SEMIDEFINITE BLOCK MATRICES[*]

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**Abstract.** This paper aims to give singular value and norm inequalities associated with $2 \times 2$ positive semidefinite block matrices.

**Key words.** Singular value, Positive semidefinite matrix, Block matrix, Unitarily invariant norm, Inequality.

**AMS subject classifications.** 15A45, 15A60.

**1. Introduction.** Let $\mathbb{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. A norm $\|.\|$ on $\mathbb{M}_n(\mathbb{C})$ is called unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n(\mathbb{C})$ and for all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. For Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we write $A \geq B$ to mean $A - B$ is positive semidefinite, particularly, $A \geq 0$ indicates that $A$ is positive semidefinite. Likewise, if A is positive definite, we write $A > 0$. For $A \in \mathbb{M}_n(\mathbb{C})$, the singular values of $A$, denoted by $s_1(A), s_2(A), \ldots, s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. If $A$ is Hermitian, we label its eigenvalues as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. Several relations between eigenvalues of Hermitian matrices can be obtained by Weyl's monotonicity principle [2], which asserts that if $A, B$ are Hermitian and $A \geq B$, then

$$\lambda_j(A) \geq \lambda_j(B) \quad \text{for } j = 1, \ldots, n.$$

The direct sum of $A$ and $B$, denoted by $A \oplus B$, is defined to be the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Positive semidefinite block matrices play an important role in deriving matrix inequalities. A survey of results about $2 \times 2$ positive semidefinite block matrices and related inequalities can be found in Section 7.7 of [8].

It is evident that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then $A$ and $C$ are positive semidefinite.

A norm inequality due to Horn and Mathias [7] says that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then for all $p > 0$ and for every unitarily invariant norm, we have

$$(1.1) \qquad \qquad \||B|^p\|^2 \leq \|A^p\| \, \|C^p\|.$$

Recently, Audeh and Kittaneh [1] proved that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

(1.2)
$$s_j(B) \leq s_j(A \oplus C) \quad \text{for } j = 1, \ldots, n.$$

Bhatia and Kittaneh [3] proved that if $A$, $B$ are positive semidefinite, then

(1.3)
$$s_j\left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right) \leq s_j\left(\frac{A+B}{2}\right) \quad \text{for } j = 1, \ldots, n.$$

On the other hand, Hirzallah and Kittaneh have proved in [6] that if $A$, $B \in \mathbb{M}_n(\mathbb{C})$, then

(1.4)
$$s_j\left(\frac{A+B}{2}\right) \leq s_j(A \oplus B) \quad \text{for } j = 1, \ldots, n.$$

In this paper, we are interested in finding singular value versions of the inequality $(1.1)$. More singular value inequalities and norm inequalities involving products, sums, and direct sums of matrices based on the positivity of certain block matrices will be considered.

**2. Lemmas.** The following lemmas are essential in our analysis. The first lemma is a consequence of the min-max principle (see [2, p. 75]). The second and third lemmas have been proved in [7]. The fourth lemma follows from the unitary equivalence of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ and $\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}$. In fact, if $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$, then $U$ is unitary and $\begin{bmatrix} A & B \\ B & A \end{bmatrix} = U \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} U^*$ (see [5]). The fifth lemma has been proved in [4]. The sixth lemma is a weak log majorization result that is known as the Weyl majorant theorem (see [2, p. 42]).

LEMMA 2.1. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$s_j(ACB) \leq s_1(A) s_j(C) s_1(B) \quad for \ j = 1, \ldots, n.$$

LEMMA 2.2. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $A > 0$ *and* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$B^* A^{-1} B \leq C.$$

LEMMA 2.3. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$B^* B \leq C^{\frac{1}{2}} U^* A U C^{\frac{1}{2}} \quad for \ some \ unitary \ matrix \ U.$$

LEMMA 2.4. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian. Then*

$$\pm B \leq A \quad if \ and \ only \ if \quad \begin{bmatrix} A & B \\ B & A \end{bmatrix} \geq 0.$$

LEMMA 2.5. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be Hermitian and such that $\pm B \leq A$. Then*

$$(2.5) \qquad s_j(B) \leq s_j(A \oplus A) \quad for \ j = 1, \ldots, n$$

*and*

$$(2.6) \qquad \|B\| \leq \|A\| \quad for \ every \ unitarily \ invariant \ norm.$$

Note that the inequality $(2.5)$ is equivalent to saying that

$$s_j(B) \leq s_{\left[\frac{j+1}{2}\right]}(A) \quad \text{for } j = 1, \ldots, n,$$

where $[k]$ denotes the integer part of $k$. In view of Lemma 2.4, this inequality also follows from the inequality $(1.2)$.

LEMMA 2.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$ have eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ arranged in such a way that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$. Then*

$$\prod_{j=1}^{k} |\lambda_j(A)| \leq \prod_{j=1}^{k} s_j(A) \quad for \ k = 1, \ldots, n-1,$$

*with equality for $k = n$.*

**3. Main results.** In view of the inequality $(1.1)$, one might wonder if the singular value inequality

$$s_j^2(B) \leq s_j(A)s_j(C) \quad \text{for } j = 1, \ldots, n$$

holds true. However, this is refuted by considering $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and $B = A^{\frac{1}{2}}C^{\frac{1}{2}}$. In this case,

$$\begin{bmatrix} A & A^{\frac{1}{2}}C^{\frac{1}{2}} \\ C^{\frac{1}{2}}A^{\frac{1}{2}} & C \end{bmatrix} = \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ C^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ C^{\frac{1}{2}} & 0 \end{bmatrix}^{*} \geq 0$$

and $s_2^2\left(A^{\frac{1}{2}}C^{\frac{1}{2}}\right) \approx 0.17 > 0.14 \approx s_2(A)s_2(C)$.

A weaker inequality follows from the fact that if $\begin{bmatrix} A & B \\ B^{*} & C \end{bmatrix} \geq 0$, then $B = A^{\frac{1}{2}}KC^{\frac{1}{2}}$ for some contraction matrix $K$ (see [2, p. 269]). Using this and Lemma 2.1, we have

$$s_j^2(B) \leq \min\left\{s_j(A)s_1(C), s_j(C)s_1(A)\right\} \quad \text{for } j = 1, \ldots, n.$$

Our next singular value inequality, which involves a commutativity condition, can be stated as follows.

THEOREM 3.1. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ and $AB = BA$. Then*

$$s_j(B) \leq s_j\left(A^{\frac{1}{2}}C^{\frac{1}{2}}\right) \quad for \ j = 1, \ldots, n.$$

*Proof.* First assume that $A > 0$. The general case follows by a continuity argument. Using Lemma 2.2 and the commutativity of $A$ and $B$, we have

$$\begin{aligned}
B^*B &= B^*A^{-1}AB \\
&= A^{\frac{1}{2}}B^*A^{-1}BA^{\frac{1}{2}} \\
&\leq A^{\frac{1}{2}}CA^{\frac{1}{2}}.
\end{aligned}$$

Now, by the Weyl monotonicity principle, we have

$$\begin{aligned}
s_j(B) &= \lambda_j^{\frac{1}{2}}(B^*B) \\
&\leq \lambda_j^{\frac{1}{2}}\left(A^{\frac{1}{2}}CA^{\frac{1}{2}}\right) \\
&= \lambda_j^{\frac{1}{2}}\left((A^{\frac{1}{2}}C^{\frac{1}{2}})(A^{\frac{1}{2}}C^{\frac{1}{2}})^*\right) \\
&= s_j\left(A^{\frac{1}{2}}C^{\frac{1}{2}}\right) \quad \text{for } j = 1, \ldots, n. \quad \square
\end{aligned}$$

If $AB = BA$, the singular value inequality in Theorem 3.1 is a refinement of the inequality (1.2). By the inequalities (1.3) and (1.4), we have

$$s_j(B) \leq s_j\left(A^{\frac{1}{2}}C^{\frac{1}{2}}\right) \leq \frac{1}{2}s_j(A+C) \leq s_j(A \oplus C) \quad \text{for } j = 1, \ldots, n.$$

Theorem 3.1 is not true if the hypothesis of commutativity of $A$ and $B$ is omitted. To see this, let $A = C = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$. Then $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, but $s_2(B) \approx 0.70 > 0.30 \approx s_2(A^{\frac{1}{2}}C^{\frac{1}{2}})$.

THEOREM 3.2. *Let $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then*

(3.7) $$X^*AX + Y^*CY \geq \pm(X^*BY + Y^*B^*X).$$

*Proof.* Since

$$\begin{bmatrix} X^*AX + Y^*B^*X + X^*BY + Y^*CY & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix}^* \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} X & 0 \\ Y & 0 \end{bmatrix} \geq 0$$

and

$$\begin{bmatrix} X^*AX - Y^*B^*X - X^*BY + Y^*CY & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ -Y & 0 \end{bmatrix}^* \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} X & 0 \\ -Y & 0 \end{bmatrix} \geq 0,$$

it follows that $X^*AX + Y^*B^*X + X^*BY + Y^*CY \geq 0$ and $X^*AX - Y^*B^*X - X^*BY + Y^*CY \geq 0$. Thus,

$$X^*AX + Y^*CY \geq \pm(X^*BY + Y^*B^*X). \qquad \square$$

Using Lemma 2.5 and Theorem 3.2, we have the following singular value and norm inequalities involving sums and direct sums of matrices.

COROLLARY 3.3. *Let* $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$(3.8) \qquad s_j(X^*BY + Y^*B^*X) \leq s_j((X^*AX + Y^*CY) \oplus (X^*AX + Y^*CY)) \ \text{for } j = 1, \dots, n$$

*and*

$$(3.9) \qquad \|X^*BY + Y^*B^*X\| \leq \|X^*AX + Y^*CY\| \ \text{for every unitarily invariant norm.}$$

Letting $X = Y = I$ in Corollary 3.3, we have the following result.

COROLLARY 3.4. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$(3.10) \qquad s_j(B + B^*) \leq s_j((A + C) \oplus (A + C)) \ \text{for } j = 1, \dots, n$$

*and*

$$(3.11) \qquad \|B + B^*\| \leq \|A + C\| \ \text{for every unitarily invariant norm.}$$

The inequality (3.11) has been recently obtained in [9] using a different argument.

If $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then $\begin{bmatrix} C & B^* \\ B & A \end{bmatrix} \geq 0$, and so $\begin{bmatrix} A+C & B+B^* \\ B^*+B & A+C \end{bmatrix} \geq 0$. Thus, applying Theorem 3.1, we have the following result.

COROLLARY 3.5. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, $AB = BA$, *and* $BC = CB$. *Then*

$$s_j(B + B^*) \leq s_j(A + C) \ \text{for } j = 1, \dots, n.$$

As an application of Corollary 3.5, we have the following singular value inequality for normal matrices.

COROLLARY 3.6. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be normal and such that* $AB = BA$. *Then*

$$s_j(A^*B + B^*A) \leq s_j(A^*A + B^*B) \ \text{for } j = 1, \dots, n.$$

*Proof.* First observe that $\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \geq 0$. Since $AB = BA$, it follows by the Fuglede-Putnam theorem (see [2, p. 235]) that $A^*B$ commutes with $A^*A$ and $B^*B$. Thus, by Corollary 3.5, we have

$$s_j(A^*B + B^*A) \leq s_j(A^*A + B^*B) \ \text{for } j = 1, \dots, n. \qquad \square$$

We remark here that Corollary 3.6 is not true if the hypothesis of commutativity of $A$ and $B$ is omitted. To see this, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $A$ and $B$ are normal, but $s_2(A^*B + B^*A) = 2 > 1 = s_2(A^*A + B^*B)$. Moreover, Corollary 3.6 is not true if the hypothesis of normality of $A$ and $B$ is omitted. To see this, let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AB = BA$, but $s_2(A^*B + B^*A) \approx 0.83 > 0.81 \approx s_2(A^*A + B^*B)$.

In spite of the failure of Theorem 3.1 without the hypothesis of commutativity of $A$ and $B$, using Lemma 2.3 and the Weyl monotonicity principle, one can prove the following related result.

THEOREM 3.7. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$s_j(B) \leq s_j(A^{\frac{1}{2}}UC^{\frac{1}{2}}) \ \text{for some unitary matrix } U \text{ and for } j = 1, \dots, n.$$

Employing Theorem 3.7 and the inequalities (1.3) and (1.4), we can give a different proof of the inequality (1.2) as follows:

$$\begin{aligned} s_j(B) &\leq s_j(A^{\frac{1}{2}}UC^{\frac{1}{2}}) \\ &= s_j\left(\left|A^{\frac{1}{2}}U\right|C^{\frac{1}{2}}\right) \\ &\leq \frac{1}{2}s_j(U^*AU + C) \\ &\leq s_j(U^*AU \oplus C) \\ &= s_j(A \oplus C) \ \text{ for } j = 1, \dots, n. \end{aligned}$$

**4. On the inequality $\pm B \leq A$.** If $A, B \in \mathbb{M}_n(\mathbb{C})$ are Hermitian and if $\pm B \leq A$, then the inequality

(4.12)                              $s_j(B) \leq s_j(A) \ \text{for } j = 1, \dots, n,$

which is stronger than the inequalities (2.5) and (2.6), need not be true. To see this, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\pm B \leq A$, but $s_2(B) = 1 > 0.70 \approx s_2(A)$.

In spite of the failure of the inequality (4.12), the inequality (2.6) can be strengthened to the following weak log majorization relation, which can also be concluded from the inequality (2.4) in [7].

THEOREM 4.1. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian and such that* $\pm B \leq A$. *Then*

$$\prod_{j=1}^{k} s_j(B) \leq \prod_{j=1}^{k} s_j(A) \ \text{ for } k = 1, \dots, n.$$

*Proof.* By Lemma 2.4, $\begin{bmatrix} A & B \\ B & A \end{bmatrix} \geq 0$. Thus, $B = A^{\frac{1}{2}}KA^{\frac{1}{2}}$ for some contraction matrix $K$, and so for

$k = 1, \ldots, n$, we have

$$\prod_{j=1}^{k} s_j(B) = \prod_{j=1}^{k} |\lambda_j(B)|$$

$$= \prod_{j=1}^{k} \left| \lambda_j \left( A^{\frac{1}{2}} K A^{\frac{1}{2}} \right) \right|$$

$$= \prod_{j=1}^{k} |\lambda_j(AK)|$$

$$\leq \prod_{j=1}^{k} s_j(AK) \qquad \text{(by Lemma 2.6)}$$

$$\leq \prod_{j=1}^{k} s_j(A) s_1(K) \qquad \text{(by Lemma 2.1)}$$

$$\leq \prod_{j=1}^{k} s_j(A). \qquad \square$$

As an application of Theorem 4.1, in view of the inequality (3.7), we have the following result, which is stronger than the inequality (3.9).

COROLLARY 4.2. *Let* $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$\prod_{j=1}^{k} s_j(X^*BY + Y^*B^*X) \leq \prod_{j=1}^{k} s_j(X^*AX + Y^*CY) \ \text{for } k = 1, \ldots, n.$$

Letting $X = Y = I$ in Corollary 4.2, we have the following corollary (see [9]), which is stronger than the inequality (3.11).

COROLLARY 4.3. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$\prod_{j=1}^{k} s_j(B + B^*) \leq \prod_{j=1}^{k} s_j(A + C) \ \text{for } k = 1, \ldots, n.$$

We conclude the paper by observing that the inequalities (2.5) and (3.10) are equivalent.

THEOREM 4.4. *The following statements are equivalent.*

(*i*) *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian and such that* $\pm B \leq A$. *Then*

$$s_j(B) \leq s_j(A \oplus A) \ \text{for } j = 1, \ldots, n.$$

(*ii*) *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0.$ *Then*

$$s_j(B + B^*) \leq s_j((A + C) \oplus (A + C)) \ \text{for } j = 1, \ldots, n.$$

*Proof.* (i) $\Longrightarrow$ (ii). Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then it follows by the inequality (3.7) that $\pm(B + B^*) \leq A + C$.

Thus, by (i), we have

$$s_j(B + B^*) \leq s_j((A + C) \oplus (A + C)) \ \text{ for } j = 1, \ldots, n.$$

(ii)$\Longrightarrow$(i). Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be Hermitian and such that $\pm B \leq A$. Then, by Lemma 2.4,

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \geq 0.$$

Thus, by (ii), we have

$$s_j(2B) \leq s_j(2A \oplus 2A),$$

and so

$$s_j(B) \leq s_j(A \oplus A) \ \text{ for } j = 1, \ldots, n. \quad \square$$

In a similar fashion, one can prove the following theorem, which asserts that the inequalities (2.6) and (3.11) are equivalent.

THEOREM 4.5. *The following statements are equivalent.*

(*i*) *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be Hermitian and such that* $\pm B \leq A$. *Then*

$$\|B\| \leq \|A\| \ \text{ for every unitarily invariant norm.}$$

(*ii*) *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *be such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0.$ *Then*

$$\|B + B^*\| \leq \|A + C\| \ \text{ for every unitarily invariant norm.}$$

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