Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

SELF-INTERLACING POLYNOMIALS II: MATRICES WITH SELF-INTERLACING SPECTRUM*

MIKHAIL TYAGLOV†

Abstract. An $n\times n$ matrix is said to have a self-interlacing spectrum if its eigenvalues λ_k, $k=1,\dots,n$, are distributed as follows:

$$\lambda_1>-\lambda_2>\lambda_3>\cdots>(-1)^{n-1}\lambda_n>0.$$

A method for constructing sign definite matrices with self-interlacing spectrum from totally nonnegative ones is presented. This method is applied to bidiagonal and tridiagonal matrices. In particular, a result by O. Holtz on the spectrum of real symmetric anti-bidiagonal matrices with positive nonzero entries is generalized.

Key words. Self-interlacing polynomials, Totally nonnegative matrices, Tridiagonal matrices, Anti-Bidiagonal matrices, Oscillatory matrices.

AMS subject classifications. 15A18, 15B05, 12D10, 15B35, 15B48.

1. Introduction. In [7], there were introduced the so-called self-interlacing polynomials. A polynomial $p(z)$ is called self-interlacing if all its roots are real, simple and interlacing the roots of the polynomial $p(-z)$. It is easy to see that if λ_k, $k=1,\dots,n$, are the roots of a self-interlacing polynomial, then they are distributed as follows

$$\lambda_1>-\lambda_2>\lambda_3>\cdots>(-1)^{n-1}\lambda_n>0, \tag{1.1}$$

or

$$-\lambda_1>\lambda_2>-\lambda_3>\cdots>(-1)^{n}\lambda_n>0. \tag{1.2}$$

The polynomials whose roots are distributed as in (1.1) (resp., in (1.2)) are called self-interlacing of kind I (resp., of kind II). It is clear that a polynomial $p(z)$ is self-interlacing of kind I if and only if the polynomial $p(-z)$ is self-interlacing of kind II. Thus, it is enough to study self-interlacing polynomials of kind I, since all the results for self-interlacing polynomials of kind II will be obtained automatically.

DEFINITION 1.1. An $n\times n$ matrix is said to possess a *self-interlacing spectrum* if its eigenvalues λ_k, $k=1,\dots,n$, are real, simple, and distributed as in (1.1).

In [7], it was proved that a polynomial

$$p(z)=a_0z^n+a_1z^{n-1}+a_2z^{n-2}+a_3z^{n-3}+\cdots+a_n=\sum_{k=0}^{n}a_kz^{n-k} \tag{1.3}$$

is self-interlacing of kind I if, and only if, the polynomial

$$p(z)=a_0z^n-a_1z^{n-1}-a_2z^{n-2}+a_3z^{n-3}+\cdots+(-1)^{\frac{n(n+1)}{2}}a_n=\sum_{k=0}^{n}(-1)^{\frac{k(k+1)}{2}}a_kz^{n-k} \tag{1.4}$$

*Received by the editors on December 12, 2016. Accepted for publication on : February 2, 2017. Handling Editor: Raphael Lowey.

†School of Mathematical Sciences, Shanghai Jiao Tong University, China, and Faculty of Mathematics, Far East Federal University, Russia (tyaglov@sjtu.edu.cn). The author is Shanghai Oriental Scholar whose work was supported by Russian Science Foundation, grant no. 14-11-00022.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

is Hurwitz stable, that is, has all its roots in the open left half-plane of the complex plane. Thus, there is a one-to-one correspondence between the self-interlacing polynomials of kind *I* and the Hurwitz stable polynomials. Now since the set of all polynomials with *positive* roots is isomorphic to a subset of the set of Hurwitz stable polynomials (to the set of all polynomials with negative roots), we conclude that this set is isomorphic to a subset of the set of all self-interlacing polynomials of kind *I*. Consequently, it is worth to relate some classes of positive definite and totally nonnegative matrices to matrices with self-interlacing spectrum.

In this work, we consider some classes of *real* matrices with self-interlacing spectrum and develop a method of constructing such kind of matrices from a given totally positive matrix. Namely, we show how and under what conditions it is possible to relate a totally nonnegative matrix with a matrix with self-interlacing spectrum (Theorem 2.9). We apply this theorem to totally nonnegative bidiagonal and tridiagonal matrices (generalizing a result by O. Holtz) and explain how our technique can be extended for other classes of structured matrices. A part of this work was appeared first in the technical report [6].

2. Matrices with self-interlacing spectrum. At first, we recall some definitions and statements from the book [3, Chapter V].

DEFINITION 2.1 ([3]). A square matrix $A = \|a_{ij}\|_1^n$ is called *sign definite* of class n if for any $k \leq n$, all the non-zero minors of order k have the same sign ε_k. The sequence $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ is called the *signature sequence* of the matrix A.

A sign definite matrix of class n is called *strictly sign definite* of class n if all its minors are different from zero.

REMARK 2.2. Let us recall that sign definite matrices of class n are also called $n \times n$ sign regular matrices and that strictly sign definite matrices of class n are also called $n \times n$ strictly sign regular matrices (see [1, 5]).

DEFINITION 2.3 ([3]). A square sign definite matrix $A = \|a_{ij}\|_1^n$ of class n is called the *matrix of class n^+* if some its power is a strictly sign definite matrix of class n.

Note that a sign definite (strictly sign definite) matrix of class n with the signature sequence $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = 1$ is totally nonnegative (*strictly totally positive*). Also a sign definite matrix of class n^+ with the signature sequence of the form $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = 1$ is an *oscillatory* (oscillation) matrix (see [3]), that is, a totally nonnegative matrix whose certain power is strictly totally positive. It is clear from the Binet-Cauchy formula [2] that the square of a sign definite matrix is totally nonnegative.

In [3, Chapter V], the following theorem was established.

THEOREM 2.4. *Let the matrix $A = \|a_{ij}\|_1^n$ be totally nonnegative. Then the matrices $B = \|a_{n-i+1,j}\|_1^n$ and $C = \|a_{i,n-j+1}\|_1^n$ are sign definite of class n. Moreover, the signature sequence of the matrices B and C has the form:*

$$\varepsilon_k = (-1)^{\frac{k(k-1)}{2}}, \quad k = 1, 2, \dots, n. \tag{2.1}$$

Note that the matrices B and C can be represented as follows

$$B = JA \quad \text{and} \quad C = AJ,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

where

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.2}$$

It is easy to see that the matrix J is sign definite of class n (but not of class n^+) with the signature sequence of the form (2.1). So by the Binet-Cauchy formula we obtain the following statement.

THEOREM 2.5. *The matrix $A = \|a_{ij}\|_1^n$ is totally nonnegative if and only if the matrix JA (or the matrix AJ) is sign definite of class n with the signature sequence (2.1) where the matrix J is defined in (2.2).*

Obviously, the converse statement is also true.

THEOREM 2.6. *The matrix $A = \|a_{ij}\|_1^n$ is a sign definite matrix of class n with the signature sequence (2.1) if and only if the matrix JA (or the matrix AJ) is totally nonnegative.*

In the sequel, we need the following two theorems established in the book [3, Chapter V].

THEOREM 2.7. *Let the matrix $A = \|a_{ij}\|_1^n$ be a sign definite of class n^+ with the signature sequence ε_k, $k = 1,2,\dots,n$. Then all the eigenvalues λ_k, $k = 1,2,\dots,n$, of the matrix A are nonzero real and simple, and if*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0, \tag{2.3}$$

then

$$\operatorname{sign}\lambda_k = \frac{\varepsilon_k}{\varepsilon_{k-1}}, \quad k = 1,2,\dots,n, \ \ \varepsilon_0 = 1. \tag{2.4}$$

THEOREM 2.8. *A totally nonnegative matrix $A = \|a_{ij}\|_1^n$ is oscillatory if, and only if, A is nonsingular, and the following inequalities hold*

$$a_{j,j+1} > 0 \quad \textit{and} \quad a_{j+1,j} > 0, \qquad j = 1,2,\dots,n-1.$$

Now we are in a position to complement Theorem 2.4.

THEOREM 2.9. *Let all the entries of a nonsingular matrix $A = \|a_{ij}\|_1^n$ be nonnegative, and suppose that for each i, $i = 1,2,\dots,n-1$, there exist numbers r_1 and r_2, $1 \le r_1, r_2 \le n$, depending on i, such that*

$$a_{n-i,r_1} \cdot a_{n+1-r_1,i} > 0, \quad a_{n+1-i,r_2} \cdot a_{n+1-r_2,i+1} > 0 \tag{2.5}$$

$$(\textit{or} \quad a_{i,n+1-r_1} \cdot a_{r_1,n-i} > 0, \quad a_{i+1,n+1-r_2} \cdot a_{r_2,n+1-i} > 0). \tag{2.6}$$

The matrix A is totally nonnegative if, and only if, the matrix $B = JA = \|a_{n-i+1,j}\|_1^n$ (or, respectively, the matrix $C = AJ = \|a_{i,n-j+1}\|_1^n$) is sign definite of class n^+ with the signature sequence defined in (2.1). Moreover, the matrix B (or, respectively, the matrix C) possesses a self-interlacing spectrum.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

Proof. We prove the theorem in the case when the condition (2.5) holds. The case of the condition (2.6) can be established analogously.

Let A be a nonsingular totally nonnegative matrix, and let the condition (2.5) hold. From Theorem 2.5 it follows that the matrix $B = JA$ is sign definite of class n with the signature sequence (2.1). In order for the matrix to be sign definite of class n^+ it is necessary and sufficient that a certain power of this matrix B be strictly sign definite of class n. Since the entries of the matrix J have the form

$$(J)_{ij} = \begin{cases} 1, & i = n+1-j, \\ 0, & i \neq n+1-j, \end{cases}$$

the entries of the matrix B can be represented as follows:

$$b_{ij} = \sum_{k=1}^{n} (J)_{ik} a_{kj} = a_{n+1-i,j}.$$

Consider the totally nonnegative matrix B^2. Its entries have the form

$$(B^2)_{ij} = \sum_{k=1}^{n} b_{ik} b_{kj} = \sum_{k=1}^{n} a_{n+1-i,k} a_{n+1-k,j}.$$

From these formulæ and from (2.5) it follows that all the entries of the matrix B^2 above and under the main diagonal are positive, that is, $(B^2)_{i,i+1} > 0$ and $(B^2)_{i+1,i} > 0$, $i = 1,2,\dots,n-1$. By Theorem 2.8, B^2 is an oscillatory matrix. According to the definition of oscillatory matrices, a certain power of B^2 is strictly totally positive. Thus, we proved that a certain power of the matrix B is strictly sign definite, so B is a sign definite matrix of class n^+ with the signature sequence (2.1) according to Theorem 2.4. By Theorem 2.7, all eigenvalues of the matrix B are nonzero real and simple. Moreover, if we enumerate the eigenvalues in order of decreasing absolute values as in (2.3), then from (2.4) and (2.1) we obtain that the spectrum of B is of the form (1.1).

The converse assertion of the theorem follows from Theorem 2.5. □

REMARK 2.10. If the matrix $-A$ is totally nonnegative and the conditions (2.5) (or the conditions (2.6)) hold, then the matrix $B = JA$ (or, respectively, $C = AJ$) has a spectrum of the form (1.2).

REMARK 2.11. One can obtain other types of totally nonnegative matrices which result in matrices with self-interlacing spectra after multiplication by the matrix J. To do this we need to change the conditions (2.5)–(2.6) by other ones such that, for instance, the matrix B^4 (or B^6, or B^8 etc) becomes oscillatory.

Theorem 2.9 implies the following corollary.

COROLLARY 2.12. *A nonsingular matrix A with positive entries is totally nonnegative if, and only if, the matrix $B = JA$ (or the matrix $C = AJ$) is sign definite of class n^+ with the signature sequence (2.1). Moreover, the matrix B possesses a self-interlacing spectrum.*

Consider a partial case of conditions (2.5). Suppose that all diagonal entries of the matrix A are positive: $a_{jj} > 0$, $j = 1,2,\dots,n$. It is easy to see that in this case, the conditions (2.5) hold if $a_{j,j+1} > 0$, $j = 1,2,\dots,n-1$. If all remaining entries of the matrix A are nonnegative, then by Theorem 2.9 the matrices $B = JA$ and $C = AJ$ have self-interlacing spectra whenever A is totally nonnegative.

If all other entries of the matrix A (that is, all entries except a_{jj} and $a_{j,j+1}$, which are positive) equal zero, then A is a bidiagonal matrix with positive entries on and above the main diagonal. Clearly, A is totally nonnegative.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

Then by Theorem 2.9 the matrix $B = JA$ possesses a self-interlacing spectrum. Note that the matrix B in this case is anti-bidiagonal with positive entries, that is, it has the form

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & b_{n-3} & b_{n-2} \\ 0 & 0 & 0 & \cdots & b_{n-4} & b_{n-4} & 0 \\ & & & \iddots & & & \\ \vdots & \vdots & \vdots & a & \vdots & \vdots & \vdots \\ & & \iddots & & & & \\ 0 & 0 & c_{n-5} & \cdots & 0 & 0 & 0 \\ 0 & c_{n-4} & c_{n-4} & \cdots & 0 & 0 & 0 \\ c_{n-1} & c_{n-3} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad b_j > 0,\ c_j > 0,\ a > 0, \tag{2.7}$$

where all the entries b_j, $j = 1,2,\dots,n-1$, lie above the main diagonal, all the entries c_j, $j = 1,2,\dots,n-1$, lie under the main diagonal and the only entry on the main diagonal is a.

Thus, we proved the following fact.

THEOREM 2.13. *Any anti-bidiagonal matrix with positive entries as in (2.7) possesses a self-interlacing spectrum.*

In [4], the same theorem was established under the additional assumption that $b_j = c_j$, $j = 1,\dots,n-1$. In the same manner as in the work [4], it also can be shown that the spectrum of the matrix (2.7) coincides with the spectrum of the following tridiagonal matrix

$$T_n = \begin{bmatrix} a & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & 0 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1} \\ 0 & 0 & 0 & c\dots & c_{n-1} & 0 \end{bmatrix}.$$

Indeed, let us denote the matrix B defined in (2.7) by B_n in order to stress the dependence of the matrix B on n, and let us denote by $p_n(z)$ the monic characteristic polynomial of the matrix B_n:

$$p_n(z) = \det(zI_n - B_n),$$

where I_n is the $n \times n$ identity matrix. Then it is easy to see that the characteristic polynomials of matrices B_n with different sizes satisfy the following three-term recurrence relations:

$$p_n(z) = zp_{n-1}(z) - b_{n-1}c_{n-1}p_{n-2}(z), \quad n \geq 2,$$

$$p_0(z) = 1, \quad p_1(z) = z - a.$$

At the same time, the characteristic polynomials $q_n(z)$ of the matrices T_n with different sizes satisfy the same recurrence relations with the same initial polynomials: $q_0(z) = 1$ and $q_1(z) = z - a$. Therefore, $p_n(z) = q_n(z)$ for any $n \in \mathbb{N}$, and the spectra of matrices B_n and T_n coincide for each n.

It is well-known [3] that the spectrum of the matrix T_n does not depend on the entries b_j and c_j separately. It depends on products b_jc_j, $j = 1,2,\dots,n-1$. So in order for the matrices (2.7) and T_n to have a self-interlacing

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

spectra, it is sufficient the inequalities $a > 0$ and $b_jc_j > 0$, $j = 1,2,\dots,n-1$, to hold. However, the inverse spectral problem cannot be solved here uniquely unless $b_j = c_j$, $j = 1,\dots,n-1$ (the case of the work [4]).

Finally, consider a tridiagonal matrix

$$M_J = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{bmatrix}, \tag{2.8}$$

where $a_k, b_k, c_k \in \mathbb{R}$ and $c_kb_k \neq 0$. In [3], the following fact was proved.

THEOREM 2.14. *Let the matrix M_J defined in (2.8) be nonnegative. Then M_J is oscillatory if, and only if, all the entries b_k and c_k are positive and all the leading principal minors of M_J are also positive:*

$$\det \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{k-1} & b_{k-1} \\ 0 & 0 & 0 & \cdots & c_{k-1} & a_k \end{bmatrix} > 0, \quad k = 1,\dots,n. \tag{2.9}$$

This theorem together with Theorem 2.9 implies the following statement.

THEOREM 2.15. *The anti-tridiagonal matrix*

$$A_J = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & b_1 & a_1 \\ 0 & 0 & 0 & \cdots & b_2 & a_2 & c_1 \\ 0 & 0 & 0 & \cdots & a_3 & c_2 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \\ 0 & b_{n-2} & a_{n-2} & \cdots & 0 & 0 & 0 \\ b_{n-1} & a_{n-1} & c_{n-2} & \cdots & 0 & 0 & 0 \\ a_n & c_{n-1} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

where $a_j, b_j, c_j > 0$ for $j = 1,2,\dots,n-1$, is sign definite of class n^+ and possesses a self-interlacing spectrum if, and only if, the following inequalities hold:

$$(-1)^{\frac{k(k-1)}{2}} \det \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & b_1 & a_1 \\ 0 & 0 & 0 & \cdots & b_2 & a_2 & c_1 \\ 0 & 0 & 0 & \cdots & a_3 & c_2 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \\ 0 & b_{k-2} & a_{k-2} & \cdots & 0 & 0 & 0 \\ b_{k-1} & a_{k-1} & c_{k-2} & \cdots & 0 & 0 & 0 \\ a_k & c_{k-1} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} > 0, \tag{2.10}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 51-57, February 2017.

for $k = 1, 2, \dots, n$.

Proof. If the matrix A_J is sign definite of class n^+ and has a self-interlacing spectrum of the form (1.1), then according to Theorem 2.7, the signs of its nonzero minors can be calculated by the formula (2.1). This implies the inequalities (2.10).

Conversely, let the inequalities (2.10) hold. Then we have that the inequalities (2.9) hold for the matrix $M_J = JA_J$. By Theorem 2.14, the matrix M_J is oscillatory and, in particular, totally nonnegative [3]. Now notice that $(M_J)_{ii} > 0$, $i = 1, \dots, n$, and $(M_J)_{k,k+1} > 0$, $k = 1, \dots, n-1$, so M_J satisfies the condition (2.5) of Theorem 2.9. Therefore, the matrix A_J is sign definite of class n^+ and has a self-interlacing spectrum of the form (1.1). $\square$

3. Conclusion. Theorem 2.9 and Corollary 2.12 provide a method of constructing matrices with self-interlacing spectrum from given totally nonnegative and oscillatory matrices. We gave two examples of application of this method for bidiagonal and tridiagonal matrices. One can use Remark 2.11 to generalize Theorem 2.9 and to apply the generalization for other types of structured matrices.

Acknowledgment. The author thanks Prof. Raphael Loewy and an anonymous referee for helpful comments and suggestions.

REFERENCES

[1] T. Ando. Totally positive matrices. *Linear Algebra Appl.*, 90:165–219, 1987.
[2] F.R. Gantmacher. *The Theory of Matrices*, Vol. I, AMS Chelsea Publishing, Providence, RI, 1959 (translated from the Russian by K.A. Hirsch).
[3] F.R. Gantmacher and M.G. Krein. *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, revised edition. AMS Chelsea Publishing, Providence, RI, 2002 (translation based on the 1941 Russian original, edited by Alex Eremenko).
[4] O. Holtz. The inverse eigenvalue problem for symmetric anti-bidiagonal matrices. *Linear Algebra Appl.*, 408:268–274, 2005.
[5] S. Karlin. *Total Positivity*, Vol. I. Stanford University Press, California, 1968.
[6] M. Tyaglov. Generalized Hurwitz polynomials. Technical report, http://arxiv.org/abs/1005.3032, 2010.
[7] M. Tyaglov. Self-interlacing polynomials. Preprint, https://arxiv.org/abs/1611.09156, 2016