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GENERALIZED LEFT AND RIGHT WEYL SPECTRA OF UPPER TRIANGULAR OPERATOR MATRICES*

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Abstract. In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the sets of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is generalized Weyl and generalized left (right) Weyl, are completely described. Furthermore, the following intersections and unions of the generalized left Weyl spectra

$$\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma^g_{lw}(M_C) \quad \text{and} \quad \bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma^g_{lw}(M_C)$$

are also described, and necessary and sufficient conditions which two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ have to satisfy in order for M_C to be a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, are presented.

Key words. Operator matrix, Generalized left(right) Weyl, Spectrum.

AMS subject classifications. 47A10, 47A53, 47A55.

1. Introduction. Let $\mathcal{H},\mathcal{K}$ be infinite dimensional complex separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H},\mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. By $\mathcal{F}(\mathcal{H},\mathcal{K})$ we denote the set of all operators from $\mathcal{B}(\mathcal{H},\mathcal{K})$ with a finite dimensional range. For a given $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. Let $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \operatorname{codim}\mathcal{R}(A)$, and $d(A) = \dim \mathcal{R}(A)^{\perp}$.

If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then A is said to be a upper semi-Fredholm operator. If $\beta(A) < \infty$, then A is called a lower semi-Fredholm operator. A semi-Fredholm operator is one which is either upper semi-Fredholm or lower semi-Fredholm. An operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is called Fredholm if it is both lower semi-Fredholm and upper semi-Fredholm. The subset of $\mathcal{B}(\mathcal{H},\mathcal{K})$ consisting of all Fredholm operators is denoted by $\Phi(\mathcal{H},\mathcal{K})$. By $\Phi_+(\mathcal{H},\mathcal{K})$ ($\Phi_-(\mathcal{H},\mathcal{K})$) we denote the set of all upper (lower) semi-Fredholm operators from $\mathcal{B}(\mathcal{H},\mathcal{K})$.

If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) \leq d(A)$, then A is a generalized left Weyl operator. If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $d(A) \leq n(A)$, then A is a generalized right Weyl operator. Notice that in the cases of generalized left (right) Weyl operators, $n(A)$ and $d(A)$ are allowed to be infinity. An operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a generalized Weyl operator if it is both generalized right Weyl and generalized left Weyl. The set of all generalized Weyl operators from $\mathcal{B}(\mathcal{H},\mathcal{K})$ is denoted by $W^g(\mathcal{H},\mathcal{K})$.

Let $W_{gl}(\mathcal{H},\mathcal{K})$ ($W_{gr}(\mathcal{H},\mathcal{K})$) denote the subset of $\mathcal{B}(\mathcal{H},\mathcal{K})$ consisting of all generalized left (right) Weyl operators. For an operator $C \in \mathcal{B}(\mathcal{H})$, the generalized left (right) Weyl spectrum $\sigma^g_{lw}(C)$ ($\sigma^g_{rw}(C)$) is defined by

$$\sigma^g_{lw}(C)(\sigma^g_{rw}(C)) = \{\lambda \in \mathbb{C} : C - \lambda I \ \text{ is not generalized left (right) Weyl}\}.$$

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The generalized Weyl spectrum is defined by

$$\sigma_w^g(C) = \{\lambda \in \mathbb{C} : C - \lambda I \text{ is not generalized Weyl}\}.$$

In this paper, we address the question for which operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, there exists an operator $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that an upper-triangular operator matrix

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

is generalized left (right) Weyl. There are many papers which consider some types of invertibility, regularity and some other properties of an upper-triangular operator matrix M_C (see [1]–[17] and references therein) as well as various types of spectra of M_C. This paper is a continuation of the work presented in [10], where the sets $\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_w^g(M_C)$ and $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_w^g(M_C)$ are described and some necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl are given, but the set of all such operators C is not described. As a corollary of our main results we obtain a description of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl, and we denote this set by $S_{GW}(A,B)$. The sets $\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{lw}^g(M_C)$ and $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{lw}^g(M_C)$ are described for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ as well as the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl which is denoted by $S_{GLW}(A,B)$. In an analogous way, similar results can be provided for $\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{rw}^g(M_C)$ and $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{rw}^g(M_C)$.

2. Results. In this section, by $\mathcal{H}$, $\mathcal{K}$ we denote complex separable Hilbert spaces. For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, by M_C we denote

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

where $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Evidently, for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, arbitrary $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ can be represented by

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{pmatrix}. \tag{2.1}$$

First, we will state some auxiliary lemmas which will be used in the proof of the main result.

LEMMA 2.1. *If $A \in \mathcal{B}(\mathcal{H})$ and $D \in \mathcal{F}(\mathcal{H})$, then $\mathcal{R}(A+D)$ is closed if and only if $\mathcal{R}(A)$ is closed.*

LEMMA 2.2. *Let $S \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $R \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be given operators.*

(i) If $\mathcal{R}(S)$ is non-closed and $\mathcal{R}(\begin{pmatrix} S & T \end{pmatrix})$ is closed, then $n(\begin{pmatrix} S & T \end{pmatrix}) = \infty$.

(ii) If $\mathcal{R}(S)$ is non-closed and $\mathcal{R}(\begin{pmatrix} S \\ R \end{pmatrix})$ is closed, then $d(\begin{pmatrix} S \\ R \end{pmatrix}) = \infty$.

Proof. (i) Suppose that $\mathcal{R}(S)$ is non-closed, $\mathcal{R}(\begin{pmatrix} S & T \end{pmatrix})$ is closed and $n(\begin{pmatrix} S & T \end{pmatrix}) < \infty$. Then $\begin{pmatrix} S & T \end{pmatrix}$ is a left Fredholm operator which implies that there exists an operator $\begin{pmatrix} X \\ Y \end{pmatrix} : \mathcal{H} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}$ such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} S & T \end{pmatrix} = I + K,$$

for some compact operator $K \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Hence, $XS = I_{\mathcal{H}} + K_1$, for some compact operator $K_1 \in \mathcal{B}(\mathcal{H})$ which implies that S is left Fredholm and so $\mathcal{R}(S)$ is closed, which is a contradiction.

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(ii) The proof follows by taking adjoints in (i). □

In the following theorem, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we present necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is a generalized left Weyl operator, and we completely describe the set of all such $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

THEOREM 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. There exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl if and only if one of the following conditions is satisfied:*

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A)+n(B) \leq d(A)+d(B)$. In this case,

$$S_{GLW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \text{ is given by } (2.1), C_3 \text{ has closed range}, \\ n(A)+n(C_3) \leq d(C_3)+d(B)\Big\}.$$

(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A)=\infty$. In this case,

$$S_{GLW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathcal{R}(B^*)+R(C^*P_{\mathcal{R}(A)^\perp}) \text{ is closed}\Big\}.$$

(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $n(B)=d(A)+d(B)=\infty$. In this case,

$$S_{GLW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathcal{R}(A)+\mathcal{R}(CP_{\mathcal{N}(B)}) \text{ is closed}, \\ d(B)+\operatorname{codim}(\mathcal{R}(A)+\mathcal{R}(CP_{\mathcal{N}(B)}))=\infty\Big\}.$$

(iv) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed and $n(B)=d(A)=\infty$. In this case,

$$S_{GLW}(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathcal{R}(M_C) \text{ is closed}\}.$$

For simplicity, we will divide the statement of this theorem into four propositions and prove each of them separately.

PROPOSITION 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. There exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl if and only if $n(A)+n(B) \leq d(A)+d(B)$. In this case,*

$$S_{GLW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \text{ is given by } (2.1), C_3 \text{ has closed range}, \\ n(A)+n(C_3) \leq d(C_3)+d(B)\Big\}.$$

Proof. If $n(A)+n(B) \leq d(A)+d(B)$, then M_0 is a generalized left Weyl operator. Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is a generalized left Weyl operator and that C is given by (2.1). Then M_C has a matrix representation

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix},$$

where $A_1 : \mathcal{H} \longrightarrow \mathcal{R}(A)$ is right invertible and $B_1 : \mathcal{N}(B)^\perp \longrightarrow \mathcal{K}$ is left invertible. Evidently, there exists invertible $U,V \in \mathcal{B}(\mathcal{H}\oplus\mathcal{K})$ such that

$$UM_CV = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix}. \tag{2.2}$$

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Hence, UM_CV is a generalized left Weyl which implies that

$$n(A_1)+n(C_3) \leq d(C_3)+d(B_1). \tag{2.3}$$

Since,

$$n(A_1)=n(A), \quad n(B)=n(C_3)+\dim \mathcal{N}(C_3)^{\perp},$$

$$d(B_1)=d(B) \quad \text{and} \quad d(A)=d(C_3)+\dim \mathcal{R}(C_3),$$

having in mind that $\dim \mathcal{N}(C_3)^{\perp}=\dim \mathcal{R}(C_3)$ and (2.3), we get

$$n(A)+n(B) \leq d(A)+d(B).$$

To describe the set of all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is a generalized left Weyl, notice that for arbitrary C given by (2.1), there exists invertible $U,V \in \mathcal{B}(\mathcal{H}\oplus\mathcal{K})$ such that UM_CV is given by (2.10). Hence, M_C is a generalized left Weyl if and only if UM_CV is a generalized left Weyl which is equivalent with the fact that $\mathcal{R}(C_3)$ is closed and that (2.3) holds. □

PROPOSITION 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ is closed and $\mathcal{R}(B)$ is non-closed. There exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl if and only if $d(A)=\infty$. In this case,*

$$S_{GLW}(A,B)=\left\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathcal{R}(B^*)+\mathcal{R}(C^*P_{\mathcal{R}(A)^{\perp}}) \text{ is closed}\right\}.$$

Proof. Suppose that $d(A)=\infty$. Then M_{C_0} is a generalized left Weyl operator for C_0 given by

$$C_0=\begin{pmatrix} 0 \\ J \end{pmatrix} : \mathcal{K} \longrightarrow \mathcal{R}(A)\oplus\mathcal{R}(A)^{\perp},$$

where $J:\mathcal{K}\longrightarrow\mathcal{R}(A)^{\perp}$ is unitary. Evidently, M_{C_0} is represented by

$$M_{C_0}=\begin{pmatrix} A_1 & 0 \\ 0 & J \\ 0 & B \end{pmatrix} : \mathcal{H}\oplus\mathcal{K} \longrightarrow \mathcal{R}(A)\oplus\mathcal{R}(A)^{\perp}\oplus\mathcal{K},$$

where $A_1:\mathcal{H}\longrightarrow\mathcal{R}(A)$ is right invertible. Since J is invertible, there exists an invertible operator $U \in \mathcal{B}(\mathcal{H}\oplus\mathcal{K})$ such that

$$UM_{C_0}=\begin{pmatrix} A_1 & 0 \\ 0 & J \\ 0 & 0 \end{pmatrix} : \mathcal{H}\oplus\mathcal{K} \longrightarrow \mathcal{R}(A)\oplus\mathcal{R}(A)^{\perp}\oplus\mathcal{K}.$$

Now, it is clear that UM_{C_0} is a generalized left Weyl operator, and so M_{C_0} is a generalized left Weyl operator.

Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl. Then M_C has a matrix representation

$$M_C=\begin{pmatrix} A_1 & C_1 \\ 0 & C_2 \\ 0 & B \end{pmatrix} : \mathcal{H}\oplus\mathcal{K} \longrightarrow \mathcal{R}(A)\oplus\mathcal{R}(A)^{\perp}\oplus\mathcal{K}, \tag{2.4}$$

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where $A_1 : \mathcal{H} \longrightarrow \mathcal{R}(A)$ is right invertible. Thus, there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$M_C V = \begin{pmatrix} A_1 & 0 \\ 0 & C_2 \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp} \oplus \mathcal{K}. \tag{2.5}$$

Now we will show that $d(A) = \infty$: Indeed, if $d(A) < \infty$, then $\mathcal{R}(C_2^*)$ is finite dimensional. Since $\mathcal{R}(M_C V)$ is closed, we have that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed, which implies that $\mathcal{R}(B^*) + \mathcal{R}(C_2^*)$ is closed. This, together with $\dim \mathcal{R}(C_2^*) < \infty$, implies that $\mathcal{R}(B)$ is closed. This is a contradiction. Hence, $d(A) = \infty$.

In order to describe the set $S_{GLW}(A,B)$, notice that for arbitrary $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, M_C has a form (2.4) and that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that $M_C V$ is given by (2.5). Hence, M_C is generalized left Weyl if and only if C is such that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed and that

$$n(A_1) + n\left(\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)\right) \leq d(A_1) + d\left(\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)\right). \tag{2.6}$$

Notice that by Lemma 2.2, we have that for each $C_2 \in \mathcal{B}(\mathcal{K}, \mathcal{R}(A)^{\perp})$ such that $\mathcal{R}\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right)$ is closed, it follows that $d\left(\begin{pmatrix} C_2 \\ B \end{pmatrix}\right) = \infty$. Thus,

$$S_{GLW}(A,B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^{\perp}}) \text{ is closed} \right\}. \quad \square$$

PROPOSITION 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ is non-closed and $\mathcal{R}(B)$ is closed. There exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is generalized left Weyl if and only if $n(B) = d(A) + d(B) = \infty$. In this case,*

$$S_{GLW}(A,B) = \Big\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A) + R(CP_{\mathcal{N}(B)}) \text{ is closed}, \\ d(B) + \operatorname{codim}(\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)})) = \infty \Big\}.$$

Proof. Suppose that $n(B) = d(A) + d(B) = \infty$. Then there exists a left invertible operator $C_1 : \mathcal{N}(B) \longrightarrow \mathcal{H}$ such that $\mathcal{R}(C_1) = \overline{\mathcal{R}(A)}$. We will prove that M_C is a generalized left Weyl operator for C given by

$$C = \begin{pmatrix} C_1 & 0 \end{pmatrix} : \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{H}.$$

Evidently, M_C is represented by

$$M_C = \begin{pmatrix} A & C_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{H} \oplus \mathcal{K},$$

where $B_1 : \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{K}$ is left invertible and

$$\mathcal{R}(M_C) = (\mathcal{R}(A) + \mathcal{R}(C_1)) \oplus \mathcal{R}(B_1) = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(B).$$

Thus, $\mathcal{R}(M_C)$ is closed and

$$d(M_C) = d(A) + d(B) = \infty,$$

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i.e., M_C is a generalized left Weyl operator.

Conversely, suppose that there exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl. It follows that M_C has a matrix representation

$$M_C = \begin{pmatrix} A & C_1 & C_2 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{H} \oplus \mathcal{K}, \tag{2.7}$$

where $B_1 : \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{K}$ is left invertible and there exists an invertible operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_C = \begin{pmatrix} A & C_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{H} \oplus \mathcal{K}. \tag{2.8}$$

Since UM_C has a closed range, by Lemma 2.1 and the fact that $\mathcal{R}(A)$ is non-closed, we have that $n(B) = \infty$. Also, applying Lemma 2.2, we get that $n(\begin{pmatrix} A & C_1 \end{pmatrix}) = \infty$ which implies that $d(UM_C) = d(B) + d(\begin{pmatrix} A & C_1 \end{pmatrix}) = \infty$. Since $d(\begin{pmatrix} A & C_1 \end{pmatrix}) \le d(A)$, it follows that $d(A) + d(B) = \infty$.

In order to describe the set $S_{GLW}(A,B)$, notice that for arbitrary $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, M_C has a form (2.7) and that there exists an invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that UM_C is given by (2.8). Hence, M_C is generalized left Weyl if and only if C is such that $\mathcal{R}\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right)$ is closed and that

$$n\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) + n(B_1) \le d\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) + d(B_1). \tag{2.9}$$

Notice that if $\mathcal{R}\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right)$ is closed, then by Lemma 2.2, we have that $n\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) = \infty$. Hence, M_C is a generalized left Weyl operator for $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if $\mathcal{R}\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right)$ is closed and $d\left(\begin{pmatrix} A & C_1 \end{pmatrix}\right) + d(B_1) = \infty$. Obviously, $d(B_1) = d(B)$. ☐

PROPOSITION 2.7. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed. There exists $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized left Weyl if and only if $n(B) = d(A) = \infty$. In this case,*

$$S_{GLW}(A,B) = \left\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : R(M_C) \text{ is closed}\right\}.$$

Proof. Since $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are non-closed, by Lemma 2.2, we conclude that if $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is such that $\mathcal{R}(M_C)$ is closed, then $n(M_C) = d(M_C) = \infty$. Hence, M_C is generalized left Weyl if and only if $R(M_C)$ is closed. Now, the proof directly follows by Theorem 2.6 of [4]. ☐

REMARK 1. It is interesting to notice that the condition $d(B) + \operatorname{codim}(\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)})) = \infty$ from Proposition 2.6, appearing also in item (iii) of Theorem 2.3, can be replaced by the condition $d(C_3) + d(B) = \infty$, where C_3 is the block-operator defined by (2.1). So, if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are such that $\mathcal{R}(A)$ is non-closed and $\mathcal{R}(B)$ is closed, then

$$M_C = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \\ \mathcal{K} \end{pmatrix},$$

where $A_1 : \mathcal{H} \longrightarrow \overline{\mathcal{R}(A)}$ is with a dense range and $B_1 : \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{K}$ is left invertible. There exists an invertible $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ such that

$$UM_C = \begin{pmatrix} A_1 & C_1 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^{\perp} \\ \mathcal{K} \end{pmatrix}. \tag{2.10}$$

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Now, it is evident that $\mathcal{R}(M_C)$ is closed if and only if $\begin{pmatrix} A_1 & C_1 \\ 0 & C_3 \end{pmatrix}$ is closed which is equivalent with the fact that $\mathcal{R}(A)+R(CP_{\mathcal{N}(B)})$ is closed. Also,

$$d\left(\begin{pmatrix} A_1 & C_1 \\ 0 & C_3 \end{pmatrix}\right) = n\left(\begin{pmatrix} A_1^* & 0 \\ C_1^* & C_3^* \end{pmatrix}\right) = n(C_3^*) = d(C_3),$$

since A_1^* is injective ($\mathcal{R}(A_1)=\mathcal{R}(A)$). Hence, in this case, the set S_{GLW} can also be described by

$$S_{GLW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \text{ is given by (2.1)},\ \mathcal{R}(A)+R(CP_{\mathcal{N}(B)}) \text{ is closed},\ d(C_3)+d(B)=\infty\Big\}.$$

As a corollary of the previous theorem, we get the description of the set $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma^g_{lw}(M_C)$:

COROLLARY 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. Then*

$$\begin{aligned}\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma^g_{lw}(M_C) = &\big\{\lambda\in\mathbb{C} : \mathcal{R}(A-\lambda I) \textit{ is not closed},\ n(B-\lambda I)<\infty\big\}\\ &\cup\big\{\lambda\in\mathbb{C} : \mathcal{R}(B-\lambda I) \textit{ is not closed},\ d(A-\lambda I)<\infty\big\}\\ &\cup\big\{\lambda\in\mathbb{C} : \mathcal{R}(A-\lambda I) \textit{ is not closed},\ \mathcal{R}(B-\lambda I) \textit{ is closed},\\ &\quad d(A-\lambda I)+d(B-\lambda I)<\infty\big\}\\ &\cup\big\{\lambda\in\mathbb{C} : \mathcal{R}(A-\lambda I),\mathcal{R}(B-\lambda I) \textit{ are closed},\\ &\quad n(A-\lambda I)+n(B-\lambda I)>d(A-\lambda I)+d(B-\lambda I)\big\}.\end{aligned}$$

Using Theorem 2.3, Remark 1 and the fact that A is generalized left Weyl if and only if A^* is generalized right Weyl, we can give the description of the set $S_{GW}(A,B)$ which consists of all $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl. Notice that necessary and sufficient conditions for the existence of $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl are given in [10].

THEOREM 2.9. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. There exists $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl if and only if one of the following conditions is satisfied:*

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A)+n(B)=d(A)+d(B)$. In this case,

$$S_{GW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \textit{ is given by } (2.1), C_3 \textit{ has closed range},\ n(A)+n(C_3)=d(C_3)+d(B)\Big\}.$$

(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A)=n(A)+n(B)=\infty$. In this case,

$$S_{GW}(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \textit{ is given by } (2.1), n(A)+n(C_3)=\infty,\ \mathcal{R}(B^*)+R(C^*P_{\mathcal{R}(A)^\perp}) \textit{ is closed}\}.$$

(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed and $n(B)=d(A)+d(B)=\infty$. In this case,

$$S_{GW}(A,B) = \Big\{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : C \textit{ is given by } (2.1), d(B)+d(C_3)=\infty,\ \mathcal{R}(A)+\mathcal{R}(CP_{\mathcal{N}(B)}) \textit{ is closed}\Big\}.$$

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(iv) $\mathcal{R}(A)$ *and* $\mathcal{R}(B)$ *are non-closed and* $n(B) = d(A) = \infty$. *In this case,*

$$S_{GW}(A,B) = \{C \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathcal{R}(M_C) \text{ is closed}\}.$$

Proof. Since necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl are given in [10], we need only prove that the set $S_{GW}(A,B)$ is given as claimed in each of the four possible cases appearing above.

(i) Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed and $n(A)+n(B) = d(A)+d(B)$. Using Theorem 2.3, we have that $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is such that M_C is generalized left Weyl if and only if C is given by (2.1), where C_3 has closed range and

$$n(A)+n(C_3) \leq d(C_3)+d(B).$$

Since we are looking for $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that M_C is generalized Weyl, we are asking for which $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying the previously mentioned condition, M_C is generalized right Weyl i.e $(M_C)^*$ is generalized left Weyl. Since,

$$(M_C)^* = \begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix} : \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathcal{K} \\ \mathcal{H} \end{pmatrix}$$

and for C given by (2.1), C^* is given by

$$C^* = \begin{pmatrix} C_4^* & C_2^* \\ C_3^* & C_1^* \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^*) \\ \mathcal{N}(A^*)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(B^*)} \\ \mathcal{R}(B^*)^\perp \end{pmatrix}, \tag{2.11}$$

applying Theorem 2.3 we get that $(M_C)^*$ is a generalized left Weyl operator if and only if $\mathcal{R}(C_3^*)$ is closed and

$$n(B^*)+n(C_3^*) \leq d(C_3^*)+d(A^*)$$

which is equivalent with $\mathcal{R}(C_3)$ being closed and the inequality $d(C_3)+d(B) \leq n(A)+n(C_3)$. Hence, M_C is a generalized Weyl operator if and only if C is given by (2.1), where C_3 has closed range and $n(A)+n(C_3) = d(C_3)+d(B)$.

(ii) Suppose that $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed and $d(A) = n(A)+n(B) = \infty$. Using Theorem 2.3, we have that $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is such that M_C is generalized left Weyl if and only if $\mathcal{R}(B^*)+\mathcal{R}(C^*P_{\mathcal{R}(A)^\perp})$ is closed. By item (iii) of Theorem 2.3, using the representations of $(M_C)^*$ given above, we get that $(M_C)^*$ is a generalized left Weyl operator if and only if $\mathcal{R}(B^*)+\mathcal{R}(C^*P_{\mathcal{R}(A)^\perp})$ is closed and $d(A^*)+\mathrm{codim}(\mathcal{R}(B^*)+\mathcal{R}(C^*P_{\mathcal{N}(A^*)})) = \infty$. By Remark 1, we have that the last condition is equivalent with $d(C_3^*)+d(A^*) = \infty$, i.e., $n(A)+n(C_3) = \infty$, where C_3 is the block operator in the representation (2.1) of C.

Hence, M_C is a generalized Weyl operator if and only if C is given by (2.1), where $\mathcal{R}(B^*)+\mathcal{R}(C^*P_{\mathcal{R}(A)^\perp})$ is closed and $n(A)+n(C_3) = \infty$.

Items (iii) and (iv) can be proved in a similar manner. □

In the next theorem, we present necessary and sufficient conditions which two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ have to satisfy in order for M_C to be a generalized left Weyl operator for each $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

THEOREM 2.10. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})$. *Then* M_C *is a generalized left Weyl operator for each* $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ *if and only if* $\mathcal{R}(A)$ *and* $\mathcal{R}(B)$ *are closed and one of the following conditions is satisfied:*

(1) $d(A) < \infty,\ n(B) = \infty,\ d(B) = \infty,$

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(2) $d(A)=\infty,\ n(B)<\infty$,
(3) $d(A),n(B)<\infty$, $n(A)+n(B)\leq d(A)+d(B)$.

Proof. Suppose that M_C is a generalized left Weyl operator for each $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$. If at least one of $\mathcal{R}(A)$ and $\mathcal{R}(B)$ is not closed, we have that M_0 is not a generalized left Weyl operator since its range is not closed. So, it follows that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed subspaces.

Notice that for arbitrary $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$, M_C is given by

$$M_C=\begin{pmatrix} 0 & A_1 & C_1 & C_2\\ 0 & 0 & C_3 & C_4\\ 0 & 0 & 0 & B_1\\ 0 & 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{N}(A)\\ \mathcal{N}(A)^{\perp}\\ \mathcal{N}(B)\\ \mathcal{N}(B)^{\perp}\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{R}(A)^{\perp}\\ \mathcal{R}(B)\\ \mathcal{R}(B)^{\perp}\end{pmatrix}, \tag{2.12}$$

where A_1, B_1 are invertible operators and that there exist invertible $U,V\in\mathcal{B}(\mathcal{H}\oplus\mathcal{K})$ such that

$$UM_CV=\begin{pmatrix} 0 & A_1 & 0 & 0\\ 0 & 0 & C_3 & 0\\ 0 & 0 & 0 & B_1\\ 0 & 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{N}(A)\\ \mathcal{N}(A)^{\perp}\\ \mathcal{N}(B)\\ \mathcal{N}(B)^{\perp}\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{R}(A)^{\perp}\\ \mathcal{R}(B)\\ \mathcal{R}(B)^{\perp}\end{pmatrix}. \tag{2.13}$$

So, for any $C_3\in\mathcal{B}(\mathcal{N}(B),\mathcal{R}(A)^{\perp})$, we have that $\mathcal{R}(C_3)$ is closed and

$$n(A)+n(C_3)\leq d(B)+d(C_3).$$

Hence, at least one of $d(A)$ and $n(B)$ is finite. So, we will consider all possible cases (there are 3 in total) when at least one of $d(A)$ and $n(B)$ is finite.

Suppose first that $d(A)<\infty$ and $n(B)=\infty$. Since for any $C_3\in\mathcal{B}(\mathcal{N}(B),\mathcal{R}(A)^{\perp})$, it follows that $n(C_3)=\infty$, and since there exists $C_3\in\mathcal{B}(\mathcal{N}(B),\mathcal{R}(A)^{\perp})$ such that $d(C_3)=0$, we conclude that $n(M_C)\leq d(M_C)$, for each $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if $d(B)=\infty$.

If $d(A)=\infty$ and $n(B)<\infty$ then for any $C_3\in\mathcal{B}(\mathcal{N}(B),\mathcal{R}(A)^{\perp})$, we have that $d(C_3)=\infty$, so $n(M_C)\leq d(M_C)$ is satisfied for any $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$.

If $d(A),n(B)<\infty$ then for any $C_3\in\mathcal{B}(\mathcal{N}(B),\mathcal{R}(A)^{\perp})$, we have that $n(B)-n(C_3)=d(A)-d(C_3)$, so $n(M_C)\leq d(M_C)$ if and only if $n(A)+n(B)\leq d(A)+d(B)$.

The converse implication can be proved in the same manner. □

As a corollary of the previous theorem, we also get the description of the set $\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma^g_{lw}(M_C)$:

COROLLARY 2.11. *For given operators $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{K})$ we have*

$$\begin{aligned}\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma^g_{lw}(M_C)=&\big\{\lambda\in\mathbb{C}:\mathcal{R}(A-\lambda I)\ \textit{is not closed}\big\}\\ &\cup\big\{\lambda\in\mathbb{C}:\mathcal{R}(B-\lambda I)\ \textit{is not closed}\big\}\\ &\cup\big\{\lambda\in\mathbb{C}:d(A-\lambda I)=n(B-\lambda I)=\infty\big\}\\ &\cup\big\{\lambda\in\mathbb{C}:d(A-\lambda I),n(B-\lambda I)<\infty,\\ &\quad n(A-\lambda I)+n(B-\lambda I)>d(A-\lambda I)+d(B-\lambda I)\big\}\\ &\cup\big\{\lambda\in\mathbb{C}:d(B-\lambda I)<n(B-\lambda I)=\infty\big\}.\end{aligned}$$

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REMARK 2. Throughout the paper, we have used the following fact: For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ in the each of following three cases:

(i) $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed,
(ii) $\mathcal{R}(A)$ is closed, $\mathcal{R}(B)$ is non-closed,
(iii) $\mathcal{R}(A)$ is non-closed, $\mathcal{R}(B)$ is closed,

we have that $\mathcal{R}(M_C)$ is closed if and only if the respective condition below is satisfied:

(1) $\mathcal{R}(C_3)$ is closed,
(2) $\mathcal{R}(B^*) + \mathcal{R}(C^* P_{\mathcal{R}(A)^\perp})$ is closed,
(3) $\mathcal{R}(A) + R(CP_{\mathcal{N}(B)})$ is closed.

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