



## THE HYPERPLANES OF $DW(5, \mathbb{F})$ ARISING FROM THE GRASSMANN EMBEDDING\*

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**Abstract.** The hyperplanes of the symplectic dual polar space  $DW(5, \mathbb{F})$  that arise from the Grassmann embedding have been classified in [B.N. Cooperstein and B. De Bruyn. Points and hyperplanes of the universal embedding space of the dual polar space  $DW(5, q)$ ,  $q$  odd. *Michigan Math. J.*, 58:195–212, 2009.] in case  $\mathbb{F}$  is a finite field of odd characteristic, and in [B. De Bruyn. Hyperplanes of  $DW(5, \mathbb{K})$  with  $\mathbb{K}$  a perfect field of characteristic 2. *J. Algebraic Combin.*, 30:567–584, 2009.] in case  $\mathbb{F}$  is a perfect field of characteristic 2. In the present paper, these classifications are extended to arbitrary fields. In the case of characteristic 2 however, it was not possible to provide a complete classification. The main tool in the proof is the classification of the quasi- $Sp(V, f)$ -equivalence classes of trivectors of a 6-dimensional symplectic vector space  $(V, f)$  obtained in [B. De Bruyn and M. Kwiatkowski. A 14-dimensional module for the symplectic group: orbits on vectors. *Comm. Algebra*, 43:4553–4569, 2015.].

**Key words.** Symplectic dual polar space, Hyperplane, Grassmann embedding, Trivector, Symplectic group, Exterior power.

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**1. Introduction.** Let  $V$  be a vector space of dimension  $2n \geq 4$  over a field  $\mathbb{F}$  equipped with a non-degenerate alternating bilinear form  $f$ . With  $(V, f)$ , there is associated a symplectic dual polar space  $DW(2n - 1, \mathbb{F})$ . This is the point-line geometry whose points are the  $n$ -dimensional subspaces of  $V$  that are totally isotropic with respect to  $f$  and whose lines are the  $(n - 1)$ -dimensional totally isotropic subspaces, with incidence being reverse containment. The geometry  $DW(2n - 1, \mathbb{F})$  has a full projective embedding  $e_n$  in a subspace  $\text{PG}(W)$  of  $\text{PG}(\bigwedge^n V)$ , where a point  $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_n \rangle$  of  $DW(2n - 1, \mathbb{F})$  is mapped to the 1-space  $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \dots \wedge \bar{v}_n \rangle$  of the  $n$ -th exterior power  $\bigwedge^n V$  of  $V$  (see e.g. [2, Proposition 5.1]). This embedding is called the *Grassmann embedding* of  $DW(2n - 1, \mathbb{F})$ . The subspace  $W$  is the subspace of  $\bigwedge^n V$  generated by all vectors of the form  $\bar{v}_1 \wedge \bar{v}_2 \wedge \dots \wedge \bar{v}_n$ , where  $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_n \rangle$  is an  $n$ -dimensional totally isotropic subspace, and has dimension  $\binom{2n}{n} - \binom{2n}{n-2}$ .

If  $\Pi$  is a hyperplane of  $\text{PG}(W)$ , then the set of all points of  $DW(2n - 1, \mathbb{F})$  that are mapped by  $e_n$  into  $\Pi$  is a *hyperplane* of  $DW(2n - 1, \mathbb{F})$ , that is a set of points of  $DW(2n - 1, \mathbb{F})$  distinct from the whole point set meeting each line in either a singleton or the whole line. Hyperplanes of  $DW(2n - 1, \mathbb{F})$  that can be obtained in this way are said to *arise from the Grassmann embedding*. The aim of the present paper is to give a classification of these hyperplanes. Such a classification has already been obtained (using other techniques) in [3] for finite fields of odd characteristic, and in [5] for perfect fields of characteristic 2. The proof which we will give in the present paper relies on the classification of the so-called quasi- $Sp(V, f)$ -equivalence classes of trivectors obtained in [8], which itself resulted from the lengthy classification of the  $Sp(V, f)$ -equivalence classes of trivectors (see [7]).

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**2. Preliminaries.** We denote by  $\widetilde{W}$  the subspace of  $\bigwedge^n V$  consisting of all  $n$ -vectors  $\chi$  of  $V$  (i.e., vectors  $\chi$  of  $\bigwedge^n V$ ) such that  $\chi \wedge \chi' = 0$  for all  $\chi' \in W$ . If  $\chi \in \bigwedge^n V \setminus \widetilde{W}$ , then the set of all points  $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_n \rangle$  of  $DW(2n-1, \mathbb{F})$  for which  $\chi \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \dots \wedge \bar{v}_n = 0$  is a hyperplane  $H(\chi)$  of  $DW(2n-1, \mathbb{F})$ . The hyperplanes of  $DW(2n-1, \mathbb{F})$  which arise in this way are precisely the hyperplanes of  $DW(2n-1, \mathbb{F})$  arising from the Grassmann embedding. If  $H$  is a hyperplane of  $DW(2n-1, \mathbb{F})$  arising from the Grassmann embedding, then every  $n$ -vector  $\chi$  for which  $H = H(\chi)$  is called an *associated  $n$ -vector*. Such an associated  $n$ -vector is not unique. If  $\chi_1, \chi_2 \in \bigwedge^n V \setminus \widetilde{W}$ , then  $H(\chi_1) = H(\chi_2)$  if and only if  $\chi_2 = \lambda \cdot \chi_1 + \tilde{\chi}$  for some  $\lambda \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$  and some  $\tilde{\chi} \in \widetilde{W}$ .

If  $n \geq 3$  and  $\bar{v}$  is a nonzero vector of  $V$ , then the set of all  $n$ -dimensional totally isotropic subspaces of  $V$  containing  $\bar{v}$  is a subspace of  $DW(2n-1, \mathbb{F})$  on which the induced subgeometry is isomorphic to  $DW(2n-3, \mathbb{F})$ . Such a subgeometry is called a *max*. In the special case  $n = 3$ , it is also called a *quad*. Such a quad is isomorphic to the dual polar space  $DW(3, \mathbb{F})$ , which itself is isomorphic to the generalized quadrangle  $Q(4, \mathbb{F})$  defined by the points and lines of  $PG(4, \mathbb{F})$  that are contained in a given nonsingular quadric of Witt index 2.

If  $p$  is a point of  $DW(2n-1, \mathbb{F})$ , i.e., an  $n$ -dimensional totally isotropic subspace of  $V$ , then the set of all totally isotropic subspaces meeting  $p$  is a hyperplane of  $DW(2n-1, \mathbb{F})$ . This hyperplane is called the *singular hyperplane with deepest point  $p$* .

Suppose  $n \geq 3$ . Let  $\bar{v}$  be a nonzero vector of  $V$ , and let  $M$  denote the corresponding max. If  $H'$  is a hyperplane of  $M$ , then the set of points of  $DW(2n-1, \mathbb{F})$  at distance at most 1 from  $H'$  is a hyperplane  $H$  of  $DW(2n-1, \mathbb{F})$ , called the *extension of  $H'$* . If  $H'$  is a singular hyperplane, then  $H$  is also a singular hyperplane (with the same deepest point).

Every hyperplane of the generalized quadrangle  $Q(4, \mathbb{F})$  is either a singular hyperplane, a full subgrid or an *ovoid*, an ovoid being a set of points meeting each line in a singleton.

Among the hyperplanes of  $DW(5, \mathbb{F})$  that we will meet, there are four that we will address with specific names: the singular hyperplanes, the extensions of the ovoids of the quads, the extensions of the full subgrids of the quads and the semi-singular hyperplanes. A *semi-singular hyperplane* of  $DW(5, \mathbb{F})$  is of the form  $p^\perp \cup O$ , where  $p^\perp$  denotes the set of points equal to or collinear with the point  $p$ , and  $O$  is a set of points at distance 3 from  $p$  (in the collinearity graph) such that every line at distance 2 from  $p$  contains a unique point of  $O$ .

Assume now that  $n = 3$ . Then  $V$  is a 6-dimensional vector space over  $\mathbb{F}$ . For every  $\theta \in GL(V)$ , there exists a unique  $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$  such that  $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \theta(\bar{v}_3)$  for all  $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$ . Two trivectors  $\chi_1$  and  $\chi_2$  are called  *$GL(V)$ -equivalent* if there exists a  $\theta \in GL(V)$  such that  $\chi_2 = \bigwedge^3(\theta)(\chi_1)$ . Revoy [13] obtained a complete classification of all  $GL(V)$ -equivalence classes of trivectors of  $V$ .

**PROPOSITION 2.1 ([13]).** *Let  $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_6\}$  be a basis of  $V$ . Then every nonzero trivector of  $V$  is  $GL(V)$ -equivalent with at least one of the following:*

- (A)  $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$ ;
- (B)  $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{v}_4 \wedge \bar{v}_5$ ;
- (C)  $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6$ ;
- (D)  $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_4 + \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{v}_5 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{v}_6$ ;
- (E)  $\lambda \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \mu \cdot \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_4) \wedge (\bar{v}_2 + \bar{v}_5) \wedge (\bar{v}_3 + \bar{v}_6)$  for some  $\lambda, \mu \in \mathbb{F}^*$  such that the quadratic polynomial  $\lambda X^2 + (\lambda\mu + \lambda + \mu)X + \mu$  of  $\mathbb{F}[X]$  is irreducible.

Moreover, if  $(X), (Y) \in \{(A), (B), (C), (D), (E)\}$  with  $(X) \neq (Y)$ , then *a/the trivector mentioned in  $(X)$  is never  $GL(V)$ -equivalent with a/the trivector mentioned in  $(Y)$ .*

A nonzero trivector of  $V$  is said to be of *Type  $(X)$* , where  $(X) \in \{(A), (B), (C), (D), (E)\}$  if it is  $GL(V)$ -equivalent with (one of) the trivector(s) described in  $(X)$  of Proposition 2.1. The description of the trivectors of Type  $(E)$  in terms of the parameters  $\lambda$  and  $\mu$  is a modification due to the first author of the present paper of the description given in [13]. We also note here that a complete classification of all  $GL(V)$ -equivalence classes of trivectors of  $V$  was also obtained by a number of other people under certain assumptions of the underlying field, see for instance Cohen and Helminck [1], and Reichel [12].

**3. The classification result.** We denote by  $\bar{\mathbb{F}}$  a fixed algebraic closure of  $\mathbb{F}$ . For every quadratic extension  $\mathbb{F}'$  of  $\mathbb{F}$  contained in  $\bar{\mathbb{F}}$ , we choose  $a_{\mathbb{F}'}, b_{\mathbb{F}'} \in \mathbb{F}$  such that  $\mathbb{F}'$  is the quadratic extension of  $\mathbb{F}$  defined by the polynomial  $X^2 - a_{\mathbb{F}'}X - b_{\mathbb{F}'}$ . In general, there are several possibilities for  $(a_{\mathbb{F}'}, b_{\mathbb{F}'})$ , but in this paper,  $(a_{\mathbb{F}'}, b_{\mathbb{F}'})$  will be a fixed choice among all these possibilities. Put

$$\Psi := \{(a_{\mathbb{F}'}, b_{\mathbb{F}'}) \mid \mathbb{F}' \subseteq \bar{\mathbb{F}} \text{ is a quadratic extension of } \mathbb{F}\}.$$

Suppose  $(\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$  is a *hyperbolic basis* of  $(V, f)$ , i.e., an ordered basis of  $V$  satisfying  $f(\bar{e}_i, \bar{f}_i) = 1$  for every  $i \in \{1, 2, \dots, n\}$  and  $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = f(\bar{e}_i, \bar{f}_j) = 0$  for all  $i, j \in \{1, 2, \dots, n\}$  with  $i \neq j$ .

In the remainder of this section,  $V$  will be 6-dimensional (i.e.,  $n = 3$ ). We define the following trivectors of  $V$ :

- $\chi_{A1} := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$ ;
- $\chi_{A2} := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$ ;
- $\chi_{B4}(\lambda) := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$  for some  $\lambda \in \mathbb{F}^*$ ;
- $\chi_{B5}(\lambda) := \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge (\bar{e}_2 - \bar{e}_3) \wedge (\bar{f}_2 + \bar{f}_3)$  for some  $\lambda \in \mathbb{F}^*$ ;
- $\chi_{C1}(\lambda) := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$  for some  $\lambda \in \mathbb{F}^*$ ;
- $\chi_{D2}(\lambda) := \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_2 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_1 \wedge \bar{f}_2$  for some  $\lambda \in \mathbb{F}^*$ ;
- $\chi_{D3}(\lambda_1, \lambda_2) := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \lambda_1 \cdot \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_1 + \lambda_2 \cdot \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_2$  for some  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ ;
- $\chi_{D4}(\lambda_1, \lambda_2) := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \lambda_1 \cdot \bar{e}_2 \wedge \bar{e}_3 \wedge (\bar{f}_1 + \bar{f}_3) + \lambda_2 \cdot \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_2$  for some  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ ;
- $\chi_{E1}(a, b; h_1, h_2, h_3) := 2 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + a \cdot \left( h_1 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + h_2 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_3 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 \right) + (a^2 + 2b) \cdot \left( h_1 h_2 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_1 h_3 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + h_2 h_3 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 \right) + h_1 h_2 h_3 a(a^2 + 3b) \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$  for some  $(a, b) \in \Psi$  and some  $h_1, h_2, h_3 \in \mathbb{F}^*$ .

The notation used for these trivectors is taken from earlier papers of the authors, see [7]. The subscripts  $A1, A2, \dots, E1$  indicate the type of the defined trivector  $((A), (B), (C), (D)$  or  $(E))$ .

If  $\text{char}(\mathbb{F}) \neq 2$ , then we define the following hyperplanes of  $DW(5, \mathbb{F})$ . All of them arise from the Grassmann embedding.

- I: Let  $H_1$  denote the hyperplane  $H(\chi_{A1})$ . Then  $H_1$  is the singular hyperplane of  $DW(5, \mathbb{F})$  with deepest point  $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ .
- II: Let  $H_2$  denote the hyperplane  $H(\chi_{B4}(1))$ . As we will see in Section 5,  $H_2$  is the extension of a full subgrid of the quad of  $DW(5, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ .
- III: For every nonsquare  $\lambda$  of  $\mathbb{F}$ , let  $H_3(\lambda)$  denote the hyperplane  $H(\chi_{B4}(\lambda))$ . As we will see in Section 5,  $H_3(\lambda)$  is the extension of an ovoid of the quad of  $DW(5, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ .

- IV: For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , let  $H_4(\lambda_1, \lambda_2)$  denote the hyperplane  $H(\chi_{D3}(\lambda_1, \lambda_2))$ . As we will see in Section 5, the hyperplane  $H_4(\lambda_1, \lambda_2)$  is a semi-singular hyperplane if and only if the equation  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ .
- V: Let  $H_5$  denote the hyperplane  $H(\chi_{C1}(1))$ .
- VI: For every  $(a, b) \in \Psi$  and all  $h_1, h_2, h_3 \in \mathbb{F}^*$ , let  $H_6(a, b; h_1, h_2, h_3)$  denote the hyperplane  $H(\chi_{E1}(a, b; h_1, h_2, h_3))$

If  $\text{char}(\mathbb{F}) \neq 2$ , then we call a hyperplane of  $DW(5, \mathbb{F})$  of *Type X*, where  $X \in \{I, II, \dots, VI\}$ , if it is isomorphic to (one of) the hyperplane(s) defined in X above. The following theorem classifies those hyperplanes of  $DW(5, \mathbb{F})$ ,  $\text{char}(\mathbb{F}) \neq 2$ , that arise from the Grassmann embedding. We will prove it in Section 4.3.

**THEOREM 3.1.** *Suppose  $\text{char}(\mathbb{F}) \neq 2$ . Then:*

- *Every hyperplane of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding has Type I, II, III, IV, V or VI.*
- *Let  $X, Y \in \{I, II, \dots, VI\}$  with  $X \neq Y$ . Then no hyperplane of Type X is isomorphic to a hyperplane of Type Y.*
- *Suppose  $\lambda$  and  $\lambda'$  are two nonsquares of  $\mathbb{F}$ . Then the hyperplanes  $H_3(\lambda)$  and  $H_3(\lambda')$  are isomorphic if and only if there exists an automorphism  $\tau$  of  $\mathbb{F}$  such that  $\frac{\lambda'}{\lambda^\tau}$  is a square of  $\mathbb{F}$ .*
- *Let  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$ . Then the hyperplanes  $H_4(\lambda_1, \lambda_2)$  and  $H_4(\lambda'_1, \lambda'_2)$  are isomorphic if and only if there exists an automorphism  $\tau$  of  $\mathbb{F}$  such that the diagonal matrices  $\text{diag}(\lambda_1^\tau, \lambda_2^\tau, \lambda_1^\tau \lambda_2^\tau)$  and  $\text{diag}(\lambda'_1, \lambda'_2, \lambda'_1 \lambda'_2)$  are congruent (regarded as matrices over  $\mathbb{F}$ ).*
- *Let  $(a, b), (a', b') \in \Psi$  and let  $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in \mathbb{F}^*$ . Then the hyperplanes  $H_6(a, b; h_1, h_2, h_3)$  and  $H_6(a', b'; h'_1, h'_2, h'_3)$  are isomorphic if and only if there exists an automorphism  $\tau$  of  $\mathbb{F}$  such that*
  - *the polynomials  $X^2 - aX - b$  and  $X^2 - (a')^\tau X - (b')^\tau$  define the same quadratic extension  $\mathbb{F}'$  of  $\mathbb{F}$  in  $\overline{\mathbb{F}}$ ;*
  - *there exist  $\mu \in \mathbb{F}^*$  and a nonsingular  $(3 \times 3)$ -matrix  $A$  over  $\mathbb{F}'$  with determinant belonging to  $\mathbb{F}^*$  such that  $\text{diag}(\mu(h'_1)^\tau, \mu(h'_2)^\tau, \mu(h'_3)^\tau) = A \cdot \text{diag}(h_1, h_2, h_3) \cdot (A^\psi)^T$ , where  $\psi$  is the unique nontrivial element in  $\text{Gal}(\mathbb{F}'/\mathbb{F})$ .*

The result presented in Theorem 3.1 generalizes a result of Cooperstein and De Bruyn [3] who obtained a complete classification of all hyperplanes of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding in the special case where  $\mathbb{F}$  is a finite field of odd characteristic. We have chosen the terminology of hyperplanes of Type I, II, ..., VI in the present paper such that it is consistent with the terminology used in [3]. Let us now discuss a few special cases of Theorem 3.1 (see also Section 4.3).

- (1) Suppose  $\mathbb{F}$  is an algebraically closed field of characteristic distinct from 2. Then there are up to isomorphism four hyperplanes of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding. Each such hyperplane is isomorphic to either  $H_1, H_2, H_4(1, 1)$  or  $H_5$ .
- (2) There are up to isomorphism eight hyperplanes of  $DW(5, \mathbb{R})$  arising from the Grassmann embedding. Each such hyperplane is isomorphic to either  $H_1, H_2, H_3(-1), H_4(1, 1), H_4(-1, -1), H_5, H_6(0, -1; 1, 1, 1)$  or  $H_6(0, -1; -1, 1, 1)$ .
- (3) Suppose  $q$  is an odd prime power and  $\nu$  is a given nonsquare in  $\mathbb{F}_q$ . Then there are up to isomorphism six hyperplanes of  $DW(5, \mathbb{F}_q)$  arising from the Grassmann embedding. Each such hyperplane is isomorphic to either  $H_1, H_2, H_3(\nu), H_4(1, 1), H_5$  or  $H_6(0, \nu; 1, 1, 1)$ .

Next, we discuss the case where the underlying field  $\mathbb{F}$  has characteristic 2. If  $\text{char}(\mathbb{F}) = 2$ , then we define the following hyperplanes of  $DW(5, \mathbb{F})$ . All of them arise from the Grassmann embedding.

- I: Let  $H_1$  denote the hyperplane  $H(\chi_{A1})$ . Then again  $H_1$  is the singular hyperplane of  $DW(5, \mathbb{F})$  with deepest point  $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ .
- II: Let  $H_2$  denote the hyperplane  $H(\chi_{A2})$ . We will see in Section 5 that  $H_2$  is the extension of a full subgrid of the quad of  $DW(5, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ .
- III: For every  $\lambda \in \mathbb{F}$  such that the polynomial  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  is irreducible, let  $H_3(\lambda)$  denote the hyperplane  $H(\chi_{B5}(\lambda))$ . We will see in Section 5 that the hyperplane  $H_3(\lambda)$  is the extension of an ovoid of the quad of  $DW(5, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ .
- III': For every nonsquare  $\lambda$  of  $\mathbb{F}$ , let  $H'_3(\lambda)$  denote the hyperplane  $H(\chi_{B4}(\lambda))$ . We will see in Section 5 that  $H'_3(\lambda)$  is the extension of an ovoid of the quad of  $DW(5, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ .
- IV: For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , let  $H_4(\lambda_1, \lambda_2)$  denote the hyperplane  $H(\chi_{D4}(\lambda_1, \lambda_2))$ . We will see in Section 5 that the hyperplane  $H_4(\lambda_1, \lambda_2)$  is a semi-singular hyperplane if and only if the equation  $X^2 + \lambda_1 Y^2 + \lambda_1 XY + \lambda_2 Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ .
- IV': For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$  such that the equation  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ , let  $H'_4(\lambda_1, \lambda_2)$  denote the hyperplane  $H(\chi_{D3}(\lambda_1, \lambda_2))$ . We will see in Section 5 that  $H'_4(\lambda_1, \lambda_2)$  is a semi-singular hyperplane.
- V: Let  $H_5$  denote the hyperplane  $H(\chi_{C1}(1))$ .
- VI: For every  $\lambda \in \mathbb{F}^*$ , let  $H_6(\lambda)$  denote the hyperplane  $H(\chi_{D2}(\lambda))$ .

If  $\text{char}(\mathbb{F}) = 2$ , then we call a hyperplane of  $DW(5, \mathbb{F})$  of *Type X*, where  $X \in \{I, II, III, III', IV, IV', V, VI\}$ , if it is isomorphic to (one of) the hyperplane(s) defined in X above. We will prove the following theorem in Section 4.4.

**THEOREM 3.2.** *Suppose  $\text{char}(\mathbb{F}) = 2$ . Then:*

- Every hyperplane of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding which has an associated trivector of Type (A), (B), (C) or (D) has Type I, II, III, III', IV, IV', V or VI.
- Let  $X, Y \in \{I, II, III, III', IV, IV', V, VI\}$  with  $X \neq Y$ . Then no hyperplane of Type X is isomorphic to a hyperplane of Type Y.
- Let  $\lambda, \lambda' \in \mathbb{F}$  such that the polynomials  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  and  $X^2 + \lambda' X + 1 \in \mathbb{F}[X]$  are irreducible. Then the hyperplanes  $H_3(\lambda)$  and  $H_3(\lambda')$  are isomorphic if and only if there exists an automorphism  $\tau$  of  $\mathbb{F}$  such that the polynomials  $X^2 + \lambda^\tau X + 1$  and  $X^2 + \lambda' X + 1$  define the same quadratic extension of  $\mathbb{F}$  in  $\bar{\mathbb{F}}$ .
- Let  $\lambda$  and  $\lambda'$  be two nonsquares of  $\mathbb{F}$ . Then the hyperplanes  $H'_3(\lambda)$  and  $H'_3(\lambda')$  are isomorphic if and only if there exists an automorphism  $\tau$  of  $\mathbb{F}$  such that the polynomials  $X^2 + \lambda^\tau$  and  $X^2 + \lambda'$  define the same quadratic extension of  $\mathbb{F}$  in  $\bar{\mathbb{F}}$ .
- Let  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$ . Then the hyperplanes  $H_4(\lambda_1, \lambda_2)$  and  $H_4(\lambda'_1, \lambda'_2)$  are isomorphic if and only if there exist  $\mu \in \mathbb{F}^*$  and an automorphism  $\tau$  of  $\mathbb{F}$  such that the matrices

$$\mu \cdot \begin{bmatrix} \lambda_1^\tau & 0 & \lambda_1^\tau \\ 0 & \lambda_2^\tau & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda'_1 & 0 & \lambda'_1 \\ 0 & \lambda'_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are pseudo-congruent (regarded as matrices over  $\mathbb{F}$ ).

- Let  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$  such that none of the equations  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  and  $\lambda'_1 X^2 + \lambda'_2 Y^2 + Z^2 = 0$  has solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ . Then the hyperplanes  $H'_4(\lambda_1, \lambda_2)$  and  $H'_4(\lambda'_1, \lambda'_2)$  are isomorphic if and only if there exist  $\mu \in \mathbb{F}^*$  and an automorphism  $\tau$  of  $\mathbb{F}$  such that the diagonal matrices  $\mu \cdot \text{diag}(\lambda_1^\tau, \lambda_2^\tau, 1)$  and  $\text{diag}(\lambda'_1, \lambda'_2, 1)$  are pseudo-congruent (regarded as matrices over  $\mathbb{F}$ ).
- Let  $\lambda, \lambda' \in \mathbb{F}^*$ . Then the hyperplanes  $H_6(\lambda)$  and  $H_6(\lambda')$  are isomorphic if and only if there exists an

automorphism  $\tau$  of  $\mathbb{F}$  such that  $\frac{\lambda^\tau}{\lambda}$  is a square of  $\mathbb{F}$ .

In Theorem 3.2, two  $(3 \times 3)$ -matrices  $A_1$  and  $A_2$  over  $\mathbb{F}$  are called *pseudo-congruent* if there exists a nonsingular  $(3 \times 3)$ -matrix  $M$  over  $\mathbb{F}$  such that the matrix  $A_1 - MA_2M^T$  is *alternating*, i.e., skew-symmetric and having all diagonal elements equal to 0. The relation of being pseudo-congruent defines an equivalence relation of the set of all  $(3 \times 3)$ -matrices over  $\mathbb{F}$ .

In the literature, there are already some classification results for hyperplanes of  $DW(5, \mathbb{F})$ ,  $\text{char}(\mathbb{F}) = 2$ , arising from the Grassmann embedding. Pralle [11] classified all such hyperplanes with the aid of a computer (using a backtrack search) in the case  $|\mathbb{F}| = 2$ . De Bruyn [4, 5] extended these results to perfect fields of characteristic 2. We have chosen the terminology of hyperplanes of Type I, II, ..., VI in the present paper such that it is consistent with the terminology used in [4].

Suppose now that  $\mathbb{F}$  is a perfect field of characteristic 2. It follows then from [5, Theorem 1.1] that every hyperplane of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding has an associated trivector of Type (A), (B) or (C), and from [5, Theorem 1.2] that there are up to isomorphism unique hyperplanes of Type I, II, IV, V and VI. Each hyperplane arising from the Grassmann embedding is therefore isomorphic to either  $H_1$ ,  $H_2$ ,  $H_3(\lambda)$  with  $\lambda \in \mathbb{F}$  such that  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  is irreducible,  $H_4(1, 1)$ ,  $H_5$  or  $H_6(1)$ .

**4. Proofs of Theorems 3.1 and 3.2.** We suppose here that  $V$  is a 6-dimensional vector space over a field  $\mathbb{F}$  equipped with a nondegenerate alternating bilinear form  $f$ . We call two trivectors  $\chi_1$  and  $\chi_2$  of  $V$  *Sp(V, f)-equivalent* if there exists a  $\theta \in Sp(V, f)$  such that  $\chi_2 = \bigwedge^3(\theta)(\chi_1)$ , and *quasi-Sp(V, f)-equivalent* if there exists a  $\theta \in Sp(V, f)$  and a  $\tilde{\chi} \in \widetilde{W}$  such that  $\chi_2 = \bigwedge^3(\theta)(\chi_1) + \tilde{\chi}$ .

**4.1. The classification of the quasi-Sp(V, f)-equivalence classes.** In [8], the authors obtained a classification of the quasi-Sp(V, f)-equivalence classes of trivectors of  $V$ . We will use this classification here to obtain our desired classification results about the hyperplanes of  $DW(5, \mathbb{F})$ . The classification of the quasi-Sp(V, f)-equivalence classes is summarized in the following four propositions.

**PROPOSITION 4.1 ([8]).** *If  $\text{char}(\mathbb{F}) \neq 2$ , then every trivector of  $V$  is quasi-Sp(V, f)-equivalent with (at least) one of the following trivectors:*

- (1) *the zero vector of  $\bigwedge^3 V$ ;*
- (2)  $\chi_{A1}$ ;
- (3)  $\chi_{B4}(\lambda)$  *for some  $\lambda \in \mathbb{F}^*$ ;*
- (4)  $\chi_{C1}(\lambda)$  *for some  $\lambda \in \mathbb{F}^*$ ;*
- (5)  $\chi_{D3}(\lambda_1, \lambda_2)$  *for some  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ ;*
- (6)  $\chi_{E1}(a, b; h_1, h_2, h_3)$  *for some  $(a, b) \in \Psi$  and some  $h_1, h_2, h_3 \in \mathbb{F}^*$ .*

Note that the trivectors quasi-Sp(V, f)-equivalent with the zero vector of  $\bigwedge^3 V$  are precisely the vectors of the subspace  $\widetilde{W}$ .

**PROPOSITION 4.2 ([8]).** *Suppose  $\text{char}(\mathbb{F}) \neq 2$ .*

- *Let  $i, j \in \{1, 2, 3, 4, 5, 6\}$  with  $i \neq j$ . Then no trivector mentioned in (i) of Proposition 4.1 is quasi-Sp(V, f)-equivalent with a trivector mentioned in (j) of Proposition 4.1.*
- *Let  $\lambda, \lambda' \in \mathbb{F}^*$ . Then the two trivectors  $\chi_{B4}(\lambda)$  and  $\chi_{B4}(\lambda')$  of  $V$  are quasi-Sp(V, f)-equivalent if and only if  $\frac{\lambda}{\lambda'}$  is a square in  $\mathbb{F}$ .*



- Let  $\lambda, \lambda' \in \mathbb{F}^*$ . Then the two trivectors  $\chi_{C1}(\lambda)$  and  $\chi_{C1}(\lambda')$  of  $V$  are quasi- $Sp(V, f)$ -equivalent if and only if  $\lambda' \in \{\lambda, -\lambda\}$ .
- Let  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$ . Then the two trivectors  $\chi_{D3}(\lambda_1, \lambda_2)$  and  $\chi_{D3}(\lambda'_1, \lambda'_2)$  of  $V$  are quasi- $Sp(V, f)$ -equivalent if and only if the matrices  $\text{diag}(\lambda_1, \lambda_2, \lambda_1\lambda_2)$  and  $\text{diag}(\lambda'_1, \lambda'_2, \lambda'_1\lambda'_2)$  are congruent (regarded as matrices over  $\mathbb{F}$ ).
- Let  $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in \mathbb{F}^*$  and  $(a, b), (a', b') \in \Psi$ . Then the two trivectors  $\chi_{E1}(a, b; h_1, h_2, h_3)$  and  $\chi_{E1}(a', b'; h'_1, h'_2, h'_3)$  of  $V$  are quasi- $Sp(V, f)$ -equivalent if and only if  $(a, b) = (a', b')$  and there exists a  $3 \times 3$ -matrix  $A$  over  $\mathbb{F}'$  with determinant equal to 1 such that  $A \cdot \text{diag}(h_1, h_2, h_3) \cdot (A^\psi)^T$  is equal to  $\text{diag}(h'_1, h'_2, h'_3)$  or  $\text{diag}(-h'_1, -h'_2, -h'_3)$ . Here,  $\mathbb{F}' \subseteq \overline{\mathbb{F}}$  is the quadratic extension of  $\mathbb{F}$  determined by the irreducible quadratic polynomial  $X^2 - aX - b$  of  $\mathbb{F}[X]$  and  $\psi$  is the unique nontrivial element of the Galois group  $\text{Gal}(\mathbb{F}'/\mathbb{F})$ .

PROPOSITION 4.3 ([8]). Suppose  $\text{char}(\mathbb{F}) = 2$  and  $\chi$  is a trivector of  $V$  which is quasi- $Sp(V, f)$ -equivalent with a trivector of Type (A), (B), (C) or (D). Then  $\chi$  is quasi- $Sp(V, f)$ -equivalent with (at least) one of the following trivectors:

- (1) the zero vector of  $\bigwedge^3 V$ ;
- (2)  $\chi_{A1}$ ;
- (3)  $\chi_{A2}$ ;
- (4)  $\chi_{B4}(\lambda)$  for some nonsquare  $\lambda$  of  $\mathbb{F}$ ;
- (5)  $\chi_{B5}(\lambda)$  for some  $\lambda \in \mathbb{F}$  such that the polynomial  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  is irreducible;
- (6)  $\chi_{C1}(\lambda)$  for some  $\lambda \in \mathbb{F}^*$ ;
- (7)  $\chi_{D2}(\lambda)$  for some  $\lambda \in \mathbb{F}^*$ ;
- (8)  $\chi_{D3}(\lambda_1, \lambda_2)$  for some  $\lambda_1, \lambda_2 \in \mathbb{F}^*$  such that the equation  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ ;
- (9)  $\chi_{D4}(\lambda_1, \lambda_2)$  for some  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ .

PROPOSITION 4.4 ([8]). Suppose  $\text{char}(\mathbb{F}) = 2$ .

- Let  $i, j \in \{1, 2, \dots, 9\}$  with  $i \neq j$ . Then no trivector mentioned in (i) of Proposition 4.3 is quasi- $Sp(V, f)$ -equivalent with a trivector mentioned in (j) of Proposition 4.3.
- Let  $\lambda$  and  $\lambda'$  be two nonsquares of  $\mathbb{F}$ . Then the two trivectors  $\chi_{B4}(\lambda)$  and  $\chi_{B4}(\lambda')$  are quasi- $Sp(V, f)$ -equivalent if and only if the polynomials  $X^2 + \lambda$  and  $X^2 + \lambda'$  define the same quadratic extension of  $\mathbb{F}$  in  $\overline{\mathbb{F}}$ .
- Let  $\lambda$  and  $\lambda'$  be two elements of  $\mathbb{F}$  such that the polynomials  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  and  $X^2 + \lambda' X + 1 \in \mathbb{F}[X]$  are irreducible. Then the two trivectors  $\chi_{B5}(\lambda)$  and  $\chi_{B5}(\lambda')$  are quasi- $Sp(V, f)$ -equivalent if and only if the polynomials  $X^2 + \lambda X + 1$  and  $X^2 + \lambda' X + 1$  define the same quadratic extension of  $\mathbb{F}$  in  $\overline{\mathbb{F}}$ .
- Let  $\lambda, \lambda' \in \mathbb{F}^*$ . Then the two trivectors  $\chi_{C1}(\lambda)$  and  $\chi_{C1}(\lambda')$  are quasi- $Sp(V, f)$ -equivalent if and only if  $\lambda = \lambda'$ .
- Let  $\lambda, \lambda' \in \mathbb{F}^*$ . Then the two trivectors  $\chi_{D2}(\lambda)$  and  $\chi_{D2}(\lambda')$  are quasi- $Sp(V, f)$ -equivalent if and only if  $\lambda = \lambda'$ .
- Let  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$  such that neither of the equations  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  and  $\lambda'_1 X^2 + \lambda'_2 Y^2 + Z^2 = 0$  has solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ . Then the two trivectors  $\chi_{D3}(\lambda_1, \lambda_2)$  and  $\chi_{D3}(\lambda'_1, \lambda'_2)$  are quasi- $Sp(V, f)$ -equivalent if and only if there exists a  $\mu \in \mathbb{F}^*$  such that the matrices  $\text{diag}(\mu\lambda_1, \mu\lambda_2, \mu)$  and  $\text{diag}(\lambda'_1, \lambda'_2, 1)$  are pseudo-congruent (regarded as matrices over  $\mathbb{F}$ ).
- Let  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$ . Then the two trivectors  $\chi_{D4}(\lambda_1, \lambda_2)$  and  $\chi_{D4}(\lambda'_1, \lambda'_2)$  are quasi- $Sp(V, f)$ -

equivalent if and only if there exists a  $\mu \in \mathbb{F}^*$  such that the matrices

$$\begin{bmatrix} \mu\lambda_1 & 0 & \mu\lambda_1 \\ 0 & \mu\lambda_2 & 0 \\ 0 & 0 & \mu \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda'_1 & 0 & \lambda'_1 \\ 0 & \lambda'_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are pseudo-congruent (regarded as matrices over  $\mathbb{F}$ ).

The proofs of Propositions 4.1, 4.2, 4.3 and 4.4 given in [8] heavily relied on the lengthy classification of the  $Sp(V, f)$ -equivalence classes of trivectors obtained by the authors (see [7] for an overview). In the special case the underlying field is algebraically closed and of characteristic distinct from 2, a classification of the  $Sp(V, f)$ -equivalence classes of trivectors was also obtained by Popov [10, Section 3]. In the case  $\text{char}(\mathbb{F}) \neq 2$ , the classification problem for the quasi- $Sp(V, f)$ -equivalence classes of trivectors (whose solution is presented in Propositions 4.1 and 4.2) is equivalent with the classification problem for those  $Sp(V, f)$ -equivalence classes of trivectors that are contained in the subspace  $W \subseteq \bigwedge^3 V$ . A classification of those  $Sp(V, f)$ -equivalence classes was also obtained by Igusa [9, Proposition 7, p.1026] in the special case the underlying field is algebraically closed and of characteristic distinct from 2.

**4.2. Quasi- $Sp(V, f)$ -equivalence and isomorphism of hyperplanes.** For every element  $\theta$  of  $\Gamma L(V)$ , we denote by  $\tau_\theta$  the accompanying automorphism of  $\mathbb{F}$ . Let  $G \cong \Gamma Sp(6, \mathbb{F})$  denote the group of all  $\theta \in \Gamma L(V)$  for which there exists an  $a_\theta \in \mathbb{F}^*$  such that  $f(\bar{x}^\theta, \bar{y}^\theta) = a_\theta \cdot f(\bar{x}, \bar{y})^{\tau_\theta}$ ,  $\forall \bar{x}, \bar{y} \in V$ , i.e.,  $G$  is the *semi-similarity group* of  $f$ . The elements of  $G$  can be described as follows.

- Every element of  $Sp(V, f)$  is also an element of  $G$ .
- For every  $\mu \in \mathbb{F}^*$ , the element  $\theta_\mu$  of  $GL(V)$  mapping the hyperbolic basis  $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$  of  $(V, f)$  to the ordered basis  $(\mu\bar{e}_1, \bar{f}_1, \mu\bar{e}_2, \bar{f}_2, \mu\bar{e}_3, \bar{f}_3)$  of  $V$  belongs to  $G$ .
- For every automorphism  $\tau$  of  $\mathbb{F}$ , let  $\theta_\tau$  denote the element of  $\Gamma L(V)$  mapping the vector  $a_1\bar{e}_1 + a_2\bar{e}_2 + a_3\bar{e}_3 + b_1\bar{f}_1 + b_2\bar{f}_2 + b_3\bar{f}_3$  to the vector  $a_1^\tau\bar{e}_1 + a_2^\tau\bar{e}_2 + a_3^\tau\bar{e}_3 + b_1^\tau\bar{f}_1 + b_2^\tau\bar{f}_2 + b_3^\tau\bar{f}_3$ . Then  $\theta_\tau \in G$ .

Every element of  $G$  can be written in a unique way as  $\theta \circ \theta_\mu \circ \theta_\tau$ , where  $\theta \in Sp(V, f)$ ,  $\mu \in \mathbb{F}^*$  and  $\tau$  an automorphism of  $\mathbb{F}$ .

If  $\theta \in G$ , then the permutation on the point set of  $DW(5, \mathbb{F})$  defined by  $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \mapsto \langle \bar{v}_1^\theta, \bar{v}_2^\theta, \bar{v}_3^\theta \rangle$  is an automorphism  $\Omega_\theta$  of  $DW(5, \mathbb{F})$ . Every automorphism of  $DW(5, \mathbb{F})$  can be obtained in this way. If  $\theta \in \Gamma L(V)$ , then there exists a unique  $\bigwedge^3(\theta) \in \Gamma L(\bigwedge^3 V)$  such that  $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \bar{v}_1^\theta \wedge \bar{v}_2^\theta \wedge \bar{v}_3^\theta$ ,  $\forall \bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$ .

**LEMMA 4.5.** *Let  $\chi_1$  and  $\chi_2$  be elements of  $\bigwedge^3 V \setminus \widetilde{W}$ . Then the hyperplanes  $H(\chi_1)$  and  $H(\chi_2)$  of  $DW(5, \mathbb{F})$  are isomorphic if and only if there exist  $\theta \in G$ ,  $\mu \in \mathbb{F}^*$  and  $\tilde{\chi} \in \widetilde{W}$  such that  $\chi_2 = \mu \cdot \bigwedge^3(\theta)(\chi_1) + \tilde{\chi}$ .*

*Proof.* This follows from the following facts.

- If  $\chi \in \bigwedge^3 V \setminus \widetilde{W}$  and  $\theta \in G$ , then also  $\bigwedge^3(\theta)(\chi) \in \bigwedge^3 V \setminus \widetilde{W}$  and  $\Omega_\theta(H(\chi)) = H(\bigwedge^3(\theta)(\chi))$ .
- If  $\chi'_1, \chi'_2 \in \bigwedge^3 V \setminus \widetilde{W}$ , then  $H(\chi'_1) = H(\chi'_2)$  if and only if there exists a  $\mu' \in \mathbb{F}^*$  and a  $\tilde{\chi} \in \widetilde{W}$  such that  $\chi'_2 = \mu' \cdot \chi'_1 + \tilde{\chi}$ .  $\square$

**COROLLARY 4.6.** *Let  $\chi_1$  and  $\chi_2$  be elements of  $\bigwedge^3 V \setminus \widetilde{W}$ . Then the hyperplanes  $H(\chi_1)$  and  $H(\chi_2)$  of  $DW(5, \mathbb{F})$  are isomorphic if and only if there exist  $\mu_1, \mu_2 \in \mathbb{F}^*$  and an automorphism  $\tau$  of  $\mathbb{F}$  such that the trivector  $\mu_2 \cdot \bigwedge^3(\theta_{\mu_1}) \circ \bigwedge^3(\theta_\tau)(\chi_1)$  is quasi- $Sp(V, f)$ -equivalent with  $\chi_2$ .*



### 4.3. The case $\text{char}(\mathbb{F}) \neq 2$ .

LEMMA 4.7. *Suppose  $\text{char}(\mathbb{F}) \neq 2$  and  $\mu \in \mathbb{F}^*$ .*

- *The trivector  $\mu \cdot \chi_{A1}$  is  $Sp(V, f)$ -equivalent with  $\chi_{A1}$ .*
- *For every  $\lambda \in \mathbb{F}^*$ , the trivector  $\mu \cdot \chi_{B4}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{B4}(\lambda)$ .*
- *For every  $\lambda \in \mathbb{F}^*$ , the trivector  $\mu \cdot \chi_{C1}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\mu^2 \lambda)$ .*
- *For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , the trivector  $\mu \cdot \chi_{D3}(\lambda_1, \lambda_2)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D3}(\lambda_1, \lambda_2)$ .*
- *For every  $(a, b) \in \Psi$  and all  $h_1, h_2, h_3 \in \mathbb{F}^*$ , the trivector  $\mu \cdot \chi_{E1}(a, b; h_1, h_2, h_3)$  is  $Sp(V, f)$ -equivalent with  $\chi_{E1}(a, b; \mu^2 h_1, h_2, h_3)$ .*

*Proof.* For every  $\lambda \in \mathbb{F}^*$ , we have  $\mu \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3$ ,  $\mu \cdot (\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + \lambda \cdot \bar{e}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3$  and  $\mu \cdot (\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + \lambda \mu^2 \cdot \bar{f}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3$ , where  $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$  is the hyperbolic basis  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$  of  $(V, f)$ . The first three claims of the lemma follow.

For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , we have  $\mu \cdot \chi_{D3}(\lambda_1, \lambda_2) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + \lambda_1 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{f}'_1 + \lambda_2 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_2$ , where  $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$  is the hyperbolic basis  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \mu \bar{e}_2, \frac{\bar{f}_2}{\mu}, \mu \bar{e}_3, \frac{\bar{f}_3}{\mu})$  of  $(V, f)$ . Hence,  $\mu \cdot \chi_{D3}(\lambda_1, \lambda_2)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D3}(\lambda_1, \lambda_2)$ .

For every  $(a, b) \in \Psi$  and all  $h_1, h_2, h_3 \in \mathbb{F}^*$ , we have  $\mu \cdot \chi_{E1}(a, b; h_1, h_2, h_3) = 2 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + a \cdot (\mu^2 h_1 \cdot \bar{f}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + h_2 \cdot \bar{e}'_1 \wedge \bar{f}'_2 \wedge \bar{e}'_3 + h_3 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3) + (a^2 + 2b) \cdot (\mu^2 h_1 h_2 \cdot \bar{f}'_1 \wedge \bar{f}'_2 \wedge \bar{e}'_3 + \mu^2 h_1 h_3 \cdot \bar{f}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + h_2 h_3 \cdot \bar{e}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3) + \mu^2 h_1 h_2 h_3 (a^2 + 3b) \cdot \bar{f}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3$ , where  $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$  is the hyperbolic basis  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$  of  $(V, f)$ . So,  $\mu \cdot \chi_{E1}(a, b; h_1, h_2, h_3)$  is  $Sp(V, f)$ -equivalent with  $\chi_{E1}(a, b; \mu^2 h_1, h_2, h_3)$ .  $\square$

LEMMA 4.8. *Suppose  $\text{char}(\mathbb{F}) \neq 2$  and  $\mu \in \mathbb{F}^*$ .*

- *The trivector  $\bigwedge^3(\theta_\mu)(\chi_{A1})$  is  $Sp(V, f)$ -equivalent with  $\chi_{A1}$ .*
- *For every  $\lambda \in \mathbb{F}^*$ , the trivector  $\bigwedge^3(\theta_\mu)(\chi_{B4}(\lambda))$  is  $Sp(V, f)$ -equivalent with  $\chi_{B4}(\lambda)$ .*
- *For every  $\lambda \in \mathbb{F}^*$ , the trivector  $\bigwedge^3(\theta_\mu)(\chi_{C1}(\lambda))$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\mu^3 \lambda)$ .*
- *For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , the trivector  $\bigwedge^3(\theta_\mu)(\chi_{D3}(\lambda_1, \lambda_2))$  is  $Sp(V, f)$ -equivalent with  $\chi_{D3}(\lambda_1, \lambda_2)$ .*
- *For every  $(a, b) \in \Psi$  and all  $h_1, h_2, h_3 \in \mathbb{F}^*$ , the trivector  $\bigwedge^3(\theta_\mu)(\chi_{E1}(a, b; h_1, h_2, h_3))$  is  $Sp(V, f)$ -equivalent with  $\chi_{E1}(a, b; \mu h_1, \mu h_2, \mu h_3)$ .*

*Proof.* The trivector  $\bigwedge^3(\theta_\mu)(\chi_{A1}) = (\mu^3 \bar{e}_1) \wedge \bar{e}_2 \wedge \bar{e}_3$  is  $Sp(V, f)$ -equivalent with  $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 = \chi_{A1}$ .

For every  $\lambda \in \mathbb{F}^*$ , we have  $\bigwedge^3(\theta_\mu)(\chi_{B4}(\lambda)) = (\mu^2 \bar{e}_1) \wedge (\mu \bar{e}_2) \wedge \bar{e}_3 + \lambda \cdot (\mu^2 \bar{e}_1) \wedge \frac{\bar{f}_2}{\mu} \wedge \bar{f}_3$ , and the latter trivector is  $Sp(V, f)$ -equivalent with  $\chi_{B4}(\lambda)$  since  $(\mu^2 \bar{e}_1, \frac{\bar{f}_1}{\mu^2}, \mu \bar{e}_2, \frac{\bar{f}_2}{\mu}, \bar{e}_3, \bar{f}_3)$  is a hyperbolic basis of  $(V, f)$ .

For every  $\lambda \in \mathbb{F}^*$ , the trivector  $\bigwedge^3(\theta_\mu)(\chi_{C1}(\lambda)) = (\mu \bar{e}_1) \wedge (\mu \bar{e}_2) \wedge (\mu \bar{e}_3) + (\lambda \mu^3) \cdot \frac{\bar{f}_1}{\mu} \wedge \frac{\bar{f}_2}{\mu} \wedge \frac{\bar{f}_3}{\mu}$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\lambda \mu^3)$  since  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \mu \bar{e}_2, \frac{\bar{f}_2}{\mu}, \mu \bar{e}_3, \frac{\bar{f}_3}{\mu})$  is a hyperbolic basis of  $(V, f)$ .

For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , the trivector  $\bigwedge^3(\theta_\mu)(\chi_{D3}(\lambda_1, \lambda_2)) = \mu^2 \cdot \chi_{D3}(\lambda_1, \lambda_2)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D3}(\lambda_1, \lambda_2)$  by Lemma 4.7.

For every  $(a, b) \in \Psi$  and all  $h_1, h_2, h_3 \in \mathbb{F}^*$ ,  $\bigwedge^3(\theta_\mu)(\chi_{E1}(a, b; h_1, h_2, h_3)) = 2 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + a \cdot (\mu h_1 \cdot \bar{f}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + \mu h_2 \cdot \bar{e}'_1 \wedge \bar{f}'_2 \wedge \bar{e}'_3 + \mu h_3 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3) + (a^2 + 2b) \cdot (\mu^2 h_1 h_2 \cdot \bar{f}'_1 \wedge \bar{f}'_2 \wedge \bar{e}'_3 + \mu^2 h_1 h_3 \cdot \bar{f}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + \mu^2 h_2 h_3 \cdot \bar{e}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3) + \mu^3 h_1 h_2 h_3 a(a^2 + 3b) \cdot \bar{f}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3$ , where  $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$  is the hyperbolic basis  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \mu \bar{e}_2, \frac{\bar{f}_2}{\mu}, \mu \bar{e}_3, \frac{\bar{f}_3}{\mu})$  of  $(V, f)$ . Hence,  $\bigwedge^3(\theta_\mu)(\chi_{E1}(a, b; h_1, h_2, h_3))$  is  $Sp(V, f)$ -equivalent with  $\chi_{E1}(a, b; \mu h_1, \mu h_2, \mu h_3)$ .  $\square$

LEMMA 4.9. Suppose  $a', b' \in \mathbb{F}$  such that  $X^2 - a'X - b'$  is an irreducible quadratic polynomial of  $\mathbb{F}[X]$  defining a separable quadratic extension of  $\mathbb{F}$ . Let  $(a, b)$  denote the unique element of  $\Psi$  such that  $X^2 - a'X - b'$  and  $X^2 - aX - b$  define the same quadratic extension  $\mathbb{K}$  of  $\mathbb{F}$  in  $\bar{\mathbb{F}}$ . Let  $h_1, h_2, h_3 \in \mathbb{F}^*$ . Then there exists a  $\mu \in \mathbb{F}^*$  such that the trivector  $\chi = 2 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + a' \cdot (h_1 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + h_2 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_3 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) + ((a')^2 + 2b') \cdot (h_1 h_2 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_1 h_3 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + h_2 h_3 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) + h_1 h_2 h_3 a'((a')^2 + 3b') \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$  is  $Sp(V, f)$ -equivalent with  $\chi_{E1}(a, b; \mu h_1, \mu h_2, \mu h_3)$ .

*Proof.* Let  $\psi$  denote the unique nontrivial element of  $\text{Gal}(\mathbb{K}/\mathbb{F})$ . We denote by  $(V_{\mathbb{K}}, f_{\mathbb{K}})$  a 6-dimensional symplectic space which also has  $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$  as hyperbolic basis such that  $V$  consists of all  $\mathbb{F}$ -linear combinations of the vectors of the set  $\{\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3\}$ . Then  $f$  is the restriction of  $f_{\mathbb{K}}$  to the set  $V \times V$ . If  $\bar{v} = a_1 \bar{e}_1 + b_1 \bar{f}_1 + a_2 \bar{e}_2 + b_2 \bar{f}_2 + a_3 \bar{e}_3 + b_3 \bar{f}_3 \in V_{\mathbb{K}}$  with all  $a_i$ 's and  $b_j$ 's belonging to  $\mathbb{K}$ , then we define  $\bar{v}^\psi = a_1^\psi \bar{e}_1 + b_1^\psi \bar{f}_1 + a_2^\psi \bar{e}_2 + b_2^\psi \bar{f}_2 + a_3^\psi \bar{e}_3 + b_3^\psi \bar{f}_3$ . Let  $\delta \in \mathbb{K} \setminus \mathbb{F}$  and  $\delta' \in \mathbb{K} \setminus \mathbb{F}$  be roots of the respective polynomials  $X^2 - aX - b$  and  $X^2 - a'X - b'$ . Then  $(\delta')^\psi - \delta' = \mu(\delta^\psi - \delta)$  for some  $\mu \in \mathbb{F}^*$ . By De Bruyn and Kwiatkowski [6, Section 4.2], the trivector  $\chi$  is of the form  $(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) + (\bar{v}_1^\psi \wedge \bar{v}_2^\psi \wedge \bar{v}_3^\psi)$ , where  $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$  is a 3-dimensional subspace of  $V_{\mathbb{K}}$  totally isotropic with respect to  $f_{\mathbb{K}}$  such that  $f_{\mathbb{K}}(\bar{v}_i, \bar{v}_j^\psi)$  with  $i, j \in \{1, 2, 3\}$  is equal to  $((\delta')^\psi - \delta') \cdot h_i$  if  $i = j$  and equal to 0 otherwise. Since  $((\delta')^\psi - \delta') \cdot h_i = (\delta^\psi - \delta) \cdot \mu h_i$  for every  $i \in \{1, 2, 3\}$ , Section 4.2 of [6] implies that  $\chi = (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) + (\bar{v}_1^\psi \wedge \bar{v}_2^\psi \wedge \bar{v}_3^\psi)$  is also  $Sp(V, f)$ -equivalent with  $\chi_{E1}(a, b; \mu h_1, \mu h_2, \mu h_3)$ .  $\square$

Theorem 3.1 is now an immediate consequence of Propositions 4.1 and 4.2, Corollary 4.6, and Lemmas 4.7, 4.8 and 4.9.

In the case that  $\mathbb{F}$  is an algebraically closed field of characteristic distinct from 2, the classification of the hyperplanes of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding as stated in Section 3 immediately follows from Theorem 3.1.

Also the classification of the hyperplanes of  $DW(5, \mathbb{R})$  arising from the Grassmann embedding follows immediately from Theorem 3.1 if one takes into account that every automorphism of  $\mathbb{R}$  is trivial and that every nonsingular  $3 \times 3$  diagonal matrix over  $\mathbb{R}$  is congruent to either  $\text{diag}(1, 1, 1)$ ,  $\text{diag}(1, 1, -1)$ ,  $\text{diag}(1, -1, -1)$  or  $\text{diag}(-1, -1, -1)$ .

In the case  $\mathbb{F}$  is a finite field  $\mathbb{F}_q$  of odd characteristic, the classification of the hyperplanes of  $DW(5, \mathbb{F})$  arising from the Grassmann embedding follows from Theorem 3.1, from the fact that the product of two nonsquares in  $\mathbb{F}_q$  is a square and from the following two lemmas.

LEMMA 4.10. If  $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}_q^*$ ,  $q$  odd, then the matrices  $\text{diag}(\lambda_1, \lambda_2, \lambda_1 \lambda_2)$  and  $\text{diag}(\lambda'_1, \lambda'_2, \lambda'_1 \lambda'_2)$  are congruent.

*Proof.* Let  $\nu$  be a fixed nonsquare in  $\mathbb{F}$ . Observe that if  $\sigma$  is a permutation of  $\{1, 2, 3\}$  and  $\mu_1, \mu_2, \mu_3$ ,

$x, y, z \in \mathbb{F}^*$ , then the matrices  $\text{diag}(\mu_1, \mu_2, \mu_3)$  and  $\text{diag}(\mu_1^\sigma x^2, \mu_2^\sigma y^2, \mu_3^\sigma z^2)$  are congruent. Observe also that either 1 or 3 of the numbers  $\lambda_1, \lambda_2, \lambda_1 \lambda_2$  are squares. So, in order to prove the lemma, it suffices to prove that the matrices  $\text{diag}(\nu, \nu)$  and  $\text{diag}(1, 1)$  are congruent. It is known that  $\nu$  can be written in the form  $\lambda_1^2 + \lambda_2^2$  where  $\lambda_1, \lambda_2 \in \mathbb{F}$ . Then

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix}. \quad \square$$

LEMMA 4.11. Let  $q$  be odd and  $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in \mathbb{F}_q^*$ . Then there exist  $\mu \in \mathbb{F}_q^*$  and a  $(3 \times 3)$ -matrix over  $\mathbb{F}_{q^2}$  with determinant in  $\mathbb{F}_q^*$  such that  $\text{diag}(\mu h'_1, \mu h'_2, \mu h'_3) = A \cdot \text{diag}(h_1, h_2, h_3) \cdot (\bar{A})^T$ , where  $\bar{A}$  is the matrix obtained from  $A$  by raising each of its elements to the  $q$ -th power.

*Proof.* Put  $\mu := \frac{h'_1 h'_2 h'_3}{h_1 h_2 h_3}$ . Let  $x_1, x_2, x_3 \in \mathbb{F}_{q^2}$  such that  $x_1^{q+1} = \frac{\mu h'_1}{h_1}$ ,  $x_2^{q+1} = \frac{\mu h'_2}{h_2}$  and  $x_1 x_2 x_3 = \left( \frac{h'_1 h'_2 h'_3}{h_1 h_2 h_3} \right)^2$ . Then  $x_3^{q+1} = \frac{\mu h'_3}{h_3}$ . The claim of the lemma holds if we put  $A := \text{diag}(x_1, x_2, x_3)$ .  $\square$

#### 4.4. The case $\text{char}(\mathbb{F}) = 2$ .

LEMMA 4.12. Suppose  $\text{char}(\mathbb{F}) = 2$  and  $\mu \in \mathbb{F}^*$ .

- The trivector  $\mu \cdot \chi_{A1}$  is  $Sp(V, f)$ -equivalent with  $\chi_{A1}$ .
- The trivector  $\mu \cdot \chi_{A2}$  is  $Sp(V, f)$ -equivalent with  $\chi_{A2}$ .
- Let  $\lambda$  be a nonsquare of  $\mathbb{F}$ . Then the trivector  $\mu \cdot \chi_{B4}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{B4}(\lambda)$ .
- Let  $\lambda \in \mathbb{F}$  such that the polynomial  $X^2 + \lambda X + 1$  is irreducible. Then  $\mu \cdot \chi_{B5}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{B5}(\lambda)$ .
- Let  $\lambda \in \mathbb{F}^*$ . Then the trivector  $\mu \cdot \chi_{C1}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\mu^2 \lambda)$ .
- Let  $\lambda \in \mathbb{F}^*$ . Then the trivector  $\mu \cdot \chi_{D2}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D2}(\mu^4 \lambda)$ .
- Let  $\lambda_1, \lambda_2 \in \mathbb{F}^*$  such that the equation  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ . Then  $\mu \cdot \chi_{D3}(\lambda_1, \lambda_2)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D3}(\lambda_1, \lambda_2)$ .
- Let  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ . Then  $\mu \cdot \chi_{D4}(\lambda_1, \lambda_2)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D4}(\lambda_1, \lambda_2)$ .

*Proof.* If  $\lambda \in \mathbb{F}^*$ , then  $\mu \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3$ ,  $\mu \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2$ ,  $\mu \cdot (\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + \lambda \cdot \bar{e}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3$  and  $\mu \cdot (\lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_1 \wedge (\bar{e}_2 - \bar{e}_3) \wedge (\bar{f}_2 + \bar{f}_3)) = \lambda \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \bar{e}'_1 \wedge (\bar{e}'_2 - \bar{e}'_3) \wedge (\bar{f}'_2 + \bar{f}'_3)$ , where  $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$  is the hyperbolic basis  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$  of  $(V, f)$ . The first four claims of the lemma follow.

If  $\lambda \in \mathbb{F}^*$ , then  $\mu \cdot \chi_{C1}(\lambda) = (\mu \bar{e}_1) \wedge \bar{e}_2 \wedge \bar{e}_3 + (\lambda \mu^2) \cdot \frac{\bar{f}_1}{\mu} \wedge \bar{f}_2 \wedge \bar{f}_3$ . Since  $(\mu \bar{e}_1, \frac{\bar{f}_1}{\mu}, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$  is a hyperbolic basis of  $(V, f)$ ,  $\mu \cdot \chi_{C1}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\lambda \mu^2)$ .

If  $\lambda \in \mathbb{F}^*$ , then  $\mu \cdot \chi_{D2}(\lambda) = (\lambda \mu^4) \cdot \bar{e}_1 \wedge \frac{\bar{e}_2}{\mu} \wedge \frac{\bar{f}_3}{\mu^2} + \frac{\bar{e}_2}{\mu} \wedge \bar{f}_1 \wedge (\mu^2 \bar{e}_3) + \bar{f}_1 \wedge \bar{e}_1 \wedge (\mu \bar{f}_2)$ . Since  $(\bar{e}_1, \bar{f}_1, \frac{\bar{e}_2}{\mu}, \mu \bar{f}_2, \mu^2 \bar{e}_3, \frac{\bar{f}_3}{\mu^2})$  is a hyperbolic basis of  $(V, f)$ ,  $\mu \cdot \chi_{D2}(\lambda)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D2}(\lambda \mu^4)$ .

Finally, if  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , then  $\mu \cdot \chi_{D3}(\lambda_1, \lambda_2) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + \lambda_1 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{f}'_1 + \lambda_2 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_2$  and  $\mu \cdot \chi_{D4}(\lambda_1, \lambda_2) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + \lambda_1 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge (\bar{f}'_1 + \bar{f}'_3) + \lambda_2 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_2$ , where  $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$  is the

hyperbolic basis  $(\mu\bar{e}_1, \frac{\bar{f}_1}{\mu}, \mu\bar{e}_2, \frac{\bar{f}_2}{\mu}, \mu\bar{e}_3, \frac{\bar{f}_3}{\mu})$  of  $(V, f)$ . The last two claims of the lemma follow.  $\square$

LEMMA 4.13. *Suppose  $\text{char}(\mathbb{F}) = 2$  and  $\mu \in \mathbb{F}^*$ .*

- *The trivector  $\bigwedge^3(\theta_\mu)(\chi_{A1})$  is  $Sp(V, f)$ -equivalent with  $\chi_{A1}$ .*
- *The trivector  $\bigwedge^3(\theta_\mu)(\chi_{A2})$  is  $Sp(V, f)$ -equivalent with  $\chi_{A2}$ .*
- *Let  $\lambda$  be a nonsquare of  $\mathbb{F}$ . Then the trivector  $\bigwedge^3(\theta_\mu)(\chi_{B4}(\lambda))$  is  $Sp(V, f)$ -equivalent with  $\chi_{B4}(\lambda)$ .*
- *Let  $\lambda \in \mathbb{F}$  such that the polynomial  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  is irreducible. Then  $\bigwedge^3(\theta_\mu)(\chi_{B5}(\lambda))$  is  $Sp(V, f)$ -equivalent with  $\chi_{B5}(\lambda)$ .*
- *Let  $\lambda \in \mathbb{F}^*$ . Then the trivector  $\bigwedge^3(\theta_\mu)(\chi_{C1}(\lambda))$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\mu^3\lambda)$ .*
- *Let  $\lambda \in \mathbb{F}^*$ . Then the trivector  $\bigwedge^3(\theta_\mu)(\chi_{D2}(\lambda))$  is  $Sp(V, f)$ -equivalent with  $\chi_{D2}(\mu^6\lambda)$ .*
- *Let  $\lambda_1, \lambda_2 \in \mathbb{F}^*$  such that the equation  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ . Then  $\bigwedge^3(\theta_\mu)(\chi_{D3}(\lambda_1, \lambda_2))$  is  $Sp(V, f)$ -equivalent with  $\chi_{D3}(\lambda_1, \lambda_2)$ .*
- *Let  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ . Then  $\bigwedge^3(\theta_\mu)(\chi_{D4}(\lambda_1, \lambda_2))$  is  $Sp(V, f)$ -equivalent with  $\chi_{D4}(\lambda_1, \lambda_2)$ .*

*Proof.* We have  $\bigwedge^3(\theta_\mu)(\chi_{A1}) = \mu^3 \cdot \chi_{A1}$  and  $\bigwedge^3(\theta_\mu)(\chi_{A2}) = \mu^2 \cdot \chi_{A2}$ . For every  $\lambda \in \mathbb{F}$  such that the polynomial  $X^2 + \lambda X + 1 \in \mathbb{F}[X]$  is irreducible, we have  $\bigwedge^3(\theta_\mu)(\chi_{B5}(\lambda)) = \mu^2 \cdot \chi_{B5}(\lambda)$ . For all  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ , we have  $\bigwedge^3(\theta_\mu)(\chi_{D3}(\lambda_1, \lambda_2)) = \mu^2 \cdot \chi_{D3}(\lambda_1, \lambda_2)$  and  $\bigwedge^3(\theta_\mu)(\chi_{D4}(\lambda_1, \lambda_2)) = \mu^2 \cdot \chi_{D4}(\lambda_1, \lambda_2)$ . By Lemma 4.12, the first, second, fourth, seventh and eighth claim of the lemma follow.

Let  $\lambda$  be a nonsquare of  $\mathbb{F}$ . Then the trivector  $\bigwedge^3(\theta_\mu)(\chi_{B4}(\lambda)) = (\mu^2\bar{e}_1) \wedge (\mu\bar{e}_2) \wedge \bar{e}_3 + \lambda \cdot (\mu^2\bar{e}_1) \wedge \frac{\bar{f}_2}{\mu} \wedge \bar{f}_3$  is  $Sp(V, f)$ -equivalent with  $\chi_{B4}(\lambda)$  since  $(\mu^2\bar{e}_1, \frac{\bar{f}_1}{\mu^2}, \mu\bar{e}_2, \frac{\bar{f}_2}{\mu}, \bar{e}_3, \bar{f}_3)$  is a hyperbolic basis of  $(V, f)$ .

Let  $\lambda \in \mathbb{F}^*$ . Then the trivector  $\bigwedge^3(\theta_\mu)(\chi_{C1}(\lambda)) = (\mu\bar{e}_1) \wedge (\mu\bar{e}_2) \wedge (\mu\bar{e}_3) + (\lambda\mu^3) \cdot \frac{\bar{f}_1}{\mu} \wedge \frac{\bar{f}_2}{\mu} \wedge \frac{\bar{f}_3}{\mu}$  is  $Sp(V, f)$ -equivalent with  $\chi_{C1}(\lambda\mu^3)$  since  $(\mu\bar{e}_1, \frac{\bar{f}_1}{\mu}, \mu\bar{e}_2, \frac{\bar{f}_2}{\mu}, \mu\bar{e}_3, \frac{\bar{f}_3}{\mu})$  is a hyperbolic basis of  $(V, f)$ .

Let  $\lambda \in \mathbb{F}^*$ . Then the trivector  $\bigwedge^3(\theta_\mu)(\chi_{D2}(\lambda)) = (\mu^6\lambda) \cdot \bar{e}_1 \wedge \frac{\bar{e}_2}{\mu} \wedge \frac{\bar{f}_3}{\mu^3} + \frac{\bar{e}_2}{\mu} \wedge \bar{f}_1 \wedge (\mu^3\bar{e}_3) + \bar{f}_1 \wedge \bar{e}_1 \wedge (\mu\bar{f}_2)$  is  $Sp(V, f)$ -equivalent with  $\chi_{D2}(\mu^6\lambda)$  since  $(\bar{e}_1, \bar{f}_1, \frac{\bar{e}_2}{\mu}, \mu\bar{f}_2, \mu^3\bar{e}_3, \frac{\bar{f}_3}{\mu^3})$  is a hyperbolic basis of  $(V, f)$ .  $\square$

Theorem 3.2 is an immediate consequence of Propositions 4.3, 4.4, Corollary 4.6 and Lemmas 4.12, 4.13.

**5. Structure of some of the hyperplanes.** Consider again the general case  $n \geq 3$ , and let

$$(\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$$

be a hyperbolic basis of  $(V, f)$ . Put  $V' = \langle \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n \rangle$  and let  $f'$  be the nondegenerate alternating bilinear form on  $V'$  induced by  $f$ . Let  $W'$  and  $\widetilde{W}'$  denote the subspaces of  $\bigwedge^{n-1} V'$  that are defined in a similar way as the subspaces  $W$  and  $\widetilde{W}$  of  $\bigwedge^n V$ .

LEMMA 5.1. *Let  $\chi' \in \bigwedge^{n-1} V' \setminus \widetilde{W}'$  and put  $\chi := \bar{e}_1 \wedge \chi'$ . Then  $\chi \in \bigwedge^n V \setminus \widetilde{W}$  and the hyperplane  $H(\chi)$  of  $DW(2n-1, \mathbb{F})$  is the extension of a hyperplane of the max  $M$  of  $DW(2n-1, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ .*

*Proof.* Since  $\chi' \in \bigwedge^{n-1} V' \setminus \widetilde{W}'$ , there exists an  $(n-1)$ -dimensional totally isotropic subspace

$$\langle \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle$$

of  $(V', f')$  such that  $\chi' \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n \neq 0$ . Now,  $\langle \bar{f}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle$  is an  $n$ -dimensional totally isotropic subspace of  $(V, f)$  and  $\chi \wedge \bar{f}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n = (-1)^{n-1}(\bar{e}_1 \wedge \bar{f}_1) \wedge \chi' \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n \neq 0$ . So,  $\chi \in \bigwedge^n V \setminus W$  and  $H(\chi)$  is a hyperplane of  $DW(2n-1, \mathbb{F})$ .

The set of all  $(n-1)$ -dimensional totally isotropic subspaces  $\langle \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle$  of  $(V', f')$  for which  $\chi' \wedge (\bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n) = 0$  is a hyperplane of the dual polar space isomorphic to  $DW(2n-3, \mathbb{F})$  defined by  $(V', f')$ . This implies that the set of all subspaces  $\langle \bar{e}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle$ , where  $\langle \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle$  is an  $(n-1)$ -dimensional totally isotropic subspace of  $(V', f')$  satisfying  $\chi' \wedge (\bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n) = 0$ , is a hyperplane  $G$  of the max  $M$  of  $DW(2n-1, \mathbb{F})$  corresponding to the vector  $\bar{e}_1$ . We will show that  $H(\chi)$  is the extension of  $G$ .

Since  $\chi \wedge \bar{e}_1 = 0$ , the max  $M$  is completely contained in  $H(\chi)$ . So, in order to prove that  $H(\chi)$  is the extension of the hyperplane  $G$ , we must prove that if  $u$  is a point of  $DW(2n-1, \mathbb{F})$  not contained in  $M$  and  $u'$  is the unique point of  $M$  collinear with  $u$ , then  $u \in H(\chi)$  if and only if  $u' \in G$ .

A point  $u$  of  $DW(2n-1, \mathbb{F})$  not contained in  $M$  is of the form  $\langle \bar{f}_1 + \bar{v}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle$  where  $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n \in \langle \bar{e}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3, \dots, \bar{e}_n, \bar{f}_n \rangle$ . For every  $i \in \{2, \dots, n\}$ , put  $\bar{v}_i = \bar{v}'_i + \bar{v}''_i$ , where  $\bar{v}'_2, \bar{v}'_3, \dots, \bar{v}'_n \in \langle \bar{e}_1 \rangle$  and  $\bar{v}''_2, \bar{v}''_3, \dots, \bar{v}''_n \in \langle \bar{e}_2, \bar{f}_2, \dots, \bar{e}_n, \bar{f}_n \rangle$ . Then  $u' = \langle \bar{e}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle = \langle \bar{e}_1, \bar{v}'_2, \bar{v}'_3, \dots, \bar{v}'_n \rangle$ . Now,  $\langle \bar{f}_1 + \bar{v}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \rangle \in H(\chi)$  if and only if  $\bar{e}_1 \wedge \chi' \wedge (\bar{f}_1 + \bar{v}_1) \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_n = \bar{e}_1 \wedge \chi' \wedge (\bar{f}_1 + \bar{v}_1) \wedge \bar{v}''_2 \wedge \bar{v}''_3 \wedge \cdots \wedge \bar{v}''_n = 0$ . This precisely happens when  $\chi' \wedge \bar{v}''_2 \wedge \bar{v}''_3 \wedge \cdots \wedge \bar{v}''_n = 0$ , i.e., when  $u' \in G$ .  $\square$

From now on we assume that  $n = 3$ . If  $H$  is a hyperplane of  $DW(5, \mathbb{F})$ , then a quad  $Q$  is called *deep* (with respect to  $H$ ) if  $Q \subseteq H$ . If  $H$  is a singular hyperplane with deepest point  $p$ , then the deep quads are precisely the quads containing the point  $p$ . If  $H$  is the extension of an ovoid of a quad  $Q$ , then  $Q$  is the unique deep quad. If  $H$  is the extension of a full subgrid  $G$  of a quad  $Q$ , then the deep quads are precisely the quads that contain a line of  $G$ .

By Lemma 5.1, it follows that every hyperplane of Type I, II, III or III' of  $DW(5, \mathbb{F})$  is either a singular hyperplane, the extension of an ovoid of a quad or the extension of a full subgrid of a quad. To know which of the three types of hyperplanes is occurring in each case, one can determine the configuration of the deep quads. The configuration of the deep quads follows from Lemma 4.19, Lemma 4.20(1),(2) and Lemma 4.22(1),(2) of [8]. From these lemmas, we can conclude that the hyperplanes of Type I are singular hyperplanes, the hyperplanes of Type II are extensions of full subgrids of quads and the hyperplanes of Type III or III' are extensions of ovoids of quads. In fact, it is not so hard to see that the hyperplanes of Type I are singular.

Let  $p$  be the point  $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$  of  $DW(5, \mathbb{F})$ . Then the singular hyperplane  $H_p$  with deepest point  $p$  consists of all 3-dimensional totally isotropic subspaces  $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$  such that  $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle \cap \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \neq \{\bar{o}\}$ , or equivalently  $(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) \wedge (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = 0$ . As a consequence,  $H_p = H(\chi_{A1})$ .

**LEMMA 5.2.** *Let  $H$  be a hyperplane of  $DW(5, \mathbb{F})$  and  $p$  a point such that  $p^\perp \subseteq H$ . Then  $H$  is a semi-singular hyperplane if and only if no quad through  $p$  is deep.*

*Proof.* Suppose  $H$  is a semi-singular hyperplane. Then  $H$  contains a unique point  $q$  for which  $q^\perp \subseteq H$  and this point must coincide with  $p$ . As the semi-singular hyperplane  $H$  contains no points at distance 2 from  $p$ , no quad through  $p$  can be deep.

Conversely, suppose no quad through  $p$  is deep. Then there are no points at distance 2 from  $p$  that are contained in  $H$ . So,  $H = p^\perp \cup O$ , where  $O$  is a set of points at distance 3 from  $p$ . Now, take a line  $L$  at distance 2 from  $p$ . As  $L$  contains a unique point at distance 2 from  $p$  (which is not contained in  $H$ ),  $L$

intersects  $H$  in a singleton, i.e.,  $|L \cap O| = 1$ . So,  $H$  must be a semi-singular hyperplane.  $\square$

LEMMA 5.3. *Let  $\lambda_1, \lambda_2 \in \mathbb{F}^*$ .*

- *The hyperplane  $H(\chi_{D3}(\lambda_1, \lambda_2))$  is semi-singular if and only if  $\lambda_1 X^2 + \lambda_2 Y^2 + Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ .*
- *The hyperplane  $H(\chi_{D4}(\lambda_1, \lambda_2))$  is semi-singular if and only if the equation  $X^2 + \lambda_1 Y^2 + \lambda_1 XY + \lambda_2 Z^2 = 0$  has no solutions for  $(X, Y, Z) \in \mathbb{F}^3 \setminus \{(0, 0, 0)\}$ .*

*Proof.* Let  $\chi$  be one of the trivectors  $\chi_{D3}(\lambda_1, \lambda_2)$ ,  $\chi_{D4}(\lambda_1, \lambda_2)$ . Let  $p$  be the point  $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$  of  $DW(5, \mathbb{F})$ . If  $x \in p^\perp \setminus \{p\}$ , then  $x = \langle \bar{e}'_1, \bar{e}'_2, \bar{v} \rangle$ , where  $\bar{e}'_1, \bar{e}'_2 \in \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$  and  $\bar{v} \in V \setminus \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ . This implies that  $\chi \wedge \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{v} = 0$ , i.e.,  $x \in H(\chi)$ . It follows that  $p^\perp \subseteq H$ . Lemma 5.2 then implies that  $H(\chi)$  is semi-singular if and only if no quad through  $p$  is deep.

By Lemmas 4.19, 4.20 (4) and 4.22 (4) of [8], it follows that no quad through  $p$  is deep if and only if the conditions mentioned in the lemma are satisfied.  $\square$

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