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LORENTZ TRANSFORMATION FROM AN ELEMENTARY POINT OF VIEW*

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Abstract. Elementary methods are used to examine some nontrivial mathematical issues underpinning the Lorentz transformation. Its eigen-system is characterized through the exponential of a G-skew symmetric matrix, underlining its unconnectedness at one of its extremes (the hyper-singular case). A different yet equivalent angle is presented through Pauli coding which reveals the connection between the hyper-singular case and the shear map.

Key words. Generalized Euler-Rodrigues formula, Minkowski space, Lorentz group, $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{SO}_0(3,1)$.

AMS subject classifications. 15A90, 15A57, 17B81, 22E70.

1. Introduction. The studies of Lorentz transformation usually place it within quite sophisticated theories. Yet, it stands to reason to believe that a significant part of the theory could and should be derived within the standard framework of the linear algebra and Euclidean geometry, that is, "simple from simple". A reader can see a similar approach in other publications such as [8] (cf. Subsection 1.1.2). Our title paraphrases the title of the famous book by Hans Rademacher [13].

We simplify and expand our previous work [6] (a comment on [12]). Our propositions admit simple proofs that avoid specialized terminology or advanced theories. In other words, we confine to the "lingua franca" of the common mathematical background so that standard textbooks on linear algebra and calculus suffice as a technical reference. Still, we couldn't resist ourselves from introducing a few linguistic inventions, one is the notion of a "slider" for a specific group of operators (compare [1]), and another well known action we dub a "jaws operator", for these coined names capture the essence of the underlying properties. We pay a special attention to "hyper-singular" matrices whose mere singularity is enhanced by their extreme behavior.

A more advanced setup from the point of view of mathematics or physics, based

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on tensors, Lie groups and algebras, etc., can be found in numerous and extensive literature, cf., for example [2, 9, 11, 14] and [4, Chapter 18], to name but a small sample. Exponential formulas similar to one in Proposition 2.6 below have been constantly derived by various methods while they could be traced back to 1960s (cf. [3] and references therein; [10]).

The scope of the paper is as follows.

Section 2: The matrix $G = \operatorname{diag}(1, -1, -1, -1)$ entails the G-transformation, $X \mapsto GX^{\mathsf{T}}G$ and the notions of G symmetric, G-skew symmetric, and G-orthogonal matrices. We examine in detail the eigen-system of a G-skew symmetric matrix and show that it is unconnected at the extreme of hyper-singular matrices.

Section 3: An eigen-analysis of the Lorentz transformation yields its intrinsic structure. We establish the surjectivity of the exponential map by elementary means. The fact that every Lorentz matrix is an exponential of a Maxwell matrix is well known but its derivation is typically quite involved (cf., for example [2]).

Section 4: With the help of the Pauli's axiomatic formalism, we represent Lorentz matrices as so called "jaws operators", which again leads to the exponentiation of G-skew symmetric matrices. The defective Lorentz matrices, usually omitted in the literature, appear as images of the shear transformations.

1.1. Notation and basics. We consider complex vectors and matrices in a finite dimensional real or complex Euclidean space. We use the usual $n \times 1$ matrix to represent a vector, indicated by the boldface sanserif font, e.g., x. The transpose is denoted by X^{T} and X^* marks the Hermitian transpose. For the scalar product, we write $(xy) = \mathsf{x}^{\mathsf{T}}\mathsf{y}$ and the length is written as $x = ||\mathsf{x}||$. We use the universal symbols I and O for the identity and null matrix whose dimension will be clear from the context. Two matrices X and Y are called *product orthogonal* if $XY = YX = 0$, while they are called *orthogonal* if

$$\langle\, X\,,\, Y\,\rangle \overset{\mathsf{def}}{=} \frac{1}{n}\operatorname{tr} X^*Y = 0.$$

Among a plethora of equivalent norms, we choose the normalized Frobenius (also known as Hilbert-Schmidt) norm

$$||X|| = \frac{1}{\sqrt{n}}\sqrt{\operatorname{tr} X^*X}, \quad \text{i.e., } ||X||^2 = \frac{1}{n}\sum_j\sum_k |a_{jk}|^2 = \frac{1}{n}\sum_k s_k^2,$$

where s_k are nonnegative singular values of X, i.e., s_k^2 are eigenvalues of the symmetric nonnegative definite matrix X^*X. Hence,

$$||I|| = 1 \quad \text{and} \quad ||XY|| \leq ||X|| \cdot ||Y||,$$

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which yields the well-defined exponential $e^X = \sum_{n=0} X^n/n!$ so that

$$\left\|e^X\right\| \le e^{||X||}, \qquad \left\|e^X - e^Y\right\| \le e^{\max(||X||,||Y||)}\,||X-Y||,$$

and

$$\left\|e^X - I - X\right\| \le \frac{1}{2}\,||X||^2\,e^{||X||}.$$

Hence, if a matrix $X = X(t)$ is continuous with respect to a complex or real variable t, so is $E(t) = e^{X(t)}$. Let $X(t)$ be differentiable at 0 with $F = X'(0)$ and $X(0) = I$. Then $X = I + tF + R$, where the remainder is simply $R = X - I - tF$ and $||R||/t \to 0$ as $t \to 0$. Hence, the above inequalities entail the estimate

$$\begin{aligned} \frac{\left\|e^{I+tF+R} - I - tF\right\|}{t} &\le \frac{\left\|e^{I+tF+R} - e^{I+tF}\right\|}{t} + \frac{\left\|e^{I+tF} - I - tF\right\|}{t} \\ &\le c\left(\frac{||R||}{t} + t\,||F||^2\right) \to 0, \end{aligned}$$

for some constant c, independent of t. In other words, $E(t)$ is differentiable at 0 and $E'(0) = F$. If, additionally, $X(t)$ has the property

$$X(s)X(t) = X(s+t), \tag{1.1}$$

so $X(t)$ commute, then F commutes with $X(t)$ and derivatives of arbitrary orders exist at every point. Moreover,

$$\frac{d^n X(t)}{dt^n} = F^n\,E(t), \qquad E(t) = e^{t\,F}. \tag{1.2}$$

The matrix F is called the *generator* of the matrix-valued process $X(t)$.

The spectral representation of the Hermitian square, $X^*X = UD^2U^*$, where U is a unitary matrix and $D = \operatorname{diag}(s_k)$, yields the square root $P = \sqrt{X^*X} \overset{\mathsf{def}}{=} UDU^*$, i.e., $P^2 = X^*X$. A nonsingular X (so P is also nonsingular) admits the unique polar representation $X = UP$, with the unitary matrix $U = XP^{-1}$.

1.1.1. Rodrigues formula. The matrix calculus, involving analytic functions $f(A)$ with matrix argument, has been of great interest (cf., for example [6], [12], and references therein) since the origins of linear algebra, with the exponential e^Z playing the leading role.

While the diagonalization is of utmost importance, the pattern of powers Z^n is also extremely useful. For example, for a c-idempotent matrix Z (i.e., $Z^2 = cZ$, so for $c = 1$ the matrix Z represents a projection, while Z is nilpotent of order 2 for $c = 0$) one obtains the formula

$$f(tZ) = \begin{cases} f(0)\,I + \dfrac{f(ct) - f(0)}{c}\,Z, & \text{for } c \ne 0, \\ f(0)\,I + f'(0)\,Z, & \text{for } c = 0, \end{cases}$$

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valid for $|t| < r$ for some $r \in [0, \infty]$. Similar albeit more complicated formulas result from more general properties. Suppose that Z is (c,k)-idempotent, i.e., $Z^k = c\,Z$ for a scalar $c = c_k$, where k is the smallest integer exponent with this property. Then a neat pattern of powers Z^n entails a more sophisticated version of the above formula. For example, if $k = 3$ and $c = \pm\, a^2$, then

$$(1.3) \qquad e^{tZ} = \begin{cases} I + \dfrac{\sin at}{a}\, Z + \dfrac{1-\cos at}{a^2}\, Z^2 & \text{when } c < 0 \qquad \text{(a)} \\ I + \dfrac{\sinh at}{a}\, Z + \dfrac{\cosh at - 1}{a^2}\, Z^2 & \text{when } c > 0 \qquad \text{(b)} \\ I + tZ + \dfrac{t^2}{2}\, Z^2 & \text{when } c = 0 \qquad \text{(c)} \end{cases} .$$

EXAMPLE 1.1. A vector $\mathbf{h}$ generates a singular skew symmetric matrix

$$V = V(\mathbf{h}) \overset{\mathsf{def}}{=} \begin{bmatrix} 0 & h_3 & -h_2 \\ -h_3 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{bmatrix}.$$

In other words, $V(\mathbf{h})\mathbf{u} = \mathbf{u} \times \mathbf{h}$. Let $\mathbf{h}$ be a unit vector and θ be a real scalar. Then $V^3 = -V$, so

$$e^{-\theta\, V(\mathbf{h})} = I - \sin\theta\, V(\mathbf{h}) + (1-\cos\theta)\, V^2(\mathbf{h}),$$

which is the classical Rodrigues formula for a matrix representation of the positive (according to the Right Hand Rule) rotation by an angle θ about the unit vector $\mathbf{h}$. Conversely, if R is orthogonal with $\det(R) = 1$, then R is a rotation by θ about an axis $\mathbf{h}$, so $R = e^{-\theta V(\mathbf{h})}$.

We note the basics properties of the cross product

$$(1.4) \qquad V(\mathbf{h})\mathbf{d} + V(\mathbf{d})\mathbf{h} = \mathbf{0}, \qquad V(\mathbf{h})V(\mathbf{d}) = \mathbf{h}\mathbf{d}^{\mathsf{T}} - (dh)\, I,$$

The three matrices I, V, V^2 form a basis of the span of powers of V. Since $V^2(\mathbf{h}) = \mathbf{h}\mathbf{h}^{\mathsf{T}} - I$, hence I, V, $\mathbf{h}\mathbf{h}^{\mathsf{T}}$ is a basis as well. Therefore,

$$e^{-\theta\, V(\mathbf{h})}\mathbf{u} = \cos\theta\, \mathbf{u} - \sin\theta\, \mathbf{u} \times \mathbf{h} + (1-\cos\theta)\, (uh)\, \mathbf{h}.$$

1.1.2. Maxwell equations and Lorentz transformation. The matrix

$$G \overset{\mathsf{def}}{=} \operatorname{diag}(1,-1,-1,-1) = \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix}$$

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entails the G-*transpose* $X^{\mathsf{G}} \overset{\mathsf{def}}{=} GX^{\mathsf{T}}G$. That is,

$$\begin{bmatrix} c & \mathbf{y}^{\mathsf{T}} \\ \mathbf{x} & S \end{bmatrix}^{\mathsf{G}} = \begin{bmatrix} c & -\mathbf{x}^{\mathsf{T}} \\ -\mathbf{y} & S^{\mathsf{T}} \end{bmatrix}.$$

Clearly, $(XY)^{\mathsf{G}} = Y^{\mathsf{G}}X^{\mathsf{G}}$, and we arrive at the notions of a G-symmetric ($X^{\mathsf{G}} = X$) or G-skew symmetric ($X^{\mathsf{G}} = -X$) matrix. Every G-skew symmetric 4×4 matrix has the null diagonal form

$$F = F(\mathbf{d}, \mathbf{h}) \overset{\mathsf{def}}{=} \begin{bmatrix} 0 & \mathbf{d}^{\mathsf{T}} \\ \mathbf{d} & V(\mathbf{h}) \end{bmatrix}, \quad \text{where} \quad V^{\mathsf{T}} = -V, \tag{1.5}$$

so we may perceive $(\mathbf{d}, \mathbf{h}) \mapsto F$ as a linear mapping from $(\mathbb{R}^3)^2$ into $\mathbb{R}^{4\times 4}$.

Now, let us examine the pattern of Maxwell equations. For a $\mathbb{R}^3 \mapsto \mathbb{R}^3$ vector field $\mathbf{F}$ we use the formal notation div $\mathbf{F} = \nabla \cdot \mathbf{F}$ and $\mathbf{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$. We say that two vectors fields $\mathbf{B}$ and $\mathbf{E}$ of the variable $(t; \mathbf{x})$ satisfy *Maxwell's equations* (cf. Gottlieb [5]) with a scalar ρ and a vector field $\mathbf{J}$ such that

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \tag{1.6}$$
$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}, \tag{1.7}$$
$$\nabla \cdot \mathbf{E} = \rho, \tag{1.8}$$
$$\nabla \cdot \mathbf{B} = 0. \tag{1.9}$$

Using the Reverse Polish Notation (where the operator follows the operand) we rewrite Maxwell's equations in the matrix form, using the aforementioned operator $F(\mathbf{E}, \mathbf{B})$ and its G-conjugate $\tilde{F} = F(\mathbf{B}, -\mathbf{E})$:

$$\begin{aligned} &(1.7)\ \&\ (1.8): && \begin{bmatrix} 0 & \mathbf{E}^{\mathsf{T}} \\ \mathbf{E} & V(\mathbf{B}) \end{bmatrix} \begin{bmatrix} -\partial_t \\ \nabla \end{bmatrix} = \begin{bmatrix} \rho \\ \mathbf{J} \end{bmatrix}, \\ &(1.6)\ \&\ (1.9): && \begin{bmatrix} 0 & \mathbf{B}^{\mathsf{T}} \\ \mathbf{B} & -V(\mathbf{E}) \end{bmatrix} \begin{bmatrix} -\partial_t \\ \nabla \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}. \end{aligned}$$

The fundamental invariance of Maxwell's equations under the Lorentz transformation is well known (cf., for example [8, Sect.1.8] for an elementary yet rigorous justification). Therefore, we call a nonzero matrix (1.5) a *Maxwell matrix*. On the other hand, by a *Lorentz matrix* we understand a G-orthogonal matrix X, i.e., $X^{-1} = X^{\mathsf{G}}$, and we call it *proper*, if $\det X = 1$ and $X_{00} > 0$ (the rows and columns are enumerated $0, 1, 2, 3$ to distinguish the temporal variable from the three spatial variables).

2. Eigen-system of G-skew symmetry.

2.1. Eigenvalues and skew conjugate.

PROPOSITION 2.1. *Let F be a 4×4 Maxwell matrix.*

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(a) *The eigenvalues are* $\pm\sigma$, $\pm i\,\theta$ *for some real parameters* σ, θ, *which satisfy an elementary system of quadratic equations, solvable at will:*

$$\sigma^2 - \theta^2 = d^2 - h^2, \qquad \sigma^2\,\theta^2 = (dh)^2. \tag{2.1}$$

(b) *Additionally,* $(h^2-\theta^2)(h^2+\sigma^2) = h^2 d^2 - (dh)^2 = ||\mathbf{d}\times\mathbf{h}||^2 \geq 0$. *In particular,* $\sigma^2 \leq d^2$ *and* $\theta^2 \leq h^2$, *and either equality occurs if and only if* $\mathbf{d}||\mathbf{h}$.

(c) F *is diagonalizable if and only if at least one of the eigenvalues is nonzero. The quadruple zero eigenvalue occurs if and only if* $\mathbf{d}\perp\mathbf{h}$ *and* $d = h$, *in which case we call* F *hyper-singular.*

Proof. Since F is G-skew symmetric, the eigenvalue pattern follows directly from the determinant, computable by routine:

$$\det(F) = -(dh)^2 \tag{2.2}$$

Since the double eigenvalues of F^2 are σ^2 and $-\theta^2$, so

$$det(F^2) = \sigma^4\,\theta^4, \qquad \mathrm{tr}\,(F^2) = 2(\sigma^2 - \theta^2).$$

On the other hand, a little calculation shows that $\mathrm{tr}\,(F^2) = 2(d^2 - h^2)$ and with (2.2) in mind we arrive at the stated equations.

Equations (2.1) entail the formula in part (b), which in turn yields the corollaries.

Regarding part (c), a nonsingular Maxwell matrix with four distinct eigenvalues is clearly diagonalizable. So it is when only one parameter is 0, due to the rank at least 2. Finally, a nonzero matrix with the quadruple zero eigenvalue does not posses an eigenbasis. ∎

Remark 2.2. We have $(dh) = \pm\sigma\theta$. The sign issue was present and resolved in Example 1.1. In the Rodrigues formula the negative sign in the exponent has ensured the right orientation of the 3-space, which has agreed with the order of factors in the cross product operator $V(\mathbf{h})\mathbf{u} = \mathbf{u}\times\mathbf{h}$. Of course, the inverse order could be chosen as well but then it would affect the Rodrigues formula and possibly most of the formulas in this paper. Once the choice is made, all sign set-ups must be consistent with it. Therefore, to prevent the unnecessary ambiguity that may arise in the case of nonzero eigenvalues, when associating sigma and theta with a given F we henceforth will assume that $\sigma \geq 0$ and choose the symbol θ for the eigenvalue $\pm\theta i$ for which $(dh) = \sigma\theta$ (see Example 2.3 below).

We call F *normalized* if $\theta^2 + \sigma^2 = 1$, which can be assumed without loss of generality for most of these notes. For any function Φ on $(\mathbb{R}^3)^2$, we define the skew conjugate

$$\tilde{\Phi}(\mathbf{u},\mathbf{v}) = \Phi(\mathbf{v}, -\mathbf{u}).$$

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In particular, we obtain a G-skew symmetric

$$\tilde{F} = F(\mathbf{h}, -\mathbf{d}) = \begin{bmatrix} 0 & \mathbf{h}^{\intercal} \\ \mathbf{h} & -V(\mathbf{d}) \end{bmatrix} \quad \Rightarrow \quad \tilde{\tilde{F}} = -F.$$

From the properties (1.4) of the cross product, we infer that

$$\begin{array}{lll} & \text{(a)} & F\tilde{F} = \tilde{F}F = (dh)\, I = \sigma\theta\, I, \\ (2.3) & \text{(b)} & F^2 - \tilde{F}^2 = (d^2 - h^2)\, I = (\sigma^2 - \theta^2)\, I, \\ & \text{(c)} & F^3 = (d^2 - h^2)\, F + (dh)\, \tilde{F} = (\sigma^2 - \theta^2)\, F + \sigma\theta\, \tilde{F}. \end{array}$$

Hence, if $\mathbf{d} \perp \mathbf{h}$, the algebra generated by F, i.e., the vector space spanned by the powers F^n, is 3-dimensional with a basis I, F, F^2, while for nonorthogonal vectors the span is 4-dimensional with a basis I, F, F^2, $\tilde{F}$.

EXAMPLE 2.3. Let $\mathbf{d} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\intercal}$, $\mathbf{h} = \begin{bmatrix} 1 & -2 & -1 \end{bmatrix}^{\intercal}$. Thus, $(dh) = -2$ and

$$F = \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & -2 & -1 & 0 \end{array}\right], \quad \tilde{F} = \left[\begin{array}{c|ccc} 0 & 1 & -2 & -1 \\ \hline 1 & 0 & -1 & 1 \\ -2 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{array}\right].$$

The eigenvalues are $\pm 1, \pm 2i$, so according to our convention stated in Remark 2.2 we choose $\sigma = 1$, $\theta = -2$. In particular, the product $F\tilde{F} = -2I = (dh)I$. Notice that F needs the factor $1/\sqrt{5}$ to become normalized.

PROPOSITION 2.4. *$F^3 = 0$ for a hyper-singular F, which also entails the exponential* (1.3).(c).

As noted after (2.3), a Maxwell matrix F generates a 4-dimensional algebra in which the pattern of powers quickly becomes rather intricate. Therefore, a change to an adequate basis may resolve this issue. For a normalized F with eigenvalues $\pm\,\sigma$, $\pm\, i\theta$, and choosing $(dh) = \sigma\theta$, we define

$$Z = \theta\, F - \sigma\, \tilde{F},$$

which implies that $\tilde{Z} = \sigma\, F + \theta\, \tilde{F}$, together forming the rotation

$$\begin{bmatrix} Z \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} \theta & -\sigma \\ \sigma & \theta \end{bmatrix} \begin{bmatrix} F \\ \tilde{F} \end{bmatrix}.$$

Applying the inverse rotation,

$$F = \theta Z + \sigma \tilde{Z} \quad \text{and} \quad \tilde{F} = -\sigma Z + \theta \tilde{Z}.$$

Then, from the definition and formulas (2.3) we obtain the following relations which no doubt simplify the patterns of (2.3), and can be extended easily to higher

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powers and, consequently, to the exponential in an elementary way (cf. (1.3)).

$$\begin{aligned} &\text{(a)} \quad Z^3 = -Z, \\ &\text{(b)} \quad \tilde{Z}^3 = \tilde{Z}, \\ &\text{(c)} \quad Z\tilde{Z} = \tilde{Z}Z = 0. \end{aligned} \tag{2.4}$$

Proposition 2.5. *In general, the representation $F = aX + bY$ is unique, where a, b are nonzero, $XY = 0$, $X^3 = -X$ and $Y^3 = Y$.*

Proof. The components satisfy also the linear equation $F^3 = -a^3\, X + b^3\, Y$. Hence, the underlying 2×2 matrix is invertible, resulting in the unique solutions

$$X = \frac{1}{a(a^2+b^2)}\left(b^2\, F - F^3\right) \quad \text{and} \quad Y = \frac{1}{b(a^2+b^2)}\left(a^2\, F + F^3\right). \quad \square$$

Proposition 2.6. *For a normalized F we obtain the G-orthogonal exponentials*

$$e^{tF} = e^{t\theta Z} e^{t\sigma\tilde{Z}} = I + \sin t\theta\, Z + (1 - \cos t\theta)\, Z^2 + \sinh t\sigma\, \tilde{Z} + (\cosh t\sigma - 1)\, \tilde{Z}^2.$$

Alternatively, we may substitute

$$Z^2 = F^2 - \sigma^2\, I \quad \text{and} \quad \tilde{Z}^2 = F^2 + \theta^2\, I. \tag{2.5}$$

Proof. The pattern of cubes (2.4), as noted in (1.3), entails the formulas

$$\begin{aligned} e^{t\theta Z} &= I + \sin t\theta\, Z + (1 - \cos t\theta\,)Z^2, \\ e^{t\sigma\tilde{Z}} &= I + \sinh t\sigma\, \tilde{Z} + (\cosh t\sigma - 1)\tilde{Z}^2. \end{aligned}$$

The full formula follows since the orthogonal components commute. The substitution is a consequence of (2.3). $\square$

Example 2.7. The normalization, followed by the change of basis (the replacement of F, $\tilde{F}$ by Z, $\tilde{Z}$), greatly simplify arguments and clarifies formulas. Yet, when it comes to numerical data one should expect to pay a price in regard to appearance, which we will illustrate, continuing Example 2.3. First we normalize F, dividing $\sigma = 1$ and $\theta = -2$ by $\sqrt{5} = \sqrt{\sigma^2 + \theta^2}$. Hence, the normalized F and $\tilde{F}$ become

$$F = \frac{1}{\sqrt{5}} \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & -2 & -1 & 0 \end{array}\right], \quad \tilde{F} = \frac{1}{\sqrt{5}} \left[\begin{array}{c|ccc} 0 & 1 & -2 & -1 \\ \hline 1 & 0 & -1 & 1 \\ -2 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{array}\right].$$

The normalized $\sigma = \frac{1}{\sqrt{5}}$, $\theta = -\frac{2}{\sqrt{5}}$ entail the normalized matrices Z, $\tilde{Z}$:

$$Z = \theta F - \sigma\tilde{F} = \frac{1}{5} \begin{bmatrix} 0 & -3 & 0 & -1 \\ -3 & 0 & 3 & -5 \\ 0 & -3 & 0 & -1 \\ -1 & 5 & 1 & 0 \end{bmatrix}, \quad \tilde{Z} = \sigma F + \theta\tilde{F} = \frac{1}{5} \begin{bmatrix} 0 & -1 & 5 & 3 \\ -1 & 0 & 1 & 0 \\ 5 & -1 & 0 & 3 \\ 3 & 0 & -3 & 0 \end{bmatrix}.$$

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The eigenvalues of Z and $\tilde{Z}$ are, respectively, $(i, -i, 0, 0)$ and $(-1, 1, 0, 0)$. In order to display exponentials explicitly we denote

$$c = \cos\frac{2t}{\sqrt{5}}, \quad s = \sin\frac{2t}{\sqrt{5}}, \quad S = \sinh\frac{t}{\sqrt{5}}, \quad C = \cosh\frac{t}{\sqrt{5}}.$$

The direct computations yield the formulas

$$\begin{aligned}\exp(t\theta Z) &= I + \sin(t\theta)Z + (1 - \cos(t\theta))Z^2 \\ &= \frac{1}{5}\begin{bmatrix} -2c+7 & c+3s-1 & 2c-2 & -3c+s+3 \\ -c+3s+1 & 5c & c-3s-1 & 5s \\ -2c+2 & c+3s-1 & 2c+3 & -3c+s+3 \\ 3c+s-3 & -5s & -3c-s+3 & 5c \end{bmatrix},\end{aligned}$$

$$\begin{aligned}\exp(t\sigma\tilde{Z}) &= I + \sinh(t\sigma)\tilde{Z} + \cosh(t\sigma) - I)\tilde{Z}^2 \\ &= \frac{1}{5}\begin{bmatrix} 7C-2 & -C-S+1 & -2C+5S+2 & 3C+3S-3 \\ C-S-1 & 5 & -C+S+1 & 0 \\ 2C+5S-2 & -C-S+1 & 3C+2 & 3C+3S-3 \\ -3C+3S+3 & 0 & 3C-3S-3 & 5 \end{bmatrix}.\end{aligned}$$

Without referring to the "theory", one may verify directly that product orthogonal Z and $\tilde{Z}$ commute, so do these two one-parameter subgroups of the Lorentz group:

$$\exp(t\theta Z)\exp(s\sigma\tilde{Z}) = \exp(s\sigma\tilde{Z})\exp(t\theta Z).$$

Corollary 2.8. *Let F be normalized. Then*

(a) $\det(e^{tF}) = 1$*;*
(b) *denoting by δ_t the $(0, 0)$-entry in the matrix e^{tF}, we have*

$$\delta_t = 1 + (1 - \cos t\theta)(d^2 - \sigma^2) + (\cosh t\sigma - 1)(d^2 + \theta^2) \geq 1,$$

and the equality occurs if and only if $\sigma = 0$ ($\mathbf{d} \perp \mathbf{h}$) and $t\theta = 2n\pi$, or $\mathbf{d} = \mathbf{0}$ and $\theta = 0$;
(c) $\operatorname{tr} e^{tF} = 2(\cos t\theta + \cosh t\sigma) \geq 0$, *with the equality occurring if and only if $t\theta = (2n+1)\pi$ and $\sigma = 0$.*

Proof. (a) is obvious, (b) follows from (2.5) in virtue of Proposition 2.1.(c), and (2.5) (or the shape of the eigenvalues $e^{\pm\sigma}$, $e^{\pm i\theta}$) imply (c). □

2.2. More on orthogonal decomposition. Although the orthogonal decomposition is unique yet there are alternative ways to find the components. Let m

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denote the number of nonzero distinct eigenvalues of F which we order into a sequence $\lambda_k, k = 1, \ldots, m$. For a diagonalizable matrix F, the diagonalization formula $F = VDV^{-1}$ yields the orthogonal decomposition

$$F = \sum_{k=1}^{m} \lambda_k X_k,$$

where $X_k = VE_kV^{-1}$, $k = 1, \ldots, m$, and E_k are mutually orthogonal diagonal projections. For a single nonzero eigenvalue λ_k, E_k has the single 1 at the k^{th} position. Since the components X_k commute and, like E_k, are mutually orthogonal projections, i.e., $X_k^2 = X_k$, then

$$e^F = I + \sum_{k=1}^{m} \left(e^{\lambda_k} - 1\right) X_k.$$

We now rephrase Proposition 2.5:

Proposition 2.9. *A diagonalizable G-skew symmetric matrix $F = F(\mathsf{d}, \mathsf{h})$ admits a unique orthogonal decomposition $F = X + Y$ such that*

$$X^3 = \sigma^2 X \quad \textit{and} \quad Y^3 = -\theta^2 Y, \tag{2.6}$$

where the G-skew symmetric components are given explicitly by the formulas

$$X = \frac{\theta^2}{\sigma^2 + \theta^2} F + \frac{1}{\sigma^2 + \theta^2} F^3, \qquad Y = \frac{\sigma^2}{\sigma^2 + \theta^2} F - \frac{1}{\sigma^2 + \theta^2} F^3. \tag{2.7}$$

If F is singular, then the decomposition is trivial, i.e., either $X = 0$ or $Y = 0$.

Proof. We pool the pairs of projections together

$$X = \sigma(X_1 - X_2), \quad Y = i\theta(Y_1 - Y_2),$$

and check directly their listed properties. Formulas (2.7) come as solutions of the equations

$$F = X + Y, \quad F^3 = X^3 + Y^3 = \sigma^2 X - \theta^2 Y,$$

and show that X and Y are G-skew symmetric. ☐

Corollary 2.10. *The exponential of a G-skew symmetric matrix is G-orthogonal and*

$$\Lambda = e^F = I + \frac{\sinh\sigma}{\sigma} X + \frac{\cosh\sigma - 1}{\sigma^2} X^2 + \frac{\sin\theta}{\theta} Y + \frac{1 - \cos\theta}{\theta^2} Y^2. \tag{2.8}$$

For a singular F with $\sigma^2 + \theta^2 > 0$, either the X-part or the Y-part vanishes. For a hyper-singular F, we obtain

$$e^F = I + F + \frac{1}{2} F^2.$$

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Proof. The regular pattern of the power series ensured by (2.6) yields the exponential. By the same token, $G(F^\intercal)^n G = (-1)^n F^n$. Hence, and by (2.7), where the signs are changed only at the sine and hyperbolic sine, $G\Lambda^\intercal G = Ge^{F^\intercal}G = e^{-F} = \Lambda^{-1}$.

Due to continuity, the exponentials carry over to the simplified formulas as θ or σ (or both) converge to 0. Also, we have already used the pattern $F^3 = 0$ in Proposition 2.4. ◻

If m is the number of nonzero distinct eigenvalues, then the components are solutions of the linear equations

$$\sum_{k=1}^{m} \lambda_k^p X_k = F^p, \quad p = 1, \ldots, m,$$

with the invertible Vandermonde $m \times m$ matrix $M = [\lambda_k^p]$. In particular, the exponential e^F is a linear combination of linearly independent powers F^p, $p = 0, \ldots, m$.

Let F be normalized and nonsingular, i.e., $m = 4$. Then, displaying the Vandermonde matrix

$$M = \begin{bmatrix} \sigma & -\sigma & i\theta & -i\theta \\ \sigma^2 & \sigma^2 & -\theta^2 & -\theta^2 \\ \sigma^3 & -\sigma^2 & -i\theta^3 & i\theta^3 \\ \sigma^4 & \sigma^4 & -\theta^4 & \theta^4 \end{bmatrix},$$

we arrive at its inverse, verifiable directly:

$$M^{-1} = \frac{1}{2} \begin{bmatrix} \sigma^{-2} & 0 & 0 & 0 \\ 0 & \sigma^{-2} & 0 & 0 \\ 0 & 0 & \theta^{-2} & 0 \\ 0 & 0 & 0 & \theta^{-2} \end{bmatrix} \cdot \begin{bmatrix} \sigma\theta^2 & \theta^2 & \sigma & 1 \\ -\sigma\theta^2 & \theta^2 & -\sigma & 1 \\ -i\sigma^2\theta & -\sigma^2 & i\theta & 1 \\ i\sigma^2\theta & -\sigma^2 & -i\theta & 1 \end{bmatrix}.$$

In other words,

$$\begin{aligned} X_1 &= \frac{1}{2\sigma^2} \Big\{ \left(F^4 + \theta^2 F^2\right) + \sigma \left(F^3 + \theta^2 F\right) \Big\}, \\ X_2 &= \frac{1}{2\sigma^2} \Big\{ \left(F^4 + \theta^2 F^2\right) - \sigma \left(F^3 + \theta^2 F\right) \Big\}, \\ X_3 &= \frac{1}{2\theta^2} \Big\{ \left(F^4 - \sigma^2 F^2\right) + i\theta \left(F^3 - \sigma^2 F\right) \Big\}, \\ X_4 &= \frac{1}{2\theta^2} \Big\{ \left(F^4 - \sigma^2 F^2\right) - i\theta \left(F^3 - \sigma^2 F\right) \Big\}. \end{aligned} \tag{2.9}$$

As mentioned before, instead of the basis consisting of powers up to the fourth power,

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which may be somewhat cumbersome to compute, we may switch to the simpler basis I, F, F^2, $\tilde{F}$.

Corollary 2.11. *In the new basis, formulas* (2.9) *read:*

$$\begin{aligned}
X_1 &= \frac{1}{2}\Big\{ \left(\theta^2 I + F^2\right) + \ \left(\sigma F + \theta \tilde{F}\right) \Big\},\\
X_2 &= \frac{1}{2}\Big\{ \left(\theta^2 I + F^2\right) - \ \left(\sigma F + \theta \tilde{F}\right) \Big\},\\
X_3 &= \frac{1}{2}\Big\{ \left(\sigma^2 I - F^2\right) - i \left(\theta F - \sigma \tilde{F}\right) \Big\},\\
X_4 &= \frac{1}{2}\Big\{ \left(\sigma^2 I - F^2\right) + i \left(\theta F - \sigma \tilde{F}\right) \Big\}.
\end{aligned}$$

The exact formulas (2.7) *now take an alternative form:*

$$X = \sigma^2 Z + \sigma\theta \tilde{Z}, \qquad Y = \theta^2 Z - \sigma\theta \tilde{Z},$$

which linearizes the exponential (2.8).

Proof. Use F^3 and compute F^4 from (2.3). ◻

2.3. Eigenvectors. Recall that we choose σ and θ to satisfy $(dh) = \sigma\theta$. In what follows without loss of generality we may and do assume that $h = 1$. To return to the general case (or to examine the case $h = 0$) it suffices to substitute $\sigma := \sigma/h$, $\theta := \theta/h$, then the case $h = 0$ can be handled in the limit. Relations (2.1) simplify further even already simple properties:

$$\begin{aligned}
(\mathbf{d} \times \mathbf{h}) \times \mathbf{h} \quad &= -h^2 \mathbf{d} + (dh)\,\mathbf{h} \quad &&= -\mathbf{d} + \sigma\theta\,\mathbf{h},\\
\zeta \overset{\text{def}}{=} \|\mathbf{d} \times \mathbf{h}\| \quad &= \sqrt{d^2h^2 - (dh)^2} \quad &&= \sqrt{(1-\theta^2)(1+\sigma^2)}.
\end{aligned} \tag{2.10}$$

2.3.1. The regular case. First we assume that neither $\mathbf{d}$ and $\mathbf{h}$ are parallel, nor they are orthogonal, i.e., $\zeta > 0$ and $(dh) = \sigma\theta \neq 0$. The assumption yields an orthonormal basis of $\mathbb{R}^4$:

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ \mathbf{h} \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\zeta} \begin{bmatrix} 0 \\ (\mathbf{d} \times \mathbf{h}) \times \mathbf{h} \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{\zeta} \begin{bmatrix} 0 \\ \mathbf{d} \times \mathbf{h} \end{bmatrix}.$$

We immediately find the matrix representation of F:

$$\left[\begin{array}{c|ccc}
0 & (dh) & -\zeta & 0 \\ \hline
(dh) & 0 & 0 & 0 \\
-\zeta & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{array}\right].$$

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To display eigenvectors we will use the function

$$c = c(\sigma, \theta) = \sqrt{\frac{1-\theta^2}{1+\sigma^2}}.$$

Recall that $(dh) = \sigma\theta$ and $h = 1$. Then eigenvectors are:

$$\begin{aligned} \pm\,\sigma \quad &\sim \quad \Big(\ \mathbf{v}_0 + c\,\mathbf{v}_3\Big) \pm \ \Big(\ \theta\,\mathbf{v}_1 - \sigma c\,\mathbf{v}_2\Big), \\ \pm\, i\theta \quad &\sim \quad \Big(c\,\mathbf{v}_0 + \ \mathbf{v}_3\Big) \mp i\,\Big(\sigma c\,\mathbf{v}_1 + \ \ \theta\,\mathbf{v}_2\Big). \end{aligned} \tag{2.11}$$

(Let us repeat that in order to pass from the normalized case $h = 1$ to the general case it suffices to substitute $\sigma := \sigma/h, \theta := \theta/h$.) By (2.1) and (2.10) the complex Euclidean norm of both vectors equals $\sqrt{2}$.

2.3.2. An insight into hyper-singularity. With exception of the quadruple null eigenvalue: $\sigma = \theta = 0$, the boundary cases: $\mathbf{d} \perp \mathbf{h}$ and $\mathbf{d}||\mathbf{h}$ are obtained in the limit from the above representations. We will specify these cases but first let us comment on the issue.

The matrix $F = F_p$ is a continuous function of its vector parameter $p = (\mathbf{d}, \mathbf{h}) \in \mathbb{R}^6$. Denote by E the set of all eigenvectors of F_p, for all p, and by E^0 the set of eigenvectors belonging to the open set $P^0 = \{\, (dh) \neq 0 \text{ and } \zeta > 0\,\}$. The question is whether an eigenvector $\mathbf{e}_p$ is a cluster point of E^0. If this happens, we may call the system *connected* at p. In the opposite case, not only a matrix is singular but the eigenvector is separated from the rest of potential eigenvectors, justifying again the name "hyper-singular".

1. *The connected case:*
   (a) $d = 0$, so $\theta = h, \sigma = 0$. Let $\mathbf{h}_1, \mathbf{h}, \mathbf{h}_3$ form a positively oriented orthonormal basis of $\mathbb{R}^3$. Then

   $$0 \quad \sim \quad \begin{bmatrix} 0 \\ \mathbf{h} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}, \qquad i\theta \quad \sim \quad \begin{bmatrix} 0 \\ \mathbf{h}_1 \end{bmatrix} + i \begin{bmatrix} 0 \\ \mathbf{h}_3 \end{bmatrix}.$$

   Now let $h = 1$ and let $\sigma \to 0$ in (2.11). Hence, $\theta \to 1$, so $c \to 0$ and $\zeta \to 0$. The eigenvectors for $\pm\,\sigma$ actually converge to the vectors listed above. However, the normalized vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ move within the unit circle on the plane orthogonal to $\mathbf{h}$, mapping a path along which $\mathbf{d}$ converges to $\mathbf{0}$.
   In other words, the given particular vectors may not converge but their eigenspaces do. In fact, by a routine compactness argument there exists a discrete orbit convergent to the complex vector such as one listed above.

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(b) $h = 0$, so $\theta = 0, d = \sigma$. Choose two orthonormal vectors $\mathbf{d}^{\perp}$ orthogonal do $\mathbf{d}$. Then

$$0 \quad \sim \quad \begin{bmatrix} 0 \\ \mathbf{d}^{\perp} \end{bmatrix}, \qquad \pm\sigma \quad \sim \quad \begin{bmatrix} \pm d \\ \mathbf{d} \end{bmatrix}.$$

By duality $Z \mapsto \tilde{Z}$ and the first part the system is connected here.

(c) $d > 0$, $h > 0$, $(dh) = 0$ but only one eigenvalue is 0. Formula (2.11) covers this case. Proposition that the undetermined ratio $\frac{\sigma\theta}{\lambda}$ could be interpreted by continuity if we adopt the convention $\frac{0}{0} = 1$.

(d) $\mathbf{d}||\mathbf{h}$ and $d > 0$, $h > 0$. Let us augment $\mathbf{h}$ to form a positively oriented orthonormal basis $\mathbf{h}, \mathbf{h}_2, \mathbf{h}_3$ in $\mathbb{R}^3$. Then

$$\sigma = \pm d \quad \sim \quad \begin{bmatrix} \sigma \\ \mathbf{d} \end{bmatrix}, \qquad i\theta = ih \quad \sim \quad \begin{bmatrix} 0 \\ \mathbf{h}_2 \end{bmatrix} + i \begin{bmatrix} 0 \\ \mathbf{h}_3 \end{bmatrix}.$$

By (2.11), since $c^2 = \zeta^2/(1+\sigma^2) \to 0$ as $\zeta \to 0$, the system is connnected here.

2. *The disconnected case:*
$d > 0$, $h > 0$ but $\sigma = \theta = 0$. Then

$$0 \quad \sim \quad \begin{bmatrix} 0 \\ \mathbf{h} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h^2 \\ \mathbf{d}\times\mathbf{h} \end{bmatrix}.$$

Let $h = 1$ and $\theta, \sigma \to 0$. Then $d^2 - \sigma^2\theta^2 = \zeta^2 - (1-\theta^2)(1+\sigma^2) \to 1$. Hence, $d \to 1$. Also, $c \to 1$. Thus, all eigenvectors in (2.11) converge to $\mathbf{v}_3 + \mathbf{v}_0$, the second vector listed above, up to a multiplier.
Only the second eigenvector is preserved in the limit and the first eigenvector is separated, i.e., the complex Euclidean distance between $\mathbf{v}_1$ and the span of either eigenvector is 1.

3. Eigen-system of a Lorentz matrix Λ. We will examine a Lorentz matrix by studying its components, focusing on the eigenvalues and eigenvectors. We observed previously (cf. Subsection 2.3.2) that a component of a continuous "orderly" matrix may generate singularities or discontinuities. We will encounter this phenomenon again. For example, the eigen-system of the matrix A, described below, will also be disconnected on the boundary. A hidden rotation R in the interior of the manifold allows only angles $\rho \in (-\pi/2, \pi/2)$, yet on the boundary, where $s = 1$, it will exhibit a jump to the rotation by $\rho = \pi$.

In this section, we do not restrict the dimension d of the Euclidean space until it will be required by an argument. The results are still elementary, partially due to our avoidance of the sophisticated context of Lie algebras, Lie groups, and the general matrix theory (cf. also [1]).

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3.1. ***G*-orthogonal matrix.** Again, a matrix Λ is G-orthogonal if $\Lambda^{\mathsf{G}} = G\Lambda^{\intercal}G = \Lambda^{-1}$. That is,

$$\text{(3.1)} \qquad \text{(a)} \quad G\Lambda G^{\intercal}\Lambda = I \quad \text{and/or} \quad \text{(b)} \quad \Lambda G\Lambda^{\intercal}G = I.$$

Clearly, $\det\Lambda \neq 0$ and, since an eigenvalue λ entails the eigenvalue $1/\lambda$, $|\det(\Lambda)| = 1$.

The product of G-orthogonal matrices is G-orthogonal. Let us consider the 1:3 (time-space) split of a G-orthogonal matrix:

$$\Lambda = \begin{bmatrix} s & \mathbf{q}^{\intercal} \\ \mathbf{p} & A \end{bmatrix}.$$

Definition (3.1) gives characterizing equations

$$\text{(3.2)} \qquad \begin{array}{lrclrcl} \text{(i)} & A^{\intercal}\mathbf{p} & = & s\,\mathbf{q}, & A\mathbf{q} & = & s\,\mathbf{p}, \\ \text{(ii)} & A^{\intercal}A & = & \mathbf{q}\mathbf{q}^{\intercal} + I, & AA^{\intercal} & = & \mathbf{p}\mathbf{p}^{\intercal} + I, \\ \text{(iii)} & p^2 & = & s^2 - 1, & q^2 & = & s^2 - 1, \end{array}$$

where both columns are deducible from each other. Obviously, $s^2 \geq 1$ and

$$\det^2(A) = 2s^2 - 1 \quad \text{and} \quad A^{-1} = A^{\intercal} - \frac{1}{s}\,\mathbf{q}\mathbf{p}^{\intercal}.$$

By (3.2).(ii), the eigen-system of $A^{\intercal}A$ and $AA^{\intercal}$ is

$$\begin{array}{lcccccc} A^{\intercal}A: & 1 & \sim & \mathbf{q}^{\perp}, & s^2 & \sim & \mathbf{q}, \\ AA^{\intercal}: & 1 & \sim & \mathbf{p}^{\perp}, & s^2 & \sim & \mathbf{p}. \end{array}$$

Proposition 3.1. *A is orthogonal if and only if $|s| = 1$ if and only if $\mathbf{p} = \mathbf{q} = \mathbf{0}$.*

Proof. Clearly, if $|s| = 1$ or either vector is zero (hence both), then $A^{\intercal}A = AA^{\intercal} = I$. Conversely, if A is orthogonal, then $b^2 = \mathbf{p}\mathbf{p}^{\intercal} = 0$, i.e., $\mathbf{p} = \mathbf{q} = \mathbf{0}$, and $|s| = |s\,\det(A)| = |\det(\Lambda)| = 1$. ∎

Proposition 3.2. *Further, either of the above conditions, augmented by the requirement $\det(\Lambda) = 1$ and $s > 0$, occurs if and only if $A = e^{\theta V(\mathbf{h})}$ and $s = 1$. In particular, there is a G-skew symmetric matrix F, not unique, such that $\Lambda = e^F$.*

Proof. The additional condition just states the reduction to the classical 3D Rodrigues exponential formula, cf. Example 1.1. In particular, it ensured $\det(A) = 1$. Thus, there are infinitely many choices for F as the exponent, cf. Corollary 3.13. ∎

Proposition 3.3. *If $s > 0$ then $\Lambda^{\intercal}\Lambda$ is diagonalizable: $\Lambda^{\intercal}\Lambda = V\Delta V^{-1}$ with $\Delta = \operatorname{diag}(\gamma, 1/\gamma, 1, 1)$ for some $\gamma > 0$, subject to relations*

$$\text{(3.3)} \qquad \alpha^2 = s^2 - 1, \qquad \alpha = \frac{\gamma - 1}{\gamma + 1}, \qquad \gamma = \frac{1 + \alpha}{1 - \alpha},$$

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with $s = 1$ if and only if $\alpha = 0$ if and only if $\gamma = 1$. Four independent eigenvectors, of which two are owned by the eigenvalue 1, are

$$(3.4) \qquad 1 \quad \sim \quad \begin{bmatrix} 0 \\ \mathbf{q}^{\perp} \end{bmatrix}, \qquad \gamma \quad \sim \quad \begin{bmatrix} \alpha \\ \mathbf{q} \end{bmatrix}, \qquad \frac{1}{\gamma} \quad \sim \quad \begin{bmatrix} -\alpha \\ \mathbf{q} \end{bmatrix}.$$

Proof. We check directly that

$$\Lambda^{\mathsf{T}}\Lambda = 2 \begin{bmatrix} s \\ \mathbf{q} \end{bmatrix} \begin{bmatrix} s & \mathbf{q}^{\mathsf{T}} \end{bmatrix} + G,$$

which yields two eigenvectors with $\mathbf{q}^{\perp}$, orthogonal to $\mathbf{q}$, owned by the unit. The eigen-equation

$$\Lambda^{\mathsf{T}}\Lambda \begin{bmatrix} \alpha \\ \mathbf{q} \end{bmatrix} = \gamma \begin{bmatrix} \alpha \\ \mathbf{q} \end{bmatrix}$$

entails a system of simple equations

$$\begin{aligned} 2\left(s^2 - 1 + s\alpha\right) + 1 \quad &= \gamma, \\ 2\left(s^2 - 1 + s\alpha\right)\, s - \alpha \quad &= \alpha\gamma, \end{aligned}$$

which yields immediately (3.3) and the remainder of (3.4). ∎

Henceforth, we are confining ourselves to proper Lorentz matrices, i.e., to G-orthogonal matrices such that $\det(\Lambda) = 1$ and $s > 0$.

3.2. Sliders. A unit vector $\mathbf{u}$ in a Euclidean space and $t \in \mathbb{R}$ induce a group of commuting operators

$$S_t = S(\mathbf{u}; t) = I + (t-1)\, \mathbf{u}\mathbf{u}',$$

called *sliders* for the following reason:

$$S_t\mathbf{x} = \mathbf{x} + (t-1)\,(xu)\,\mathbf{u} = \mathbf{x} + (t-1)\,\mathsf{proj}_{\mathbf{u}}\mathbf{x}.$$

The product rule $S_t\, S_s = S_{ts}$ entails the powers $S_t^n = S_{t^n}$. Hence, the inverse for $t \neq 0$ is $S_t^{-1} = S_{1/t}$ and sliders S_t are "*infinitely divisible*" for $t > 0$:

$$S_t^{1/n} \overset{\mathsf{def}}{=} S_{t^{1/n}} \quad \text{because} \quad S_{(t^{1/n})^n} = S_t.$$

We choose the positive sign when n is even although, as expected,

$$J^2_{\sqrt{t}} = S_t \quad \text{and} \quad J^2_{-\sqrt{t}} = S_t, \quad t > 0.$$

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Its eigen-system emerges immediately; the eigenvalue $t \sim \mathbf{u}$ and the eigenvalue $1 \sim \mathbf{u}^\perp$. Hence, in a d-dimensional space,

$$\det(S_t) = t, \qquad \operatorname{tr} S_t = d - 1 + t, \qquad ||S_t||^2 = 1 + \frac{t^2 - 1}{d}.$$

The product rule upon the substitution $s = e^\sigma$, $t = e^\tau$ shows that sliders are exponentials of projections $\mathbf{u}\mathbf{u}^\intercal$:

$$e^{\sigma\,\mathbf{u}\mathbf{u}^\intercal} = I + (e^\sigma - 1)\,\mathbf{u}\mathbf{u}^\intercal = S_{e^\sigma} \overset{\mathsf{def}}{=} T_\sigma.$$

That is, $T_\sigma T_\tau = T_{\sigma+\tau}$ in the additive mode.

3.3. Polar representation. Let us rewrite a G-orthogonal (or Lorentz) matrix in a normalized "1+(d−1)" form

$$\Lambda = \begin{bmatrix} s & t\,\mathbf{v}^\intercal \\ t\,\mathbf{u} & A \end{bmatrix}, \quad ||\mathbf{u}|| = ||\mathbf{v}|| = 1, \quad t^2 = s^2 - 1,$$

characterized by properties (3.2), which now take the following appearance:

$$|s| \geq 1, \quad A^\intercal A = S_{s^2}(\mathbf{v}), \quad AA^\intercal = S_{s^2}(\mathbf{u}), \quad A\mathbf{v} = s\,\mathbf{u}, \quad A^\intercal\mathbf{u} = s\,\mathbf{v}. \tag{3.5}$$

We may choose the sign of t at will, changing directions of both vectors $\mathbf{u}$ and $\mathbf{v}$, or alternatively, by switching to $G\Lambda G$. Our main interest lies in *proper Lorentz matrices*, i.e., with $\det\Lambda = 1$ and $s > 0$. We will introduce these assumptions gradually.

We infer quickly that $|\det A| = s$ and find the inverse from either formula:

$$A^{-1} = S_{s^{-2}}(\mathbf{v})\,A^\intercal = A^\intercal S_{s^{-2}}(\mathbf{u}).$$

The unique polar decomposition $A = RS$ follows immediately:

$$S \overset{\mathsf{def}}{=} \sqrt{A^\intercal A} = S_s(\mathbf{v}) = I + (s-1)\,\mathbf{v}\mathbf{v}^\intercal, \qquad R \overset{\mathsf{def}}{=} AS^{-1} = A - (s-1)\,\mathbf{u}\mathbf{v}^\intercal, \tag{3.6}$$

where R is orthogonal and $R\mathbf{v} = \mathbf{u}$. We see that $\operatorname{sign}(\det R) = \operatorname{sign}(\det A)$.

Whence the unique polar decomposition emerges:

$$\Lambda = UP, \quad \text{with} \quad U = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} s & t\,\mathbf{v}^\intercal \\ t\,\mathbf{v} & S \end{bmatrix}. \tag{3.7}$$

For a proper Lorentz matrix, when $s \geq 1$ and $\det\Lambda = 1$, we see that $1 = \operatorname{sign}(\det\Lambda) = \operatorname{sign}(\det U) = \operatorname{sign}(\det R) = \operatorname{sign}(\det A)$. Thus, $\det A = s$ and R is a rotation by an angle ρ about a direction $\mathbf{r}$.

Proposition 3.4. *The eigenvalues of $P = \sqrt{\Lambda^\intercal\Lambda}$ consist of the $(d-2)$-tuple 1 and the pair of positive mutual reciprocals γ, $1/\gamma$, with the eigen-system*

$$1 \sim \begin{bmatrix} 0 \\ \mathbf{v}^\perp \end{bmatrix}, \qquad \gamma = s + t \sim \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}, \qquad \frac{1}{\gamma} = s - t \sim \begin{bmatrix} \mathbf{v} \\ -1 \end{bmatrix}.$$

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Moreover, $\gamma = 1$ *if and only if* $s = 1$.

Proof. An eigenvector $\begin{bmatrix} s \\ \mathbf{v} \end{bmatrix}$ with $\xi = 0$ yields $\mathbf{v}^{\perp}$, and for $\xi \neq 0$ the eigen-equation

$$\gamma = s + \xi t, \quad \xi\gamma = t + s\xi,$$

is solved by $\xi^2 = 1$ and $\gamma_s = s + \xi t$, as displayed. ◻

3.4. The intrinsic pattern. It is tempting to reverse the process (3.6), given the parameters s, R, $\mathbf{v}$:

$$\mathbf{u} \stackrel{\mathsf{def}}{=} R\mathbf{v}, \qquad A \stackrel{\mathsf{def}}{=} R + (s-1)\,\mathbf{u}\mathbf{v}^{\mathsf{T}}, \qquad t = \pm\sqrt{s^2 - 1}.$$

Then one might expect to recover the G-orthogonal matrix Λ in a quite trivial way. However, the postulates $\det\Lambda = 1$ and $s \geq 1$ impose intrinsic relations between parameters. Therefore, their pattern should be described first. Let us begin with the simplest cases.

Proposition 3.5.

(a) *A is orthogonal if and only if $|s| = 1$ if and only if $t = 0$.*
(b) *A is a rotation about a direction* $\mathbf{r}$ *if and only if* $s = 1$ *and* $\det\Lambda = 1$. *Equivalently,* $A = e^{-\theta\, C(\mathbf{r})}$ *and* $s = 1$.
(c) *If* $s = 1$ *then* Λ *is diagonalizable, with the double eigenvalue 1 that owns* $\begin{bmatrix} 0 \\ \mathbf{r} \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$.

Proof. (a) follows from relations $A^{\mathsf{T}}A = I + (s^2-1)\,\mathbf{v}\mathbf{v}^{\mathsf{T}}$, $AA^{\mathsf{T}} = I + (s^2-1)\,\mathbf{u}\mathbf{u}^{\mathsf{T}}$ in (3.5).

The first part of (b) follows from the remark after (3.7) regarding the signs of determinants. The next parts simply refer to the classical Rodrigues formula, cf. Example 1.1. While a rotation is involved, the uniqueness occurs only up to the periodic translation $\theta \mapsto \theta + 2n\pi$. The last part states the obvious. ◻

Henceforth we assume that $s > 1$ but we will be monitoring the limit behavior for $s \searrow 1$ as means of control. We also confine to four dimensions, $d = 4$. Consequently, $A = RS$ has at least one real eigenvalue α that owns a real unit eigenvector $\mathbf{a}$. Let ρ denote the angle of rotation R about a direction $\mathbf{r}$, i.e., $\cos\rho = \mathbf{x}^{\mathsf{T}}R^{\mathsf{T}}\mathbf{x}$ for any vector $\mathbf{x} \neq \mathbf{r}$.

Let us rewrite the eigen-equation $RS\,\mathbf{a} = A\,\mathbf{a} = \alpha\,\mathbf{a}$:

$$\mathbf{a} + (s-1)\,(av)\,\mathbf{v} = \alpha\,R^{\mathsf{T}}\mathbf{a}. \tag{3.8}$$

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Proposition 3.6. *Let Λ be a proper Lorentz matrix.*

(a) *Then* $(av)^2 = \dfrac{\alpha^2-1}{s^2-1}$, *which implies that* $1 \le |\alpha| \le s$.
(b) *If* $|\alpha| = 1$ *then* $(av) = 0$. *In this case, eigenvalues of A belong to the set* $\{-1, 1\}$, *and:*
  (a) *either we have* $(1, 1, 1)$, *i.e.,* $A = I$ *and* $\Lambda = I$,
  (b) *or we have* $(-1, -1, 1)$, *i.e.,* $s = 1$ *and A is a rotation by π about* $\mathbf{a}$.
(c) *If* $|\alpha| \ne 1$ *then α satisfies the quadratic equation* $\alpha^2 - \alpha\,(s+1)\,\cos\rho + s = 0$, *i.e.,*

$$\alpha = \frac{(s+1)\,\cos\rho \pm \sqrt{(s+1)^2\,\cos^2\rho - 4s}}{2}. \tag{3.9}$$

Proof. Comparing the squared lengths of the vectors on both sides of (3.8),

$$1 + 2(s-1)\,(av)^2 + (s-1)^2\,(av)^2 = \alpha^2,$$

we deduce (a), and then (b). Then (c) follows by multiplying (3.8) by $\mathbf{a}^\intercal$, which yields the formula

$$1 + (s-1)\,(vx)^2 = \alpha\,\cos\rho,$$

and then the listed equation by (a). ◻

Corollary 3.7. *Let Λ be proper with $s > 1$.*

(a) *Let* $|\cos\rho| = 1$. *Then*
  (i) *either* $\rho = 0$, *i.e.,* $R = I$, *and* 1, 1, s *are the eigenvalues of A. That is, A reduces to the slider,* $A = S_s(\mathbf{v})$;
  (ii) *or* $\rho = \pi$, *i.e., A is a rotation by π as in* Proposition 3.6.2.(b).
(b) *Let* $|\cos\rho| < 1$. *Then A has only one positive real eigenvalue* $\alpha = \sqrt{s}$ *and*

$$\cos\rho = \frac{2\sqrt{s}}{s+1} > 0. \tag{3.10}$$

Proof. The first statement is evident, so assume that $|\cos\rho| < 1$. Since a real eigenvalue exists, the radicand in (3.9) must be nonnegative. If it is zero, then the statement follows. To complete the proof we must exclude the remaining case. Assume by contrary that the radicand is strictly positive. We will see that this assumption implies that $\cos\rho = 1$, which has been excluded.

If A had two more real eigenvalues, then one of them would have to be double. But their product equals $s = \det A$, hence one of the roots would be 1. This occurs if and only if $\cos\rho = 1$.

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Alternatively, a single positive real eigenvalue of A would be one of the roots while the second root would be a phantom. Imposing the restriction $1 \leq \alpha \leq s$ on either root would entail again $\cos\rho = 1$. ☐

COROLLARY 3.8. *A is normal if and only if $\mathbf{u}||\mathbf{v}$ if and only if $\mathbf{r}||\mathbf{v}$. In either case, if Λ is proper, then $s = 1$ and A reduces to a rotation.*

Proof. The first statement follows directly from (3.5). Then s becomes an eigenvalue. It is positive for a proper Λ, so $s = \sqrt{s}$, i.e., $s = 1$. Proposition 3.5 finishes the argument. ☐

3.5. Eigenvalues of Λ. We have already derived the polar representation (3.7) with $t^2 = s^2 - 1$. Since the case $s = 1$ reduces to the classical Rodrigues formula, for the remainder of the subsection we assume that $s > 1$.

Based on (3.7), we write the eigenvalue equation

$$\lambda \begin{bmatrix} \varepsilon \\ \mathbf{x} \end{bmatrix} = \Lambda \begin{bmatrix} \varepsilon \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} t\,(vx) + \varepsilon s \\ S\mathbf{x} + \varepsilon\, t\, \mathbf{v} \end{bmatrix} = \begin{bmatrix} t(vx) + \varepsilon\, s \\ RS\mathbf{x} + \varepsilon\, t\, R\mathbf{v} \end{bmatrix},$$

requesting that $x = 1$, i.e., $\mathbf{x}$ is a unit vector. As before in the case of A, we rewrite the equation in a more transparent form, using (3.6), $R^{-1} = R^{\intercal}$:

$$\mathbf{x} + \Big((s-1)(vx) + \varepsilon t\Big)\,\mathbf{v} = \lambda\, R^{\intercal}\,\mathbf{x}, \qquad t\,(vx) = \varepsilon(\lambda - s),$$

or equivalently,

$$\mathbf{x} + \frac{\varepsilon}{t}\Big((s-1)(\lambda - s) + t^2\Big)\,\mathbf{v} = \lambda\, R^{\intercal}\mathbf{x}, \qquad t\,(vx) = \varepsilon(\lambda - s),$$

or even simpler:

$$\mathbf{x} + \varepsilon\,\frac{(s-1)(\lambda+1)}{t}\,\mathbf{v} = \lambda\, R^{\intercal}\mathbf{x}, \qquad t\,(vx) = \varepsilon(\lambda - s). \tag{3.11}$$

PROPOSITION 3.9. *Let $s > 1$.*

(a) *If $\lambda \neq -1$, then $|\varepsilon| = 1$.*
(b) *Let $\lambda = -1$. Then R is the rotation by $\rho = \pi$ about some direction $\mathbf{r}$, and*
 (i) *either $\mathbf{r} = \mathbf{v}$ and $\varepsilon = 0$; so $\lambda = -1$ owns $\begin{bmatrix} 0 \\ \mathbf{v}^{\perp} \end{bmatrix}$;*
 (ii) *or $\mathbf{v} \perp \mathbf{r}$ and $\lambda = -1$ owns*

$$\begin{bmatrix} \varepsilon \\ \mathbf{v} \end{bmatrix}, \text{ with } \varepsilon = -\sqrt{\frac{s-1}{s+1}}, \qquad \text{and also owns} \quad \begin{bmatrix} 0 \\ \mathbf{r}\times\mathbf{v} \end{bmatrix}.$$

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Proof. (a): When $\lambda = s$, then $\mathbf{v} \perp \mathbf{x}$ and the resulting right triangle yields

$$\cos\rho = \frac{1}{s}, \qquad \varepsilon^2 = 1.$$

Squaring the lengths in (3.11) and the formula for (vx) give

$$1 + \varepsilon^2 \left(\frac{2(\lambda+1)(\lambda-s)}{s+1} + \frac{(\lambda+1)^2(s^2-1)}{(s+1)^2} \right) = \lambda^2,$$

and again we end up with $\varepsilon^2 = 1$.

(b): For $\lambda = -1$ formula (3.11) reads

$$R\mathbf{x} = -\mathbf{x}, \qquad (vx) = -\varepsilon\sqrt{\frac{s+1}{s-1}},$$

so R is the rotation by π about $\mathbf{r}$. $\varepsilon = 0$ yields all vectors orthogonal to $\mathbf{v}$, so $\mathbf{v} = \mathbf{r}$. If $\varepsilon \neq 0$, then we would have infinitely many eigenvectors unless $\mathbf{x} = \mathbf{v} \perp \mathbf{r}$, so the formula for ε follows. The remaining eigenvector is made by $\mathbf{r}\times\mathbf{v}$ and $\varepsilon = 0$. ∎

Corollary 3.10. *Let Λ be a Lorentz matrix (not necessarily proper). Let ρ denote the angle between $\mathbf{x}$ and $R\mathbf{x}$ (R might not be a rotation). Then there exist two positive eigenvalues $\lambda = a \pm \sqrt{a^2-1}$, reciprocal to each other, that solve the quadratic equation*

$$\lambda^2 - 2a\,\lambda + 1 = 0, \qquad \textit{where} \quad a = \frac{(s+1)\cos\rho + s - 1}{2}.$$

Of course, this occurs if and only if $a \geq 1$.

Proof. Multiply (3.11) by $\mathbf{x}^\intercal$ and substitute (vx), using $\varepsilon^2 = 1$:

$$1 + \frac{(\lambda+1)(\lambda-s)}{s+1} = \lambda\,\cos\rho.$$

In other words,

$$(\lambda\cos\rho - 1)(s+1) = (\lambda - s)\,(\lambda+1),$$

which translates to the above quadratic equation. For two positive solutions to exist, a must be positive. Thus, it is necessary and sufficient that $a \geq 1$, which is rewritten above. ∎

Corollary 3.11. *Let Λ be proper with $s > 1$. Then Λ has two distinct positive eigenvalues, mutually reciprocal.*

Proof. Corollary 3.7 concludes with the intrinsic relation (3.10). Also, the assumption ensures $s > 0$ and $\cos\rho > 0$. That then condition $a \geq 1$ translates to

$$\cos\rho \geq \frac{3-s}{s+1},$$

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which is satisfied because by (3.10)

$$\cos\rho = \frac{2\sqrt{s}}{s+1} \geq \frac{3-s}{s+1}$$

and the latter equality occurs if and only if $s = 1$. However, the double root would require $\cos\rho = (3-s)/(s+1)$, and it is excluded in virtue of the assumption $s > 1$. ☐

COROLLARY 3.12. *With the exception of the single eigenvalue 1, a proper Λ is diagonalizable.*

COROLLARY 3.13. *Every proper Lorentz matrix is an exponential $\Lambda = e^F$ of a G-skew orthogonal F. The parameter θ is unique up to a periodic shift: once a θ is given, then all $\theta + 2n\pi$ will do.*

Proof. First, consider a defective Λ. Suppose that $P = \sqrt{\Lambda^\intercal \Lambda}$ has a quadruple eigenvalue 1, i.e., $\gamma = 1$. Since $2(s+1) = \operatorname{tr} P = 4$, hence $s = 1$ and $\mathbf{u} = \mathbf{v} = 0$. So, $P = I$ and we have the classical Rodrigues formula with a rotation R by an angle ρ about a unit vector $\mathbf{r}$:

$$\Lambda = U = \begin{bmatrix} 1 & \mathbf{0}^\intercal \\ \mathbf{0} & R \end{bmatrix}, \quad \text{i.e.,} \quad R = e^{-\rho C(\mathbf{r})}.$$

Besides this case, Λ is diagonalizable with eigenvalues $s, \overline{s}, e^{\sigma}, e^{-\sigma}$, where $\sigma = \frac{1}{2}\ln\gamma$. Thus, $s = e^{i\theta}$. The real $s = \pm 1$ correspond to $\theta = 0$ or $\theta = \pi$, owning two linearly independent eigenvectors. A diagonalization $\Lambda = Ve^D V^{-1}$, where $D = \operatorname{diag}(i\theta, -i\theta, \sigma, -\sigma)$ entails the family of matrices

$$\Lambda(t) = Ve^{tD}V^{-1}, \quad t \in \mathbb{R}.$$

Since V does not depend on t, we obtain a G-skew symmetric

$$F \stackrel{\mathsf{def}}{=} \lim_{t\to 0} \frac{1}{t}\Big(\Lambda(t) - I\Big) = VDV^{-1}.$$

Clearly, $e^F = \Lambda$. ☐

4. Complex coding. We use the script font to denote classes (families, spaces, ideals, etc.) of linear operators on a complex separable Hilbert space $\mathbb{H}$ with a standard orthonormal basis $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2, \ldots$ Unless specifically stated, we consider only finite dimensional $\mathbb{H} = \mathbb{C}^d$. We may indicate the dimension of the underlying Euclidean space if necessary or skip it when it is clear from the context. $\mathscr{C} = \mathscr{C}^d = \mathscr{L}(\mathbb{C}^d)$ denotes the space of all complex $d \times d$ matrices while $\mathscr{H} = \mathscr{H}^d = \mathscr{L}_H(\mathbb{C}^d)$ marks its real subspace of Hermitian matrices. We can write $\mathscr{C} = \mathscr{H} + i\mathscr{H}$, i.e., every complex matrix C can be written as a complex combination of Hermitian matrices:

$$C = H_1 + i\,H_2, \tag{4.1}$$

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where

$$H_1 = \frac{1}{2}\Big(C + C^*\Big) \quad \text{and} \quad H_2 = \frac{1}{2i}\Big(C - C^*\Big).$$

If the Euclidean space is d-dimensional, then $n \stackrel{\mathsf{def}}{=} \dim \mathscr{H} = d^2$. In particular, if we expand both matrices with respect to a basis (σ_k) in $\mathscr{H}$, with real coefficients (u_k^j), $j = 1, 2$, then X can be coded uniquely as the pair $(\mathbf{u}_1, \mathbf{u}_2)$ of two real vectors, cf. also (4.6) below.

4.1. Operators on operators and trace. Factually, will be considering the tensor products of operators. However, in order to preserve the elementary level of the presentation we avoid the symbolics and further leads to the depths of the theory. We simply answer the question "how" rather than "why".

Since the matrix $\mathbf{x}\mathbf{y}^*$ can be viewed as an operator of rank one acting on vectors, the operators $E_{jk} = \mathbf{e}_j \mathbf{e}_k^*$ form a basis of the vector space of operators $\mathscr{L}(\mathbb{H})$. The basis is ordered lexicographically: $E_{11}, \dots, E_{1,n}$, $E_{21}, \dots, E_{2,n}$, $\dots$, $E_{n1}, \dots, E_{nn}$. We define $\operatorname{tr} E_{jk} \stackrel{\mathsf{def}}{=} \delta_j^k$, extendable by linearity. That is, for $C = \sum_{j,k} c_{jk} E_{jk}$, we put $\operatorname{tr} C = \sum_j c_{jj}$. Of course, one must show that the trace is well-defined, i.e., it is independent of the basis.

By $E_{jk,pq}$ we denote the operator acting on matrices, i.e., on operators, by its action on the spanning set $\{\, \mathbf{x}\mathbf{y}^* \,\}$ of $\mathscr{L}(\mathbb{H})$,

$$E_{jk,pq}\, \mathbf{x}\mathbf{y}^* = x_k\, y_q\, \mathbf{e}_j \mathbf{e}_p^*,$$

extendable by linearity. In other words,

$$E_{jk,pq}(X) = E_{jk} X E_{pq}.$$

These operators form a basis of $\mathscr{L}(\mathscr{L}(\mathbb{H}))$, on which the trace is defined by the formula

$$\operatorname{tr} E_{jk,pq} \stackrel{\mathsf{def}}{=} \delta_j^k \cdot \delta_p^q.$$

That is, since an operator $L \in \mathscr{L}(\mathscr{L}(\mathbb{H}))$ is a linear combination of such basic operators with coefficients, say, $l_{jk,pq}$, then

$$\operatorname{tr} L = \sum_{j,p} l_{jj,pp}. \tag{4.2}$$

The index-separated coefficients $l_{jk,pq} = c_{jk}\overline{d_{pq}}$ yield the "jaws" operators

$$L_{C,D} X = C X D^*, \quad L_C \stackrel{\mathsf{def}}{=} L_{C,C}.$$

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PROPOSITION 4.1. *Immediate properties:*

(a) *L_C preserves the Hermicity.*
(b) *The "jaws composition" holds: $L_C L_D = L_{CD}$.*
(c) *$\{ L_C : \det C \neq 0 \}$ is a group with $L_C^{-1} = L_{C^{-1}}$.*
Further, L_C and C are simultaneously nonsingular or singular.
(d) *$L_C^* = L_{C^*}$;*
(e) *L_C has a nonnegative trace and further*

$$\text{(4.3)} \qquad \operatorname{tr} L_C = \sum_{j,p} c_{jj}\overline{c_{pp}} = |\operatorname{tr} C|^2, \quad \operatorname{tr} L_{C,D} = \sum_{j,p} c_{jj}\overline{d_{pp}} = \operatorname{tr} C \cdot \overline{\operatorname{tr} D}.$$

(f) *The polarization formula is valid on $\mathscr{H}$:*

$$L_{C,D} = \frac{1}{4}\left(L_{C+D} - L_{C-D}\right) = \frac{1}{2}\left(L_{C+D} - L_C - L_D\right).$$

(g) *A change of basis or the preservation of similarity: Let S be nonsingular and $D_j = SC_jS^{-1}$, $j = 1, 2$. Then*

$$L_S\, L_{C_1,C_2}\, L_S^{-1} = L_{D_1,D_2}$$

Proof. (a) and (b) are obvious.

(c): The first part follows directly from (b). For the second part, suppose that $\det C = 0$ (or $\ker C \neq 0$), i.e., $C\mathbf{x} = 0$ for some vector $\mathbf{x} \neq 0$. Then $L\mathbf{x}\mathbf{x}^* = C\mathbf{x}\mathbf{x}^*C^* = 0$ (i.e., $\ker L_C \neq 0$).

(d): This follows from duality

$$\langle\, Y\,,\, CXC^* \,\rangle = \langle\, Y\,,\, C \,\rangle\, CX = \langle\, C^*YC\,,\, X \,\rangle\,.$$

The trace formulas (4.3) follow from (4.2).

(f): The polarization formula follows from the definition.

(g): Let $Y = C_1XC_2^*$. That is, $Y = S^{-1}D_1SXS^*D_2^*(S^{-1})^*$. Therefore, $L_S(Y) = SYS^* = D_1(SXS^*)D_2^* = L_{D_1,D_2}(L_SX)$. ∎

PROPOSITION 4.2 (The Uniqueness Theorem). *Since $C = 0$ and $L_C = 0$ simultaneously, assume that $C \neq 0$. Then $L_C = L_D$ on $\mathscr{H}$ if and only if $L_C = L_D$ on $\mathscr{C}$ if and only if $C = zD$ for some unit $z \in \mathbb{C}$. Hence*

(a) *The subgroup $\{ L_C : \det C$ is real$\}$ admits the unique double representation $L_C = L_{-C}$.*
(b) *In particular, $L_C = I$ if and only if $C = \pm I$.*

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Proof. If $L_C = 0$, then for $X = I$ we obtain $CC^* = 0$, i.e., $\operatorname{tr}(CC^*) = 0$, or $C = 0$. Let now $C \neq 0$. The equivalence follows from (4.1). Since matrices $X = \mathbf{u}\mathbf{v}^*$ form a linearly dense set in $\mathscr{C}$, the identity $L_C = L_D$ on $\mathscr{C}$ is equivalent to

$$C\mathbf{u}\,(C\mathbf{v})^* = D\mathbf{u}\,(D\mathbf{v})^*, \quad \mathbf{u} \in \mathbb{H}. \tag{4.4}$$

Applying the trace, $||C\mathbf{u}|| = ||D\mathbf{u}||$, $\mathbf{u} \in \mathbb{H}$. In particular, $\ker C = \ker D$. Multiplying on the left by $(C\mathbf{u})^*$ in (4.4),

$$||C\mathbf{u}||^2\,(C\mathbf{v})^* = (C\mathbf{u}, D\mathbf{u})\,(D\mathbf{v})^*.$$

So, $C\mathbf{v} = z(\mathbf{u})\,D\mathbf{v}$ for every $\mathbf{v}$ and $\mathbf{u}$ with $C\mathbf{u} \neq 0$, where

$$z(\mathbf{u}) = \frac{(D\mathbf{u}, C\mathbf{u})}{||C\mathbf{u}||^2}.$$

Multiplying (4.4) by $N\mathbf{v}$ from the right, we see that $z(\mathbf{u}) = z(\mathbf{v})$ on $(\ker C)^c = (\ker D)^c$, i.e., z is a constant of modulus 1. The form of $z(\mathbf{u})$ on $\ker C$ is irrelevant, so we just adopt z. In the last statement and its corollary (b), we just have a real scalar z, so it must be ± 1. ◻

Operator or just matrix properties of C and L_C may be mutually reflected. However, the "forward reflection" from C to L_C is significantly easier to formulate and prove than the "backward reflection" from an alleged L_C back to C.

PROPOSITION 4.3.

(a) *Let $\gamma_j \sim \boldsymbol{\eta}_j$ and $\delta_k \sim \boldsymbol{\zeta}_k$ be eigen-systems of C and D, respectively. Then $\gamma_j\overline{\delta_k} \sim \boldsymbol{\eta}_j\boldsymbol{\zeta}_k^*$ belong to the eigen-system of $L_{C,D}$. In particular,*
   (i) $\det L_{C,D} = \det C \cdot \overline{\det D}$, $\det L_C = |\det C|^2$*;*
   (ii) *the operator L_C preserves* $\det X$ *if and only if* $|\det C| = 1$.
   (iii) *If is C is diagonalizable so is L_C.*

(a′) *Conversely, if C is defective, so is L_C.*

(b) *If C and D are unitary, then $L_{C,D}$ is a rotation. The inverse implication is false in general. However, C is unitary if and only if L_C is a rotation. In general, C does not have to be a rotation.*

(c) *If C is positive then L_C is positive. The inverse implication is false in general, Moreover, not every positive L has the jaws form.*

Proof. (a): This and its consequences follow directly from the definition as well as the first statement in (b).

(a'): In view of Proposition 4.1(g), we may choose a basis of dimension n^2 for C as we please, so we choose a canonical Jordan form (cf. [7, XI.§6]), consisting of Jordan blocks spread along the diagonal. A Jordan block is of the form $zI + U$, where

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the only nonzero elements of U are $u_{j-1,j} = 1$. A matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is defective if and only if A or B is defective. Therefore, we may assume w.l.o.g. that $C = zI + U$, which makes L_C upper triangular, with numbers $|z|^2$ on the diagonal. Hence, for $z = 0$, $L_C \neq 0$ while for $z \neq 0$, $L_C \neq |z|^2 I$. Therefore, L_C is defective.

(b): Let C have nonzero eigenvalues with at least one non-unit eigenvalue and let eigenvalues of D be their reciprocals. This makes a counter-example for the inverse implication in (b).

Suppose that L_C is a rotation, i.e., it is unitary with $\det L_C = 1$. Then $L_C\, L_C^* = I$ implies that $CC^* = I$ by Proposition 4.1 (d).

(c): If C is positive, so $C = D^2$ for some positive D, and then $L_C = L_D^2$. By (a), the eigenvalues of L_C are of the form $\gamma_j \overline{\gamma_k}$. Let $\gamma_j = \alpha_j + i\beta_j$. Thus, L_C is positive if and only if

$$\alpha_j \beta_k = \alpha_k \beta_j,$$

which is possible even when all $\beta_j \neq 0$ or when the signs of α_j are mixed, i.e., C could have some negative or even non-real eigenvalues yet L_C would be positive.

Let L have $n = d^2$ positive eigenvalues forming the set $\mathcal{E}$. In order to represent L as L_C with a positive C, a tabulation $\gamma_j \gamma_k$ of $\mathcal{E}$ is necessary (and rare). Further, it is necessary to find real vectors $\boldsymbol{\eta}_j$ so that each eigenvalue $\gamma_j \gamma_k$ would own $\boldsymbol{\eta}_j \boldsymbol{\eta}_k^*$. If these vectors are orthogonal, then we create the matrix V with $\boldsymbol{\eta}_j$ as columns, and put $C = V^* \Gamma V$ where $\Gamma = \operatorname{diag}(\gamma_j)$. $\square$

4.2. The Pauli's coding. A vector $\mathbf{x} = [x_0, x_1, x_2, x_3]^{\intercal} \in \mathbb{R}^4$ can be coded as a 2×2 complex Hermitian matrix,

$$\mathbf{x} \quad \mapsto \quad {}^{\sigma}\mathbf{x} = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} = \sum_k x_k\, \sigma_k = X,$$

entailing the corresponding basis of Pauli matrices:

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

satisfying

$$\begin{aligned} &\text{(a)} \quad \sigma_0 = I, \quad \sigma_k^2 = I, \quad \sigma_k^* = \sigma_k, \quad \sigma_j \sigma_k = -\sigma_k \sigma_j, \\ &\text{(b)} \quad \langle\, \sigma_j\,,\, \sigma_k \,\rangle = \delta_j^k, \\ &\text{(c)} \quad \sigma_1 \sigma_2 = i\, \sigma_3. \end{aligned} \tag{4.5}$$

Therefore, we can encode the vector from a Hermitian matrix:

$$\mathbf{x} = X^{\sigma}, \quad \text{where} \quad x_k = \langle\, \sigma_k\,,\, X \,\rangle = \frac{1}{2} \operatorname{tr} \sigma_k X.$$

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In other words,

$$^{\sigma}\mathbf{e}_k = \sigma_k, \qquad \sigma_k^{\sigma} = \mathbf{e}_k.$$

In virtue of (4.1), the isomorphism between real vectors and Hermitian matrices extends to the isomorphism between complex vectors and complex matrices:

$$^{\sigma}(\mathbf{u} + i\,\mathbf{v}) \stackrel{\text{def}}{=} {}^{\sigma}\mathbf{u} + i\ {}^{\sigma}\mathbf{v}. \tag{4.6}$$

An operator $L \in \mathscr{L}(\mathbb{C}^d)$ entails its matrix representation $\Lambda = [l_{jk}] \in \mathscr{L}(\mathbb{R}^{2n})$ through the isomorphism

$$L^{\sigma} = {}^{\sigma}\Lambda, \qquad \text{i.e.,} \quad L({}^{\sigma}\mathbf{x}) = {}^{\sigma}(\Lambda\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^4.$$

In practice it suffices to assign the basic vectors to the columns of Λ, i.e., $\mathbf{e}_k \mapsto \Lambda\mathbf{e}_k = \boldsymbol{l}_k$, so the k^{th} column of Λ will appear as the Hermitian

$$^{\sigma}\boldsymbol{l}_k = \sum_j l_{jk}\,\sigma_j. \tag{4.7}$$

Proposition 4.4.

1. *An operator L_C admits the matrix representation $\Lambda = \Lambda_C = [l_{jk}]$ such that*

$$l_{jk} = \langle\, \sigma_j\,,\, C\sigma_k C^*\,\rangle\,. \tag{4.8}$$

2. *Necessarily, $l_{00} = |\operatorname{tr} C|^2 \geq 0$ and $l_{00} = 0$ if and only if $C = 0$.*

Proof. It suffices to employ the basis and the duality and then compute l_{00} from (4.8). ☐

The utility of Pauli matrices is strictly confined to four dimensional spaces.

Proposition 4.5. *If the p-dimensional complex vector space $\mathbb{H} \otimes \mathbb{H} = \mathscr{L}(\mathbb{H})$ ($p = d^2$) admits a Pauli-like basis σ_k satisfying* (4.5) (a)–(b)*, then $p = 4$, i.e., $d = 2$.*

Proof. We have

$$\sum_{k\geq 0} \sigma_k\sigma_j\sigma_k = \begin{cases} p\,\sigma_0, & \text{if } j = 0, \\ (4-p)\,\sigma_j, & \text{if } j > 0. \end{cases} \tag{4.9}$$

Indeed, let $j = 0$. Then

$$\sum_{k\geq 0} \sigma_k\sigma_0\sigma_k = \sum_{k\geq 0} \sigma_0 = p\,\sigma_0$$

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Let $j > 0$. Then

$$\sum_{k\geq 0} \sigma_k \sigma_j \sigma_k = \sigma_j + \sigma_j^3 + \sum_{0<k\neq j} \sigma_k \sigma_j \sigma_k = 2\sigma_j - (p-2)\sigma_j = (4-p)\,\sigma_j$$

Using (4.9), let us evaluate the action of the following operator on matrices $C = c_0\sigma_0 + \sum_j c_j\sigma_j = c_0\sigma_0 + C_0$:

$$P(C) \overset{\mathsf{def}}{=} \sum_{k\geq 0} \sigma_k\, C\, \sigma_k = p\, c_0 + (4-p)\, C_0.$$

Consider the jaws operator $L(X) = CXC^*$ and let $\Lambda = \Lambda_C = [l_{jk}]$ be its matrix representation with respect to (σ_k), i.e.,

$$l_{jk} = \langle\, \sigma_j\, ,\, C\sigma_k C^*\, \rangle = \frac{1}{d}\, \mathrm{tr}\, \sigma_j C \sigma_k C^*.$$

Let us compute its trace:

$$\begin{aligned} \mathrm{tr}\, \Lambda_C &= \sum_j l_{jj} = \sum_j \langle\, \sigma_j\, ,\, C\sigma_j C^*\, \rangle = \frac{1}{d}\, \mathrm{tr}\, P(C)C^* = \langle\, C\, ,\, P(C)\, \rangle \\ &= \langle\, c_0\sigma_0 + C_0\, ,\, p\, c_0\sigma_0 + (4-p)\, C_0\, \rangle = |\mathrm{tr}\, C|^2 + (4-p)\, ||C_0||^2. \end{aligned}$$

In virtue of Proposition 4.1.(4.3), necessarily $p = 4$. ∎

Once we have selected and fixed the standard bases as well as the isomorphisms, there is no need to mark them anymore. That is, instead of writing $\sigma_k = {}^{\sigma}\mathbf{e}_k$ we will simply write $\sigma_k = \mathbf{e}_k$. We will also write $1 = I = \sigma_0$. In other words, while coding $X \leftrightarrow \zeta + \mathbf{z}$, we write in the code:

$$X = \zeta + \mathbf{z} \overset{\mathsf{def}}{=} \zeta\sigma_0 + \sum_{k>0} z_k \mathbf{e}_k.$$

We now stress the typographic distinction between scalars (i.e., scalar multipliers of the identity operator) and vectors that allows their quick visual recognition. Accordingly, we denote $(cz) = \mathbf{c}^{\mathsf{T}}\mathbf{z}$ even when both vectors are complex. We verify directly that

$$\mathbf{c}\,\mathbf{z} = \Big(\sum_{j>0} c_j \mathbf{e}_j\Big)\Big(\sum_{k>0} z_k \mathbf{e}_k\Big) = (cz) + i\, \mathbf{c}\times\mathbf{z}.$$

In particular, for $\mathbf{c} = \mathbf{a} + i\,\mathbf{b}$,

$$c^2 \overset{\mathsf{def}}{=} \mathbf{c}^2 = (c\, c) = \mathbf{c}^{\mathsf{T}}\mathbf{c} = a^2 - b^2 + 2i\,(ab).$$

Note that $a^2 = ||a||^2$ for a real vector $\mathbf{a}$, so we may and do assume that $a \geq 0$, whence $a = 0$ if and only if $a^2 = 0$ if and only if $\mathbf{a} = 0$. In contrast, for a non-real complex

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vector $\mathbf{c}$, $c^2 = 0$ means that it's real and imaginary components are orthogonal vectors of the same length. However, the scalar "c" stays undefined but we still may write $|c| = ||\mathbf{c}||$.

Further, unless specifically stated, the presence of a subscript such as in $\mathbf{e}_k$ automatically will mean that $k > 0$. According to this convention, $X^* = (\zeta + \mathbf{z})^* = \overline{\zeta} + \overline{\mathbf{z}}$ and a Hermitian matrix is represented by a real vector $\zeta + \mathbf{z}$. The full multiplication tables emerge as expected:

$$(\gamma + \mathbf{c})\,(\zeta + \mathbf{z}) = \Big(\gamma\zeta + (cz)\Big) + \Big(\gamma\,\mathbf{z} + \zeta\,\mathbf{c} + i\,\mathbf{c}{\times}\mathbf{z}\Big), \tag{4.10}$$

subject to the tedious but routine split into the real and imaginary part. That is, letting $\gamma = \alpha + i\beta$, $\zeta = \xi + i\eta$ and $\mathbf{c} = \mathbf{a} + i\,\mathbf{b}$, $\mathbf{z} = \mathbf{x} + i\,\mathbf{y}$, we list the components:

$$\begin{aligned}
&\text{scalar:} && \alpha\xi - \beta\eta + (ax) - (by) + \, i\,\Big(\alpha\eta + \beta\xi + (bx) + (ay)\Big),\\
&\text{vector:} && \alpha\,\mathbf{x} - \beta\,\mathbf{y} + \xi\,\mathbf{a} - \eta\,\mathbf{b} + -\mathbf{a}{\times}\mathbf{y} - \mathbf{b}{\times}\mathbf{x}\\
& && \qquad + \, i\,\Big(\alpha\,\mathbf{y} + \beta\,\mathbf{x} + \xi\,\mathbf{b} + \eta\,\mathbf{a} + \mathbf{a}{\times}\mathbf{x} + \mathbf{b}{\times}\mathbf{y}\Big).
\end{aligned}$$

Let us gather a few immediate corollaries.

Proposition 4.6. *Let $C = \gamma + \mathbf{c} = \alpha + i\beta + \mathbf{a} + i\,\mathbf{b}$. Then*

1. $(\gamma + \mathbf{c})\,(\gamma - \mathbf{c}) = \gamma^2 - c^2$*;*
2. $\det C = \gamma^2 - c^2 = \gamma^2 - \mathbf{c}^2 = \alpha^2 - \beta^2 - a^2 + b^2 + 2i\,\big(\alpha\beta - (ab)\big)$*;*
3. $\det C$ *is real if and only if* $\mathbf{a}$ *and* $\mathbf{b}$ *are G-orthogonal;*
4. $\det C = 1$ *if and only if* $\gamma^2 - c^2 = 1$ *if and only if* $C^{-1} = \gamma - \mathbf{c}$.

Let us write explicitly how C determines the jaws operator L_C, i.e., the Lorentz matrix Λ.

Lemma 4.7. *Given $C = \gamma + \mathbf{c} = \alpha + i\,\beta + \mathbf{a} + i\,\mathbf{b}$, the operator L_C is represented by a proper Lorentz matrix*

$$\Lambda = \begin{bmatrix} s & \mathbf{q}^{\intercal} \\ \mathbf{p} & A \end{bmatrix} = \begin{bmatrix} s & q_1 & q_2 & q_3 \\ \mathbf{p} & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

as follows:

$$s = \alpha^2 + \beta^2 + a^2 + b^2, \tag{4.11}$$

$$\mathbf{p} = 2\,\big(\alpha\,\mathbf{a} + \beta\,\mathbf{b} + \mathbf{a}{\times}\mathbf{b}\big), \tag{4.12}$$

$$\mathbf{q} = 2\,\big(\alpha\,\mathbf{a} + \beta\,\mathbf{b} - \mathbf{a}{\times}\mathbf{b}\big), \tag{4.13}$$

$$\mathbf{a}_j = (\alpha^2 + \beta^2 - a^2 - b^2)\,\mathbf{e}_j + 2\,\Big(a_j\,\mathbf{a} + b_j\mathbf{b} + \mathbf{e}_j{\times}(\alpha\,\mathbf{b} - \beta\,\mathbf{a})\Big), \tag{4.14}$$

$$A = (\alpha^2 + \beta^2 - a^2 - b^2)\,I + 2\,\Big(\mathbf{a}\,\mathbf{a}^{\intercal} + \mathbf{b}\,\mathbf{b}^{\intercal} + \alpha\,V(\mathbf{b}) - \beta\,V(\mathbf{a})\,\Big), \tag{4.15}$$

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where $V(\mathbf{h})$ denotes the matrix of the cross product operator, $V(\mathbf{h})\,\mathbf{x} = \mathbf{x}\times\mathbf{h}$.

Proof. We will solve the equations that correspond to the action of L_C on the basic vectors,

$$(\gamma+\mathbf{c})\,(\overline{\gamma}+\overline{\mathbf{c}}) = s+\mathbf{p}, \tag{4.16}$$

$$(\gamma+\mathbf{c})\,\mathbf{e}_j\,(\overline{\gamma}+\overline{\mathbf{c}}) = q_j+\mathbf{a}_j. \tag{4.17}$$

We calculate

$$\mathbf{c}\,\overline{\mathbf{c}} = a^2+b^2+\mathbf{a}\times\mathbf{b} \qquad \text{and} \qquad \overline{\gamma}\,\mathbf{c}+\gamma\,\overline{\mathbf{c}} = 2\,(\alpha\,\mathbf{a}+\beta\,\mathbf{b}),$$

which entail the left hand side of (4.16) and, consequently, (4.11) and (4.12):

$$|\gamma|^2+a^2+b^2+\mathbf{c}\,\overline{\mathbf{c}}+\overline{\gamma}\,\mathbf{c}+\gamma\,\overline{\mathbf{c}} = \alpha^2+\beta^2+a^2+b^2+2\,\mathbf{a}\times\mathbf{b}+2\,(\alpha\,\mathbf{a}+\beta\,\mathbf{b}).$$

Now, the left hand side of (4.17) equals to

$$|\gamma|^2\,\mathbf{e}_j+\Big(\gamma\,\mathbf{e}_j\,\overline{\mathbf{c}}+\overline{\gamma}\,\mathbf{c}\,\mathbf{e}_j\Big)+\mathbf{c}\,\mathbf{e}_j\,\overline{\mathbf{c}} = |\gamma|^2\,\mathbf{e}_j+2\,\Re\left(\gamma\,\mathbf{e}_j\,\overline{\mathbf{c}}\right)+\mathbf{c}\,\mathbf{e}_j\,\overline{\mathbf{c}}. \tag{4.18}$$

Let us calculate the portions, denoting the j^{th} coordinate of the cross product $\mathbf{a}\times\mathbf{b}$ by $(a\times b)_j$,

$$\begin{aligned} \mathbf{e}_j\,\overline{\mathbf{c}} \;&=\; a_j+\mathbf{e}_j\times\mathbf{b}+i\,(-b_j+\mathbf{e}_j\times\mathbf{a}),\\ \Re\left(\gamma\,\mathbf{e}_j\,\overline{\mathbf{c}}\right) \;&=\; \alpha\,a_j+\beta\,b_j+\mathbf{e}_j\times(\alpha\,\mathbf{b}-\beta\,\mathbf{a}),\\ \mathbf{c}\,\mathbf{e}_j\,\overline{\mathbf{c}} \;=\; \Re\left(\mathbf{c}\,\mathbf{e}_j\,\overline{\mathbf{c}}\right) \;&=\; 2\,\Big((a\times b)_j+a_j\,\mathbf{b}+b_j\,\mathbf{a}\Big)-(a^2+b^2)\,\mathbf{e}_j. \end{aligned}$$

After we substitute these portions first into (4.18), and then to (4.17), the scalar parts yield (4.13) and the vector parts yield (4.14). The matrix form (4.15) captures all of (4.14). ◻

Remark 4.8. Sometimes it may be worth to transform equations (4.16) and (4.17) by utilizing the inverse formula $(\zeta+\mathbf{z})^{-1} = \zeta-\mathbf{z}$ which is valid when $\zeta^2-z^2=1$,

$$\gamma+\mathbf{c} = (s+\mathbf{p})\,(\overline{\gamma}-\overline{\mathbf{c}}), \tag{4.19}$$

$$(\gamma+\mathbf{c})\,\mathbf{e}_j = (q_j+\mathbf{k}_j)\,(\overline{\gamma}-\overline{\mathbf{c}}). \tag{4.20}$$

4.3. Coding jaws operators. We underline multiple roles a 2×2 matrix C may play, first as an operator on $\mathbb{C}^2$, then as an asymmetric jaws operator $L_{C,I}$ or $L_{I,C}$ acting on 2×2 complex matrices, and also as the symmetric jaws L_C, as well as their combination. These roles might be easily confused:

$C\boldsymbol{\zeta}$ (as an operator on $\mathbb{C}^2$) vs. $C\,\mathbf{z}$ (as an operator on $\mathbb{C}^4$, formally $L_{C,1}\mathbf{z}$).

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Bringing back the isomorphism mark resolves the issue. While considering the asymmetric jaws operator $L_{C,I}$ we should write

$$L_{C,I}\mathbf{z} = C\,{}^{\sigma}\mathbf{z}. \tag{4.21}$$

Yet, for the sake of clarity we may slightly abuse notation, still writing "$C\mathbf{z}$" or "$\mathbf{z}C$". By the same token, the formula

$$C\mathbf{z}\mathbf{z}^*C^* = L_C\mathbf{z}\mathbf{z}^*$$

is basically confusion free when the matrix C has been already coded. In contrast, the mark "$^{\sigma}$" might fog the transparency, for the formal script would require

$$(C\,{}^{\sigma}\mathbf{z})(C\,{}^{\sigma}\mathbf{z})^* = C\,{}^{\sigma}\mathbf{z}({}^{\sigma}\mathbf{z})^*C^*.$$

Proposition 4.3 has indicated that the jaws form of an operator is relatively rare, and even for such form L_C the transfer of properties backward from L_C to C may be difficult, even for seemingly simple operators.

Proposition 4.9. $G = \operatorname{diag}(1,-1,-1,-1)$ *does not have a jaws form.*

Proof. To code $G = L_C$ with $C = \gamma + \mathbf{c}$, we would need first

$$(\gamma+\mathbf{c})(\overline{\gamma}+\overline{\mathbf{c}}) = 1, \quad \text{i.e.,} \quad \overline{\gamma}+\overline{\mathbf{c}} = \gamma - \mathbf{c}.$$

So, $\gamma = \alpha$ would be real and $\mathbf{c} = i\,\mathbf{b}$ would be pure imaginary. Then

$$(\gamma+\mathbf{c})\mathbf{e}_k(\overline{\gamma}+\overline{\mathbf{c}}) = -\mathbf{e}_k, \quad \text{i.e.,} \quad \sigma_k(\overline{\gamma}+\overline{\mathbf{c}}) = -(\gamma-\mathbf{c})\mathbf{e}_k.$$

In other words, $\mathbf{e}_k(\alpha - i\,\mathbf{b}) = (-\alpha + i\,\mathbf{b})\,\mathbf{e}_k$. Comparing the scalars, $-b_k = b_k$, i.e., $\mathbf{b} = 0$. Then $\gamma = \alpha = 0$, i.e., $C = 0$, a contradiction. ◻

Nevertheless, the G-transpose $\Lambda^{\mathsf{G}} = G\Lambda^*G$ of a 4×4 matrix Λ, representing L_C works differently in a simple way.

Corollary 4.10. *The G-transpose allows the coding* $C^{\mathsf{G}} = (\gamma+\mathbf{c})^{\mathsf{G}} \overset{\mathsf{def}}{=} \gamma - \mathbf{c} = C^{-1}$.

Proof. The first equality follows directly from Lemma 4.7 and the second equality has been stated in Proposition 4.6.(e). ◻

In virtue of the Uniqueness Theorem, Proposition 4.2, an "educated guess" of the factor C and the choice of the scalar multiplier may lead to the representation $L = L_C$. Let us focus now on proper Lorentz matrices. With respect to a fixed basis, a Lorentz matrix admits the unique polar decomposition $\Lambda = UP$, where (cf. (3.7))

$$U = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} s & t\,\mathbf{v}^{\mathsf{T}} \\ t\,\mathbf{v} & S \end{bmatrix}, \tag{4.22}$$

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with the rotation R and the slider $S = I+(s-1)\,\mathbf{v}\mathbf{v}^{\mathsf{T}}$ such that $v = 1$, $s \geq 1$, $t^2 = s^2-1$.

THEOREM 4.11. *Let P and U be given by* (4.22). *Then there exist the unique positive C with* $\det C = 1$ *and the unique rotation D (of course,* $\det D = 1$*) such that $P = L_C$ and $U = L_D$. Therefore, every proper Lorentz matrix represents a jaws operator, $\Lambda = L_M$, with $M = DC$ and* $\det M = 1$. *M is unique up to the sign, $\pm M$.*

Proof. We can choose either the formulas from Lemma 4.7 or solve equations (4.19) and (4.20), as indicated by Remark 4.8. Let us select the second venue to illustrate this alternative.

For the rotation, we try $D = \delta + i\,\mathbf{d}$ with $\det D = \alpha^2 + b^2 = 1$, $\alpha \geq 0$, and real $\mathbf{d}$. Denote by $\mathbf{r}_j$ the columns of R, the direction of the axis by $\mathbf{r}$, and the angle of rotation by ρ. We should have

$$2 + 2\cos\rho = \operatorname{tr} U = |\operatorname{tr} D|^2 = 4\alpha^2,$$

whence

$$\delta = \cos\frac{\rho}{2}, \quad b = \sin\frac{\rho}{2}.$$

Examining the action of L_D on the basis, $(\delta + i\,\mathbf{d})(\delta - i\,\mathbf{d}) = 1$ as expected, and

$$(\delta + i\,\mathbf{d})\,\mathbf{e}_j\,(\delta - i\,\mathbf{d}) = \mathbf{r}_j.$$

Rewriting the equations,

$$\delta + i\,\mathbf{d}\,\mathbf{e}_j = \mathbf{r}_j\,(\delta + i\,\mathbf{d}).$$

Comparing the scalar parts,

$$b_j = (r_j b) \quad \Rightarrow \quad \mathbf{d} = R\mathbf{d},$$

i.e., $\mathbf{d}$ lies on the axis of R, $\mathbf{d} = \sin\frac{\rho}{2}\,\mathbf{r}$.

For $t = 0$, i.e., $s = 1$, the matrix P reduces to the identity, so we choose $C = I$, obviously. Let $t > 0$, and let us try to find a real $C = \alpha + \mathbf{a}$ with $\alpha \geq 0$ and $\det C = \alpha^2 - a^2 = 1$. Denote the columns of S by $\mathbf{s}_k$ and put $\mathbf{c} = t\mathbf{v}$. Equations (4.7) now read:

$$(\alpha + \mathbf{a})(\alpha + \mathbf{a}) = s + \mathbf{c}, \qquad (\alpha + \mathbf{a})\mathbf{e}_j(\alpha + \mathbf{a}) = c_j + \mathbf{s}_j.$$

The scalar part of the first equation yields $\alpha^2 + a^2 = s$. The assumption $\det C = 1$ gives

$$\alpha = \sqrt{\frac{s+1}{2}}, \qquad a = \sqrt{\frac{s-1}{2}}.$$

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Again, the equations can be rewritten as follows:

$$\alpha + \mathbf{a} = (s + \mathbf{c})(\alpha - \mathbf{a}), \qquad (\alpha + \mathbf{a})\mathbf{e}_j = (c_j + \mathbf{s}_j)\,(\alpha - \mathbf{a}).$$

Comparing the scalar parts,

$$\alpha = s\alpha - (ca), \qquad a_j = \alpha c_j - (s_j a).$$

Substituting back $\mathbf{c} = t\mathbf{v}$ and rewriting the equations on the right in the matrix form, we obtain

$$\begin{aligned} \alpha = s\alpha - t(va) \quad &\Rightarrow \quad (va) = \frac{(s-1)\alpha}{t} = \sqrt{\frac{s-1}{2}}, \\ \mathbf{a} = \alpha\,\mathbf{c} - S\mathbf{a} \quad &= \alpha t\,\mathbf{v} - (I + (s-1)\,\mathbf{v}\mathbf{v}^{\mathsf{T}})\,\mathbf{a} \\ &= \alpha t\,\mathbf{v} - \mathbf{a} - (s-1)\frac{(s-1)\alpha}{t}\,\mathbf{v}. \end{aligned}$$

Simplifying,

$$\alpha\left(t - \frac{(s-1)^2}{t}\right) = \frac{\sqrt{2}s}{\sqrt{s-1}}\ .$$

Thus,

$$\mathbf{a} = \frac{s}{\sqrt{2(s-1)}}\,\mathbf{v}.$$

Thus, the search for D and C has been successful. ▯

4.4. Exponentials. We will show that the components C (or more precisely, asymmetric jaws $L_{C,I}$) of the jaws operator L_C, represented by a proper Lorentz matrix Λ, admit a differentiable parametrization $C(\zeta)$, with respect to a complex or real variable ζ, fulfilling the group property (1.1). Hence, by (1.2), $C(\zeta)$ admits a generator D, so that $C(\zeta) = e^{\zeta\,D}$. Therefore, $L_\zeta = L_{C(\zeta)}$ also satisfies (1.1) with the generator F. The generators are related by the formula

$$F\,X = DXC_0^* + C_0XD^* \quad (= 2\,\Re\,DXC_0^* \text{ for a Hermitian } X), \tag{4.23}$$

which follows from the product rule,

$$\frac{dL_\zeta}{d\zeta}\,X = C_\zeta{}'XC_\zeta^* + C_\zeta XC_\zeta^{*\prime}.$$

Let us also issue the warning:

$$C = e^D \not\Rightarrow L_C = e^{L_D} \quad \text{and} \quad C_1 = e^{D_1},\, C_2 = e^{D_2} \not\Rightarrow L_{C_1C_2} = e^{L_{D_1}}e^{L_{D_2}}.$$

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4.4.1. The diagonalizable matrix. Recall that in contrast to the real case, $c^2 = 0$ does not mean that $\mathbf{c} = 0$. This case will be handled in the next subsection.

PROPOSITION 4.12. *Consider the decomposition $\gamma + \mathbf{c} = (\delta + i\mathbf{d})\,(\alpha + \mathbf{a})$, where $\mathbf{c}^2 \neq 0$. Then the following parametrization entails the corresponding exponentials,*

$$\begin{array}{lll} & \textit{(general)} & \gamma + \mathbf{c} \;=\; \cosh z + \sinh z\, \mathbf{n} \;=\; e^{z\,\mathbf{n}} \\ & \textit{(rotation)} & \delta + i\,\mathbf{d} \;=\; \cos\theta + i\sin\theta\, \mathbf{r} \;=\; e^{i\theta\,\mathbf{r}}, \\ (4.24) & \textit{(positive)} & \alpha + \mathbf{a} \;=\; \cosh\phi + \sinh\phi\, \mathbf{v} \;=\; e^{\phi\,\mathbf{v}}, \\ & & e^{(\phi+i\theta)\,\mathbf{n}} \;=\; \pm\, e^{i\theta\,\mathbf{r}}\, e^{\phi\,\mathbf{v}}, \end{array}$$

where the vectors $\mathbf{r}$ and $\mathbf{v}$ are real, $n^2 = r^2 = v^2 = 1$, $\theta = \dfrac{\rho}{2}$ and ρ has been defined in the proof of Theorem 4.11. *Note that*

$$(\phi + i\theta)\mathbf{n} \neq i\,\theta\,\mathbf{r} + \phi\,\mathbf{v}.$$

Proof. It suffices to consider the general case of the first line, for the next lines are just the special cases. Since $\det M = \gamma^2 - m^2 = 1$, we can choose a nonzero scalar $z \in \mathbb{C}$ such that

$$\gamma^2 = \cosh^2 z, \qquad m^2 = \sinh^2 z,$$

and normalize the matrix M, coded by $\mathbf{c}$:

$$\mathbf{n} \overset{\mathsf{def}}{=} \frac{\sinh \overline{z}}{|\sinh z|^2}\,\mathbf{c}. \tag{4.25}$$

Since $\mathbf{n}^2 = 1$, hence

$$e^{z\,\mathbf{n}} = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,\mathbf{n}^n = \cosh z\, + \sinh z\,\mathbf{n},$$

as stated in (4.24). In conclusion, both $C = e^{(\phi+i\theta)\,\mathbf{n}}$ and $UP = e^{i\theta\,\mathbf{r}}\, e^{\phi\,\mathbf{v}}$ induce the same jaws operator, hence they are exact up to a sign. ∎

The surjectivity of the exponential map, based upon diagonalizability, has been established before, cf. Corollary 3.13. However, it was just a nonconstructive statement of existence. In contrast, the complex coding yields a generator, or rather "the generator", precisely.

COROLLARY 4.13. *Let Λ be a proper Lorentz matrix, coded as the jaws operator L_C with $C \leftrightarrow \gamma + \mathbf{c}$ with $c^2 \neq 0$ and normalized $\mathbf{n} = \mathbf{d} + i\,\mathbf{h}$. Then $\Lambda = e^{F(\mathbf{d},\,\mathbf{h})}$.*

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Proof. Let us change slightly the parametrization, defining the asymmetric jaws operators

$$C_z = \cosh\frac{z}{2} + \sinh\frac{z}{2}\,\mathbf{n},$$

which satisfy (1.2). Computing the derivative we find the symmetric jaws generator, acting on $\mathscr{H}$ and extendable onto $\mathscr{C}$, by (1.2). Slightly abusing notation, cf. (4.21),

$$C_z' = \frac{1}{2}\,C_z\mathbf{n} \quad \Rightarrow \quad FX = \frac{1}{2}\Big(\mathbf{n}X + X\overline{\mathbf{n}}\Big) = \Re\,\mathbf{n}X.$$

We check its action on the Pauli basis,

$$F^\sigma 1 = \mathbf{d}, \qquad F^{\,\sigma}\mathbf{e}_j = d_j + \mathbf{e}_j \times \mathbf{h}, \quad \text{i.e.,} \quad F = \begin{bmatrix} 0 & \mathbf{d}^{\intercal} \\ \mathbf{d} & V(\mathbf{h}) \end{bmatrix} = F(\mathbf{d}, \mathbf{h}),$$

which is a G-antisymmetric matrix. ☐

4.4.2. The defective matrix. While the handling of diagonalizable matrices is relatively simple with the help of powerful tools, yet, in contrast, defective matrices usually cause trouble. However, in the context of proper Lorentz matrices, their behavior pattern is strikingly simple.

In this subsection, we analyze the eigen-structure of $C = \gamma + \mathbf{c} = \alpha + i\,\beta + \mathbf{a} + i\,\mathbf{b}$, focusing especially on defective matrices.

PROPOSITION 4.14.

1. *The eigenvectors of C correspond to eigenvectors of the induced asymmetric jaws operator $L_{C,I}$ through the following relation,*

$$\begin{aligned} C\boldsymbol{\zeta} = \lambda\,\boldsymbol{\zeta} \quad &\textit{if and only if} \quad C^\sigma \boldsymbol{\zeta}\boldsymbol{\zeta}^* = \lambda\,\boldsymbol{\zeta}\boldsymbol{\zeta}^* \\ &\textit{if and only if} \quad (\gamma + \mathbf{c})\,\boldsymbol{\zeta}\boldsymbol{\zeta}^* = \lambda\,\boldsymbol{\zeta}\boldsymbol{\zeta}^*, \end{aligned} \tag{4.26}$$

   which is tantamount to $(\gamma + \mathbf{c})\,\boldsymbol{\zeta} = \lambda\,\boldsymbol{\zeta}$.
2. *The eigenvectors of C correspond to eigenvectors of the induced symmetric jaws operator L_C through the following relation,*

$$\begin{aligned} C\boldsymbol{\zeta}_j = \lambda_j\,\boldsymbol{\zeta}_j \quad &\Rightarrow \quad C^\sigma \boldsymbol{\zeta}_j\boldsymbol{\zeta}_k^* C^* = \lambda_j\overline{\lambda_k}\,\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k^* \\ &\textit{if and only if} \quad (\gamma + \mathbf{c})\,\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k^* = \lambda\,\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k^*. \end{aligned}$$

3. *The invertion of the coding $\boldsymbol{\zeta}\boldsymbol{\zeta}^* \mapsto H$ or $\boldsymbol{\zeta}_j\boldsymbol{\zeta}_k^* \mapsto H + i\,G$ becomes a rather cumbersome task. Fortunately, we do not need it.*
4. *Nonzero scalar factors do not affect eigenvectors, that is, $\gamma^{-1}C^{\,\sigma}\mathbf{u} = \lambda\,\mathbf{u}$ if and only if $C^{\,\sigma}\mathbf{u} = \gamma\lambda\,\mathbf{u}$. Hence, the technicalities may be alleviated by reducing C to the form*

$$C := C = 1 + t\,\mathbf{w} = 1 + t\,(\mathbf{d} + i\,\mathbf{h}), \qquad \textit{where} \quad d = h = 1, \quad t \in \mathbb{R}. \tag{4.27}$$

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5. *For a general* $\mathbf{f} = \mathbf{g} + i\,\mathbf{h}$*, we have* $\mathbf{f}\mathbf{f}^* = g^2 + h^2 + 2\,\mathbf{g}{\times}\mathbf{h}$*. Normalizing,*

$$\mathbf{f}\mathbf{f}^* \;\mapsto\; 1 + s\,\mathbf{e},$$

where $\mathbf{e} = \mathbf{g}{\times}\mathbf{h}$*,* $0 \le s \le 1$*, and*

$$\mathbf{f} = p\,\mathbf{g} + q\,\mathbf{h}, \quad \textit{where } g = h = 1,\ p^2 + q^2 = 1,\ 2pq = s.$$

Thus, $(p \pm q)^2 = 1 \pm s$ *yields four obvious solutions.*

6. *Points 4. and 5. simplify the eigen-equation:*

$$\big(1 + t\,\mathbf{w}\big)\,(1 + s\,\mathbf{e}) = \lambda\,(1 + s\,\mathbf{e}) \quad \Leftrightarrow \quad \begin{cases} 1 + st\,(we) = \lambda \\ s\,\mathbf{e} + t\,\mathbf{w} + i\,st\,\mathbf{w}{\times}\mathbf{e} = \lambda s\,\mathbf{e}, \end{cases}$$

followed by the comparison of the real and imaginary vector parts if needed.

Proof. By inspection. ◻

For the remainder of this section we assume that $\mathbf{d} \neq 0$ or $\mathbf{h} \neq 0$, which excludes the trivial case $C = \gamma\,I$. The characteristic polynomial of C is

$$\det(C - \lambda\,I) = (\gamma - \lambda)^2 - c^2, \quad \text{where} \quad c^2 = a^2 - b^2 + 2i\,(ab).$$

There are many ways to detect defective matrices, so let us begin with the double eigenvalue.

Proposition 4.15. *Let* $\det C = 1$*. Then the following are equivalent:*

(a) $c^2 = 0$*;*

(b) *There is a double eigenvalue;*

(c) 1 *is the double eigenvalue;*

(d) $\frac{1}{2}\,\mathrm{tr}\,C = \pm\,1$*;*

(e) $C = \pm\big(1 + a\,\mathbf{a} + i\,a\,\mathbf{b}\big)$*, with orthogonal unit real vectors* $\mathbf{a}, \mathbf{b}$*.*

Proof. If $c^2 \neq 0$, then the eigenvalues are $\gamma + \sqrt{c^2}$, accounting for two values of the complex radical. The double zero occurs if and only if $c^2 = 0$ if and only if $a = b > 0$ and $\mathbf{a} \perp \mathbf{b}$. Also, $\mathrm{tr}\,C = 2\,\gamma$. Therefore, (a)–(e) are equivalent. ◻

Proposition (d) above states that the "angle" between C and I is either 0 or π. That is, the meaning of "parallel" is ambiguous in a matrix space, for it may also mean "with the same linear span".

Following Proposition 4.14.(6), we may and do choose the parametrization

$$C = C_t \overset{\mathsf{def}}{=} 1 + t\,\mathbf{w}, \qquad \mathbf{w} \overset{\mathsf{def}}{=} \mathbf{d} + i\,\mathbf{h}, \quad t \in \mathbb{R}. \tag{4.28}$$

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Note that $c^2 = 0$ in Proposition 4.15.(a) means $w^2 = 0$. Therefore,

$$(1 + t\,\mathbf{w})\,(1 + s\,\mathbf{w}) = 1 + (t + s)\,\mathbf{w}. \tag{4.29}$$

In other words, the asymmetric jaws $L_{C_t,I}$, acting on Hermitian 2×2 matrices form a commutative operator group. All operators with parameter $t \neq 0$, besides having the same eigenvalue 1, share the same eigenvectors, which follows immediately from the decomposition:

$$C_s = \left(1 - \frac{s}{t}\right) I + \frac{s}{t}\, C_t.$$

In fact, the eigen-space is one-dimensional.

Proposition 4.16. *Any condition of* Proposition 4.15 *is equivalent to either of the following:*

(a) $L_{C,I}$ *is defective with* $1 \sim 1 + \mathbf{d}\times\mathbf{h}$*;*

(b) C *is defective;*

(c) C *represents a shear transformations with the matrix* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *with respect to a basis* $(\boldsymbol{\zeta}, \boldsymbol{\eta})$.

Proof.

We may and do choose $t = 1$. First we show that (4.28) implies (a). Let us consider the eigen-equation

$$\big(1 + \mathbf{w}\big)\,(1 + s\,\mathbf{e}) = 1 + s\,\mathbf{e} \quad \Leftrightarrow \quad \begin{cases} 1 + s\,(ue) + i\,s\,(ve) = 1 \\ \mathbf{u} - s\,\mathbf{v}\times\mathbf{e} + i\,(\mathbf{v} + s\,\mathbf{u}\times\mathbf{e}) = s\,\mathbf{e}. \end{cases}$$

Then $s \neq 0$, for otherwise $\mathbf{d} = \mathbf{h} = 0$. So, $\mathbf{e} \perp \mathbf{d}$, $\mathbf{e} \perp \mathbf{h}$, and thus, $\mathbf{e} = \mathbf{d}\times\mathbf{h}$ together with $s = 1$ (from the imaginary vector part) makes the eigenvector $1 + \mathbf{d}\times\mathbf{h}$ of $L_{C,1}$, and there is no more eigenvectors independent of it.

(a) $\Leftrightarrow$ (b): By Proposition 4.14, C has two independent eigenvectors if and only if $L_{C,I}$ does.

(c) $\Rightarrow$ (b) is obvious. (b) $\Rightarrow$ (c) (cf. the proof of Proposition 4.3.(a$'$)): A defective C with unit determinant is represented by $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$, $z \neq 0$, with respect to some basis $\mathbf{u}, \mathbf{v}$, so it suffices to scale $\mathbf{u} \mapsto z\mathbf{u}$. ☐

Proposition 4.17. *Let* C *have a nonzero vector part and* $\det C = 1$. C *is defective, i.e.,* $C \leftrightarrow 1 + t\,(\mathbf{d} + i\mathbf{h})$ *with* $\mathbf{d} \perp \mathbf{h}$ *if and only if* L_C *is defective. Further,* L_C *has the single eigenvalue* 1 *owning the span of* $1 + \mathbf{d}\times\mathbf{h}$ *and* $\mathbf{h}$.

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Proof. The "if" part is implicit in Proposition 4.3.(a).

In addition to $d{\times}\mathbf{h}$, the eigenvalue 1 also owns $\mathbf{h}$, which quickly follows from the identity

$$\big(1+t\,(\mathbf{d}+i\mathbf{h})\big)\,\mathbf{h}=\mathbf{v}\,\big(1-a\,(\mathbf{d}-i\mathbf{h})\big).$$

Should L_C be diagonalizable, so would be $L_C^*L_C=L_{C^*C}$. Therefore, consider

$$C^*C=1+2t^2+2t\,\mathbf{d}-2t^2\,\mathbf{d}{\times}\mathbf{h}=\text{const}\cdot(1+\mathbf{e}),$$

where $\mathbf{e}\neq 0$. Thus, it suffices to solve

$$(1+\mathbf{e})\,(\kappa+\mathbf{k})=\lambda(\kappa+\mathbf{k})\,(1-\mathbf{e})$$

for a real λ, κ and $\mathbf{k}\neq 0$. By comparing the scalar parts, $(wk)=-(wk)$, i.e., $\mathbf{k}\perp\mathbf{e}$. Thus, there are at most two vectors $\mathbf{k}$. The comparison of real vectors yields

$$\mathbf{k}+\kappa\,\mathbf{e}=\lambda\,(\mathbf{k}-\kappa\,\mathbf{e}).$$

Crossing it with $\mathbf{e}$ yields $\lambda=1$, and crossing it with $\mathbf{k}$ yields $\kappa=0$.

That is, the eigenspace of $L_C^*L_C$, hence of L_C, is two-dimensional. ☐

Defective shears $C_t=L_{C_t,I}$ with $\det C_t=1$ follow simple operational patterns: For example,

$$C_t^p=C_{tp}=1+pt\,\mathbf{w},\quad e^{C_t}=e\,C_t=e\,(1+t\,\mathbf{w}),\text{ etc.}$$

Defective jaws $L_t\overset{\mathsf{def}}{=}L_{C(t)}$ perform similarly as well since they share the group feature, because (4.29) implies

$$L_sL_t=L_{s+t}.$$

Proposition 4.18. *The scaled parametrization $C_t=1+\frac{t}{2}\mathbf{w}$, where $\mathbf{w}=\mathbf{d}+i\mathbf{h}$, entails the generator $F=F(\mathbf{d},\mathbf{h})$ of the operator group L_t,*

$$e^{tF}=I+t\,F+\frac{t^2}{2}\,\tilde F,\quad\textit{where } F^2=\tilde F=L_W \textit{ and } W\leftrightarrow\mathbf{w}. \tag{4.30}$$

Proof. We have just proved that the assumptions (1.2) are satisfied. Hence, (4.23) gives the generator, which on $\mathscr{H}$ acts as follows. For a Hermitian $X=\xi+\mathbf{x}$, slightly abusing notation, cf. (4.21), we have

$$FX=\frac{1}{2}\left(\mathbf{w}X+X\overline{\mathbf{w}}\right)=\Re\,\mathbf{w}X.$$

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Let us compute F^2, for a Hermitian X:

$$F^2 X = \mathbf{w}\left(\mathbf{w}\, X + X\, \overline{\mathbf{w}}\right) + \left(\mathbf{w}\, X + X\, \overline{\mathbf{w}}\right)\overline{\mathbf{w}} = 2\,\mathbf{w}\, X\, \overline{\mathbf{w}} \overset{\mathsf{def}}{=} \tilde{F}\, X.$$

Since $\mathbf{w}^2 = 0$, then $F^n = 0$ for $n \geq 3$, which gives (4.30).

Alternatively, Lemma 4.7 has provided the passage from C_t to L_t:

$$s = 1 + \frac{t^2}{4}, \qquad \mathbf{p} = t\,\mathbf{d} + \frac{t^2}{4}\,\mathbf{d}\times\mathbf{h}, \qquad \mathbf{q} = t\,\mathbf{d} - \frac{t^2}{4}\,\mathbf{d}\times\mathbf{h}$$
$$A = \left(1 - \frac{t^2}{2}\right) I + \frac{t^2}{2}\,(\mathbf{d}\mathbf{d}^{\mathsf{T}} + \mathbf{d}\mathbf{d}^{\mathsf{T}}) + t\,V(\mathbf{h}).$$

Evaluating the derivatives at $t = 0$, we obtain the generator of the group L_t:

$$s_0 = 0, \quad \mathbf{p}_0 = \mathbf{q}_0 = \mathbf{d}, \quad A_0 = V(\mathbf{h}).$$

We again recognize a G-antisymmetric matrix $F = F(\mathbf{d}, \mathbf{h})$. $\square$

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