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AUTOMORPHISMS OF A COMMUTING GRAPH OF RANK ONE UPPER TRIANGULAR MATRICES*

SHIKUN OU† AND JIN ZHONG‡

Abstract. Let F be a finite field, $n \geqslant 2$ an arbitrary integer, $\mathcal{M}_n(F)$ the set of all $n \times n$ matrices over F, and $\mathcal{U}_n^1(F)$ the set of all rank one upper triangular matrices of order n. For $\mathcal{S} \subseteq \mathcal{M}_n(F)$, denote $C(\mathcal{S}) = \{X \in \mathcal{S} |\ XA = AX \text{ for all } A \in \mathcal{S}\}$. The commuting graph of $\mathcal{S}$, denoted by $\Gamma(\mathcal{S})$, is the simple undirected graph with vertex set $\mathcal{S} \setminus C(\mathcal{S})$ in which for every two distinct vertices A and B, $A \sim B$ is an edge if and only if $AB = BA$. In this paper, it is shown that any graph automorphism of $\Gamma(\mathcal{U}_n^1(F))$ with $n \geqslant 3$ can be decomposed into the product of an extremal automorphism, an inner automorphism, a field automorphism and a local scalar multiplication.

Key words. Automorphisms, Commuting graphs, Characteristic subgraphs, Groups.

AMS subject classifications. 05C25, 05C60, 11T99, 15A04.

1. Introduction. Let F be a finite field, $F^* = F \setminus \{0\}$, and $n \geqslant 2$ an arbitrary integer. We denote by $\mathcal{M}_n(F)$ the set of all $n \times n$ matrices over F, $\mathcal{U}_n(F)$, $\mathcal{U}_n^{-1}(F)$ and $\mathcal{U}_n^1(F)$, respectively, the set of all upper triangular matrices, invertible upper triangular matrices and rank one upper triangular matrices in $\mathcal{M}_n(F)$. For a matrix $A = (a_{ij}) \in \mathcal{M}_n(F)$ and a map $\tau : F \to F$, let $[A]$ and A_τ, respectively, be the subspace spanned by A and the matrix $(\tau(a_{ij}))$.

The concept of commuting graph was first introduced and studied for semisimple rings by Akbari et al. in [2], and further studied in many references and therein, see [1,3–13]. A lot of results about the diameter, the connectivity of commuting graphs and so on have been obtained. Additional information about algebraic properties of the elements can be obtained by studying the properties of a commuting graph. For example, for a finite field F, if R is a ring with identity such that $\Gamma(R) \cong \Gamma(\mathcal{M}_2(F))$, then $R \cong \mathcal{M}_2(F)$, see [13]. It was conjectured that this is also true for the full matrix ring $\mathcal{M}_n(F)$, where F is a finite field and $n \geqslant 3$.

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†Faculty of Science, Jiangxi University of Science and Technology, Ganzhou 341000, China (ok6096@126.com). Supported by the NSFC grant no. 11661040, and the Science Foundation of Jiangxi University of Science and Technology (no. NSFJ2014-K12).
‡Faculty of Science, Jiangxi University of Science and Technology, Ganzhou 341000, China, (zhongjin1984@126.com). Supported by the NSFC grants no. 11661041 and no. 11426121, the Jiangxi Provincial Science and Technology Project (no. 20161BAB211016), and the Science Foundation of Jiangxi Provincial Education Department (no. GJJ150645).

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Generally, determining the full automorphisms of a graph is an important however a difficult problem both in graph theory and in algebraic theory. It seems that little is known for the automorphisms of the commuting graphs of rings. This observation motivates us to do some work on this topic. For a finite field F, it seems very difficult to determine the full automorphisms of $\Gamma(\mathcal{M}_n(F))$, so we focus on the subgraph of $\Gamma(\mathcal{M}_n(F))$ induced by $\mathcal{U}_n^1(F)$. It's clear that the vertex set of $\Gamma(\mathcal{U}_n^1(F))$ is $\mathcal{U}_n^1(F)$. The following four types of automorphisms for $\Gamma(\mathcal{U}_n^1(F))$ are called *standard automorphisms* of $\Gamma(\mathcal{U}_n^1(F))$.

- Let $P \in \mathcal{U}_n^{-1}(F)$. We define $\sigma_P : \mathcal{U}_n^1(F) \to \mathcal{U}_n^1(F)$ by $A \mapsto P^{-1}AP$. Then it is easy to see that σ_P is an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, which is called an *inner automorphism* of $\Gamma(\mathcal{U}_n^1(F))$.
- If τ is an automorphism of the field F, then the map $\theta_\tau : \mathcal{U}_n^1(F) \to \mathcal{U}_n^1(F)$ defined by $A \mapsto A_\tau$ is an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, which is called a *field automorphism* of $\Gamma(\mathcal{U}_n^1(F))$.
- Let $\varepsilon = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathcal{M}_n(F)$. Then it is not difficult to verify that the map $\eta : A \mapsto \varepsilon A^T \varepsilon$ is an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, which is called an *extremal automorphism* of $\Gamma(\mathcal{U}_n^1(F))$. Note that $\eta^2 = 1$. For convenience, the identity automorphism of $\Gamma(\mathcal{U}_n^1(F))$ is also regarded as an extremal automorphism.
- Let ξ be a permutation on $\mathcal{U}_n^1(F)$ such that each $[A]$ is stabilized, i.e, $\xi(A) = \alpha_A A$ for any $A \in \mathcal{U}_n^1(F)$, where $\alpha_A \in F^*$ depends on A. Then ξ is an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, which is called a *local scalar multiplication* of $\Gamma(\mathcal{U}_n^1(F))$.

It is not difficult to see that the following result holds.

Lemma 1.1. *Let σ_P, θ_τ, η and ξ be defined as above. Then,*

(i) $\sigma_P \cdot \xi = \xi \cdot \sigma_P$;

(ii) $\eta^{-1} \cdot \sigma_P \cdot \eta = \sigma_{\eta(P^{-1})}$;

(iii) $\eta^{-1} \cdot \theta_\tau \cdot \eta = \theta_\tau$;

(iv) $\theta_\tau^{-1} \cdot \xi \cdot \theta_\tau$ *and* $\eta^{-1} \cdot \xi \cdot \eta$ *both are local scalar multiplications of* $\Gamma(\mathcal{U}_n^1(F))$.

In this paper, we aim to describe the full automorphisms of $\Gamma(\mathcal{U}_n^1(F))$. In order to prove the main theorem of this paper, we will follow a technique from a recent paper in which the full automorphisms of the zero-divisor graph of $\mathcal{U}_n^1(F)$ were determined.

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In [14], Wong et al. showed that any graph automorphism of the zero-divisor graph of $\mathcal{U}_n^1(F)$ with $n \geqslant 3$ can be decomposed into the product of an inner automorphism, a field automorphism and a local scalar multiplication.

Our main result is as follows.

THEOREM 1.2. *Let $n \geqslant 3$. Then θ is an automorphism of $\Gamma(\mathcal{U}_n^1(F))$ if and only if θ can be decomposed into the product of an extremal automorphism, an inner automorphism, a field automorphism and a local scalar multiplication.*

2. Notations and preliminaries. Let F^n be the n-dimensional column vector space over F. It's well known that any $A \in \mathcal{U}_n^1(F)$ can be written as $A = \alpha\beta^T$, where $\alpha, \beta \in F^n$. Let $e_1,\ e_2,\ \ldots,\ e_n$ be the elements of the standard basis of F^n. For convenience, in a vector expression $\alpha = \sum a_i e_i$ the subscript i can be less than 1 or greater than n, and we use the convention that the coefficient a_i is regarded as zero if $i \leqslant 0$ or $i \geqslant n+1$ in some term $a_i e_i$. Then for $A \in \mathcal{U}_n^1(F)$, we can write $A = \sum_{i \leqslant j} a_i b_j e_i e_j^T$ with $a_i,\ b_j \in F$. Denote by I the identity matrix.

In order to describe all automorphisms of the graph $\Gamma(\mathcal{U}_n^1(F))$, we need firstly to study the automorphisms of a related graph. For $\mathcal{S} \subseteq \mathcal{M}_n(F)$, we refer to $\overline{\Gamma}(\mathcal{S})$ as the graph with vertex set $\{[A] \mid A \in \mathcal{S} \setminus C(\mathcal{S})\}$ in which for every two distinct vertices $[A]$ and $[B]$, $[A] \sim [B]$ is an edge if and only if $AB = BA$. The graph is well defined since $AB = BA$ if and only if $(aA)(bB) = (bB)(aA)$ for any $a,\ b \in F^*$. Let V_n be the vertex set of the graph $\overline{\Gamma}(\mathcal{U}_n^1(F))$, i.e.,

$$V_n = \{[A] \mid A \in \mathcal{U}_n^1(F)\}.$$

For $[A] \in V_n$, we denote by $\mathcal{N}([A])$ the set of neighbours of $[A]$. The *degree* of $[A]$, written as $\deg([A])$, is the cardinality of $\mathcal{N}([A])$. For a nonempty subset W of V_n, let $|W|$ be the cardinality of W. The subgraph of $\overline{\Gamma}(\mathcal{U}_n^1(F))$ induced by W is denoted by $\overline{\Gamma}[W]$. For two subsets U and W of the vertex set V_n, we denote $U \sim W$ (resp., $U \perp W$) if $x \sim y$ (resp., $x \not\sim y$) for any $x \in U$ and any $y \in W$. Also $\{x\} \sim W$ (resp., $\{x\} \perp W$) is denoted by $x \sim W$ (resp., $x \perp W$). Set

$$W^{\sim}(U) = \{x \in U :\ x \sim W\},$$

$$W^{\perp}(U) = \{y \in U :\ y \perp W\},$$

and

$$\Phi_{ij} = \left\{ \left[\left(\textstyle\sum_{1 \leqslant s \leqslant i-1} a_s e_s + e_i \right) \left(e_j + \textstyle\sum_{j+1 \leqslant t \leqslant n} b_t e_t \right)^T \right] \mid a_s, b_t \in F \right\}, \quad 1 \leqslant i \leqslant j \leqslant n,$$

$$\Phi_k = \textstyle\bigcup_{1 \leqslant i \leqslant n+1-k} \Phi_{i,i+k-1}, \quad 1 \leqslant k \leqslant n,$$

$$\mathcal{W}_{kl} = \Phi_{kl} \cup \Phi_{n-l+1,n-k+1}, \quad 1 \leqslant k \leqslant \lfloor \tfrac{n+1}{2} \rfloor \text{ and } k \leqslant l \leqslant n-k+1.$$

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For $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ and $1 \leqslant t \leqslant n$, by a direct computation, we have

$$\Phi_k^{\sim}(\Phi_t) = \begin{cases} \Phi_t \setminus \bigcup_{1 \leqslant i \leqslant n-k-t+2} \mathcal{W}_{i,i+t-1}, & k+t \leqslant n+1, \\ \Phi_t, & \text{otherwise}, \end{cases} \tag{2.1}$$

$$\mathcal{W}_{kk}^{\perp}(\Phi_t) = \begin{cases} \mathcal{W}_{k,k+t-1}, & t = n-2k+2, \\ \varnothing, & \text{otherwise}, \end{cases} \tag{2.2}$$

$$\mathcal{W}_{k,n-k+1}^{\perp}(\Phi_t) = \begin{cases} \mathcal{W}_{k-t+1,k}, & k \geqslant t, \\ \varnothing, & \text{otherwise}. \end{cases} \tag{2.3}$$

In [14], Wong et al. gave an explicit description of the vertex set and the order of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

Lemma 2.1. ([14, Lemma 3.1])

(i) V_n is the disjoint union of all Φ_{ij}, $1 \leqslant i \leqslant j \leqslant n$.

(ii) The number of vertices in Φ_{ij} is $|F|^{n+i-j-1}$.

(iii) The number of vertices in $\overline{\Gamma}(\mathcal{U}_n^1(F))$ is $|V_n| = \sum_{1 \leqslant i \leqslant j \leqslant n} |F|^{n+i-j-1}$.

Now, we study the vertex degree of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

Lemma 2.2. *Let $1 \leqslant i \leqslant j \leqslant n$, and $[A] \in \Phi_{ij}$. Then deg$([A]) = |V_n| - (n+i-j-2+\delta_{ij})|F|^{n-1} - \delta_{ij}$, where δ_{ij} is defined to be 0 or 1 depending on i being equal to j or not.*

Proof. We first consider $\deg([e_i e_j^T])$ for $1 \leqslant i \leqslant j \leqslant n$. For any $[\alpha\beta^T]$ in $\mathcal{N}([e_i e_j^T])$, we may assume that $\alpha = \sum_{s \leqslant k} a_s e_s \in F^n$, $\beta = \sum_{l \leqslant t} b_t e_t \in F^n$, $1 \leqslant k \leqslant l \leqslant n$. It's easily seen that

$$\left[\alpha\beta^T\right] \sim \left[e_i e_j^T\right] \Leftrightarrow b_i \left(\alpha e_j^T\right) = a_j \left(e_i \beta^T\right) \tag{2.4}$$

$$\Leftrightarrow a_k b_i = 0 \text{ for all } k \neq j, \text{ and } a_j b_l = 0 \text{ for all } l \neq i. \tag{2.5}$$

Now, we claim that $b_i = 0$. Indeed, if $b_i \neq 0$, then by (2.5), $a_k = 0$ for $k \neq j$. Thus, it follows from (2.4) that $a_j \neq 0$. Moreover, by (2.5) again, $b_l = 0$ for $l \neq i$. Consequently, $\alpha\beta^T = (a_j b_i) e_j e_i^T$, which implies that $[e_j e_i^T] = [\alpha\beta^T] \in \mathcal{N}([e_i e_i^T])$ (note that $[e_i e_i^T] \notin \mathcal{N}([e_i e_i^T])$), a contradiction. Since $b_i = 0$, there exists $l \neq i$ such that $b_l \neq 0$. Then by (2.5) we have $a_j = 0$. Therefore, $[\alpha\beta^T] \sim [e_i e_j^T]$ if and only if $a_j = 0$ and $b_i = 0$.

For any $1 \leqslant k \leqslant l \leqslant n$ and $1 \leqslant s \leqslant n$, denote

$$\Psi_{kl}^{(s*)} = \left\{ \left[\alpha\beta^T\right] \in \Phi_{kl} \mid \alpha, \beta \in F^n, \alpha^T e_s \neq 0 \right\},$$

$$\Psi_{kl}^{(*s)} = \left\{ \left[\alpha\beta^T\right] \in \Phi_{kl} \mid \alpha, \beta \in F^n, \beta^T e_s \neq 0 \right\},$$

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and

$$\Psi_1 = \Big\{ \left[\alpha\beta^T\right] \in \mathcal{U}_n^1(F) \mid \alpha, \beta \in F^n, \alpha^T e_j \neq 0 \Big\},$$

$$\Psi_2 = \Big\{ \left[\alpha\beta^T\right] \in \mathcal{U}_n^1(F) \mid \alpha, \beta \in F^n, \beta^T e_i \neq 0 \Big\}.$$

Then we can see that

$$\Psi_1 = \bigcup_{j \leqslant k \leqslant l \leqslant n} \Phi_{kl}^{(j*)}, \quad \Psi_2 = \bigcup_{1 \leqslant k \leqslant l \leqslant i} \Phi_{kl}^{(*i)},$$

and

$$\mathcal{N}\left(\left[e_i e_j^T\right]\right) = V_n \setminus \left(\Psi_1 \cup \Psi_2 \cup \left\{\left[e_i e_j^T\right]\right\}\right).$$

Notice that $\Psi_1 \cap \Psi_2 = \Phi_{ii}$ or $\varnothing$ depending on i being equal to j or not. Hence

$$\begin{aligned}
\deg\left(\left[e_i e_j^T\right]\right) &= |V_n| - |\Psi_1 \cup \Psi_2| - \delta_{ij} \\
&= |V_n| - |\Psi_1| - |\Psi_2| + |\Psi_1 \cap \Psi_2| - \delta_{ij} \\
&= |V_n| - \sum_{j \leqslant k \leqslant l \leqslant n} |\Phi_{kl}^{(j*)}| - \sum_{1 \leqslant k \leqslant l \leqslant i} |\Phi_{kl}^{(*i)}| + (1 - \delta_{ij})|\Phi_{ii}| - \delta_{ij}.
\end{aligned}$$

Now a direct computation shows that

$$|\Psi_{kl}^{(j*)}| = \begin{cases} (|F|-1)\,|F|^{n+k-l-2}, & j+1 \leqslant k \leqslant n, \\ |F|^{n+k-l-1}, & k = j, \end{cases}$$

and

$$|\Psi_{kl}^{(*i)}| = \begin{cases} (|F|-1)\,|F|^{n+k-l-2}, & 1 \leqslant l \leqslant i-1, \\ |F|^{n+k-l-1}, & l = i. \end{cases}$$

Thus,

$$\begin{aligned}
\deg([e_i e_j^T]) &= |V_n| - i|F|^{n-1} - (n-j-1)|F|^{n-1} + (1-\delta_{ij})|F|^{n-1} - \delta_{ij} \\
&= |V_n| - (n+i-j-2+\delta_{ij})|F|^{n-1} - \delta_{ij}.
\end{aligned}$$

Next, we will show that $\deg([A]) = \deg([e_i e_j^T])$ for $[A] \in \Phi_{ij}$, $1 \leqslant i \leqslant j \leqslant n$. Suppose that $[A] = [(\sum_{1 \leqslant s \leqslant i-1} a'_s e_s + e_i)(e_j + \sum_{j+1 \leqslant t \leqslant n} b'_t e_t)^T] \in \Phi_{ij}$, where $a'_s,\ b'_t \in F$. Set $P = I - \sum_{1 \leqslant s \leqslant i-1} a'_s e_s e_i^T + \sum_{j+1 \leqslant t \leqslant n} b'_t e_j e_t^T \in \mathcal{U}_n^{-1}(F)$. Then $[P^{-1}AP] = [e_i e_j^T]$. We define the map σ from $\mathcal{N}([A])$ to $\mathcal{N}([e_i e_j^T])$ by $[X] \mapsto [PXP^{-1}]$, and notice that σ is bijective. Hence, $\deg([A]) = \deg([e_i e_j^T])$ for any $[A] \in \Phi_{ij}$, $1 \leqslant i \leqslant j \leqslant n$. ☐

In order to classify automorphisms of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, it is necessary to investigate a class of special subgraphs, the characteristic subgraphs of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

Definition 2.3. For an automorphism ψ of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, a nonempty subset W of the vertex set V_n is called *stable* under ψ if $\psi(W) = W$. The subgraph $\overline{\Gamma}_0$ of

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$\overline{\Gamma}(\mathcal{U}_n^1(F))$ is called a *characteristic subgraph* if the vertex set of $\overline{\Gamma}_0$ is stable under each automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

It's obvious that the intersection, union and difference of two characteristic subgraphs are also such subgraphs, and it is not difficult to see that the following result holds.

LEMMA 2.4. *Let U and W be subsets of V_n. If $\overline{\Gamma}[W]$ and $\overline{\Gamma}[U]$ are characteristic subgraphs of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, then so are $\overline{\Gamma}[W^{\sim}(U)]$ and $\overline{\Gamma}[W^{\perp}(U)]$.*

Clearly, V_n is the disjoint union of $\Phi_1,\ \Phi_2,\ \ldots,\ \Phi_n$. For $1 \leqslant k \leqslant n$, if $[A] \in \Phi_{i,i+k-1}$, $1 \leqslant i \leqslant n+1-k$, by Lemma 2.2 we see that $\deg([A]) = |V_n| - (n-k-1+\delta_{i,i+k-1})|F|^{n-1} - \delta_{i,i+k-1}$, which shows

$$\deg([A]) = |V_n| - (n-k-1+\delta_{1k})|F|^{n-1} - \delta_{1k},\ \forall\ [A] \in \Phi_k,\ 1 \leqslant k \leqslant n, \tag{2.6}$$

where δ_{ij} is defined as in Lemma 2.2. In the following, we will introduce some characteristic subgraphs of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, which will play an important role in the studying of the automorphisms of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

LEMMA 2.5. *For any $1 \leqslant k \leqslant n$, $\overline{\Gamma}[\Phi_k]$ is a characteristic subgraph of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.*

Proof. Let ψ be an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. For any $[A] \in \Phi_k$, assume that $\psi([A]) \in \Phi_l$, $1 \leqslant k, l \leqslant n$. We can see from (2.6) that

$$\deg(\psi([A])) = |V_n| - (n-l-1+\delta_{1l})|F|^{n-1} - \delta_{1l}. \tag{2.7}$$

Since $\deg([A]) = \deg(\psi([A]))$ for any $[A] \in V_n$, then by (2.6) and (2.7) we get $k = l$, which implies that $\psi(\Phi_k) \subseteq \Phi_k$. Similarly, we have $\psi^{-1}(\Phi_k) \subseteq \Phi_k$. By considering the action of ψ on $\psi^{-1}(\Phi_k) \subseteq \Phi_k$ we get $\Phi_k = \psi(\psi^{-1}(\Phi_k)) \subseteq \psi(\Phi_k)$. Therefore, $\psi(\Phi_k) = \Phi_k$. ◻

LEMMA 2.6. *For any $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ and $k \leqslant l \leqslant n-k+1$, $\overline{\Gamma}[\mathcal{W}_{kl}]$ is a characteristic subgraph of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.*

Proof. Denote by $\mathscr{C}$ the set of all characteristic subgraphs of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. We first proceed by induction on k to show that $\overline{\Gamma}[\mathcal{W}_{kk}] \in \mathscr{C}$ for any $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$. For $k = 1$, $\Phi_n^{\perp}(\Phi_1) = \mathcal{W}_{11}$, we can see from Lemma 2.4 and Lemma 2.5 that $\overline{\Gamma}[\mathcal{W}_{11}] \in \mathscr{C}$. Assume that $\overline{\Gamma}[\mathcal{W}_{11}]$, $\overline{\Gamma}[\mathcal{W}_{22}]$, $\ldots$, $\overline{\Gamma}[\mathcal{W}_{k-1,k-1}] \in \mathscr{C}$, $2 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$. Observe that $\Phi_{n-k+1}^{\sim}(\Phi_1) = \Phi_1 \setminus \bigcup_{1\leqslant i\leqslant k} \mathcal{W}_{ii}$ (see (2.1)), which implies that $\overline{\Gamma}[\Phi_1 \setminus \bigcup_{1\leqslant i\leqslant k} \mathcal{W}_{ii}] \in \mathscr{C}$. Hence, $\overline{\Gamma}[\mathcal{W}_{kk}] = \overline{\Gamma}[\Phi_1] - \overline{\Gamma}[\Phi_1 \setminus \bigcup_{1\leqslant i\leqslant k} \mathcal{W}_{ii}] - \sum_{1\leqslant i\leqslant k-1} \overline{\Gamma}[\mathcal{W}_{ii}] \in \mathscr{C}$.

Next, we show that $\overline{\Gamma}[\mathcal{W}_{kl}] \in \mathscr{C}$ for $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ and $k+1 \leqslant l \leqslant n-k+1$ by considering the following three cases.

CASE 1. $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ *and* $l = n-k+1$.

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It's clear that $\mathcal{W}_{1n} = \Phi_{1n} = \Phi_n$, then by Lemma 2.5 we know that $\mathcal{W}_{1n} \in \mathscr{C}$. For $2 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ and $l = n-k+1$, since $\mathcal{W}_{kk}^{\perp}(\Phi_{l-k+1}) = \mathcal{W}_{kl}$ (see (2.2)), then Lemma 2.4, Lemma 2.5 and the above argument imply that $\overline{\Gamma}[\mathcal{W}_{kl}] \in \mathscr{C}$.

Case 2. $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ *and* $k+1 \leqslant l \leqslant \lfloor \frac{n+1}{2} \rfloor$.

By Case 1 we have $\overline{\Gamma}[\mathcal{W}_{l,n-l+1}] \in \mathscr{C}$. Then we can see from $\mathcal{W}_{l,n-l+1}^{\perp}(\Phi_{l-k+1}) = \mathcal{W}_{kl}$ (see (2.3)) that $\overline{\Gamma}[\mathcal{W}_{kl}] \in \mathscr{C}$.

Case 3. $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ *and* $\lfloor \frac{n+1}{2} \rfloor + 1 \leqslant l \leqslant n-k$.

We proceed by induction on t to show that $\overline{\Gamma}[\mathcal{W}_{t,l-k+t}] \in \mathscr{C}$ for $1 \leqslant t \leqslant k$. By Case 2 we have $\mathcal{W}_{1,n-l+1} \in \mathscr{C}$. For $k=1$, $\mathcal{W}_{1,n-l+1}^{\sim}(\Phi_{l-k+1}) = \Phi_{l-k+1} \setminus \mathcal{W}_{1,l-k+1}$ (see (2.1)), which implies that $\overline{\Gamma}[\Phi_{l-k+1} \setminus \mathcal{W}_{1,l-k+1}] \in \mathscr{C}$. Thus, $\overline{\Gamma}[\mathcal{W}_{1,l-k+1}] = \overline{\Gamma}[\Phi_{l-k+1}] - \overline{\Gamma}[\Phi_{l-k+1} \setminus \mathcal{W}_{1,l-k+1}] \in \mathscr{C}$. Now assume that $\overline{\Gamma}[\mathcal{W}_{1,l-k+1}]$, $\overline{\Gamma}[\mathcal{W}_{2,l-k+2}]$, $\ldots$, $\overline{\Gamma}[\mathcal{W}_{t-1,l-k+t-1}] \in \mathscr{C}$, and denote

$$\Psi = \Phi_{l-k+1} \setminus \bigcup_{1 \leqslant i \leqslant t-1} \mathcal{W}_{i,l-k+i}.$$

Obviously, $\overline{\Gamma}[\Psi] \in \mathscr{C}$. Then by $\mathcal{W}_{1,n-l+k-t}^{\sim}(\Psi) = \Psi \setminus \mathcal{W}_{t,l-k+t}$, we have $\overline{\Gamma}[\Psi \setminus \mathcal{W}_{t,l-k+t}] \in \mathscr{C}$, which implies that $\overline{\Gamma}[\mathcal{W}_{t,l-k+t}] = \overline{\Gamma}[\Psi] - \overline{\Gamma}[\Psi \setminus \mathcal{W}_{t,l-k+t}] \in \mathscr{C}$.

Therefore, $\overline{\Gamma}[\mathcal{W}_{kl}] \in \mathscr{C}$ for all $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ and $k \leqslant l \leqslant n-k+1$. ☐

3. Automorphisms of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. In this section, we construct three standard automorphisms of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, based on which we can describe any automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$ for $n \geqslant 3$.

- Let $P \in \mathcal{U}_n^{-1}(F)$. Define $\overline{\sigma}_P: V_n \to V_n$ by $[A] \mapsto [P^{-1}AP]$. Then it is easy to verify that $\overline{\sigma}_P$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, which is called an *inner automorphism* of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.
- Let τ be an automorphism of the field F. Then the map $\overline{\theta}_\tau: V_n \to V_n$ defined by $[A] \mapsto [A_\tau]$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, which is called a *field automorphism* of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.
- Define the map $\overline{\eta}: V_n \to V_n$ by $[A] \mapsto [\varepsilon A^T \varepsilon]$, where $\varepsilon = \sum_{1 \leqslant i \leqslant n} e_i e_{n+1-i}^T$. Then it is not difficult to see that $\overline{\eta}$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, which is called an *extremal automorphism* of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

The main result of this section is the following theorem.

Theorem 3.1. *Let $n \geqslant 3$. Then ψ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$ if and only if*

$$\psi = \overline{\eta}^{\delta} \cdot \overline{\sigma}_P \cdot \overline{\theta}_\tau,$$

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where $\overline{\eta}$, $\overline{\sigma}_P$ and $\overline{\theta}_\tau$ respectively are an extremal automorphism, an inner automorphism and a field automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, $\delta = 0$ or 1.

Proof. The sufficiency is obvious, we only prove the necessity. Let ψ be an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. Since $\overline{\Gamma}[\mathcal{W}_{11}]$ is a characteristic subgraph of $\overline{\Gamma}(\mathcal{U}_n^1(F))$ (see Lemma 2.6), we have $\psi([e_ne_n^T]) \in \mathcal{W}_{11}$ $(= \Phi_{11} \cup \Phi_{nn})$. If $\psi([e_ne_n^T]) \in \Phi_{11}$, then $\overline{\eta} \cdot \psi([e_ne_n^T]) \in \Phi_{nn}$. Denote $\psi_1 = \overline{\eta}^\delta \cdot \psi$ with $\delta = 0$ or 1 such that $\psi_1([e_ne_n^T]) \in \Phi_{nn}$. We complete the proof by verifying the following ten claims.

Claim 1. *There exists an inner automorphism $\overline{\sigma}_Q$ with $Q \in \mathcal{U}_n^{-1}(F)$ such that $[e_1e_j^T]$ is fixed by $\overline{\sigma}_Q \cdot \psi_1$, $1 \leqslant j \leqslant n$.*

Suppose that $\psi_1([e_ne_n^T]) = [(\sum_{1\leqslant s\leqslant n-1} a_s^{(n)} e_s + e_n)e_n^T] \in \Phi_{nn}$, where $a_s^{(n)} \in F$. Set $Q_1 = I - \sum_{1\leqslant s\leqslant n-1} a_s^{(n)} e_s e_n^T \in \mathcal{U}_n^{-1}(F)$, then $\overline{\sigma}_{Q_1} \cdot \overline{\phi_1}([e_ne_n^T]) = [e_ne_n^T]$.

For $1 \leqslant j \leqslant n-1$, we assert that $\overline{\sigma}_{Q_1} \cdot \psi_1([e_1e_j^T]) \in \Phi_{1j}$. Actually, if $\overline{\sigma}_{Q_1} \cdot \psi_1([e_1e_j^T]) \notin \Phi_{1j}$, then $\overline{\sigma}_{Q_1} \cdot \psi_1([e_1e_j^T]) \in \Phi_{n-j+1,n}$. By applying $(\overline{\sigma}_{Q_1} \cdot \psi_1)^{-1}$ to $\overline{\sigma}_{Q_1} \cdot \psi_1([e_1e_j^T]) \not\sim [e_ne_n^T]$, we have $[e_1e_j^T] \not\sim [e_ne_n^T]$, a contradiction. Now, we may assume that

$$\overline{\sigma}_{Q_1} \cdot \psi_1\left(\left[e_1e_j^T\right]\right) = \left[e_1\left(e_j + \sum_{j+1\leqslant t\leqslant n} a_t^{(j)} e_t\right)^T\right], \quad 1 \leqslant j \leqslant n-1,$$

where $a_t^{(j)} \in F$. Set $Q_2 = I - \sum_{1\leqslant s<t\leqslant n} a_t^{(s)} e_s e_t^T \in \mathcal{U}_n^{-1}(F)$, then $\overline{\sigma}_{Q_2} \cdot \overline{\sigma}_{Q_1} \cdot \psi_1([e_1e_j^T]) = [e_1e_j^T]$, $1 \leqslant j \leqslant n-1$. Observe that $\overline{\Gamma}[\Phi_n]$ is a characteristic subgraph of $\overline{\Gamma}(\mathcal{U}_n^1(F))$, then it follows that $\overline{\sigma}_{Q_2} \cdot \overline{\sigma}_{Q_1} \cdot \psi_1([e_1e_n^T]) = [e_1e_n^T]$. Denote $\psi_2 = \overline{\sigma}_Q \cdot \psi_1$ with $Q = Q_1Q_2$, then $\psi_2([e_1e_j^T]) = [e_1e_j^T]$ for all $1 \leqslant j \leqslant n$.

Claim 2. *Each $[e_ie_j^T]$ is fixed by ψ_2, $1 \leqslant i \leqslant j \leqslant n$.*

It is clear that $\psi_2([e_ie_j^T]) \subseteq \mathcal{W}_{ij} = \Phi_{ij}$ for $i = n-j+1$. For $i \neq n-j+1$, if $\psi_2([e_ie_j^T]) \notin \Phi_{ij}$, then $\psi_2([e_ie_j^T]) \in \Phi_{n-j+1,n-i+1}$. Applying ψ_2^{-1} to $[e_1e_{n-j+1}^T] \not\sim \psi_2([e_ie_j^T])$ yields $[e_1e_{n-j+1}^T] \not\sim [e_ie_j^T]$ by Claim 1, a contradiction. Thus, $\psi_2([e_ie_j^T]) \in \Phi_{ij}$, $2 \leqslant i \leqslant j \leqslant n$. Now we assume that

$$\psi_2\left(\left[e_ie_j^T\right]\right) = \left[\left(\sum_{1\leqslant s\leqslant i-1} a_s^{(i)} e_s + e_i\right)\left(e_j + \sum_{j+1\leqslant t\leqslant n} a_t^{(j)} e_t\right)^T\right], \quad 2 \leqslant i \leqslant j \leqslant n,$$

where $a_s^{(i)}, a_t^{(j)} \in F$. For $1 \leqslant s \leqslant i-1$, by applying ψ_2 to $[e_1e_s^T] \sim [e_ie_j^T]$ we have $[e_1e_s^T] \sim \psi_2([e_ie_j^T])$, which implies that $a_s^{(i)} = 0$. Similarly, $a_t^{(j)} = 0$ for $j+1 \leqslant t \leqslant n$. Thus, $\psi_2([e_ie_j^T]) = [e_ie_j^T]$ for all $2 \leqslant i \leqslant j \leqslant n$.

Claim 3. *Φ_{11} and Φ_{nn} are stable under ψ_2.*

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Denote

$$\Phi_{11}^{(k)} = \left\{ \left[e_1\alpha^T\right] \in \Phi_{11} \mid \alpha \in F^n \text{ satisfies } \alpha^T e_k = 0 \right\},\ 2 \leqslant k \leqslant n,$$

$$\Phi_{nn}^{(l)} = \left\{ \left[\beta e_n^T\right] \in \Phi_{nn} \mid \beta \in F^n \text{ satisfies } e_l^T \beta = 0 \right\},\ 1 \leqslant l \leqslant n-1,$$

$$\Phi_{11}^{*} = \left\{ \left[e_1\alpha^T\right] \in \Phi_{11} \mid \alpha \in F^n \text{ satisfies } \alpha^T e_k \neq 0 \text{ for all } 1 \leqslant k \leqslant n \right\},$$

$$\Phi_{nn}^{*} = \left\{ \left[\beta e_n^T\right] \in \Phi_{nn} \mid \beta \in F^n \text{ satisfies } e_l^T \beta \neq 0 \text{ for all } 1 \leqslant l \leqslant n \right\}.$$

Clearly, $\Phi_{11} = \Phi_{11}^{(2)} \cup \cdots \cup \Phi_{11}^{(n)} \cup \Phi_{11}^{*}$ and $\Phi_{nn} = \Phi_{nn}^{(1)} \cup \cdots \cup \Phi_{nn}^{(n-1)} \cup \Phi_{nn}^{*}$. Next, we prove the following three statements.

(S_1) $\psi_2(\Phi_{11}^{(k)}) \subseteq \Phi_{11}$, $\psi_2(\Phi_{nn}^{(l)}) \subseteq \Phi_{nn}$, $2 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant n-1$.

For $2 \leqslant k \leqslant n$, if $\psi_2(\Phi_{11}^{(k)}) \not\subseteq \Phi_{11}$, then there exists $[e_1\alpha^T] \in \Phi_{11}^{(k)}$ with $\alpha \in F^n$ such that $\psi_2([e_1\alpha^T]) \in \Phi_{nn}$. Suppose that

$$\psi_2([e_1\alpha^T]) = [\beta e_n^T] \in \Phi_{nn} \quad \text{with} \quad \beta = \sum b_i e_i \in F^n.$$

By applying ψ_2 to $[e_1\alpha^T] \sim [e_k e_n^T]$ (notice that $\alpha^T e_k = 0$) we get $[\beta e_n^T] \sim [e_k e_n^T]$, which implies that

$$(e_n^T e_k) \sum b_i e_i e_n^T = b_n e_k e_n^T. \tag{3.1}$$

When $k \neq n$, we see that $e_n^T e_k = 0$. Then by (3.1), $b_n = 0$, which shows that $[\beta e_n^T] \notin \Phi_{nn}$, a contradiction. When $k = n$, (3.1) can be rewritten as $\sum b_i e_i e_n^T = b_n e_n e_n^T$, which follows that $b_i = 0$ for $1 \leqslant i \leqslant n-1$, and $b_n \neq 0$. Then $\psi_2([e_1\alpha^T]) = [b_n e_n e_n^T] = [e_n e_n^T]$, which contradicts the result $\psi_2([e_n e_n^T]) = [e_n e_n^T]$ (see Claim 2). Hence, $\psi_2(\Phi_{11}^{(k)}) \subseteq \Phi_{11}$ for $2 \leqslant k \leqslant n$. The proof of $\psi_2(\Phi_{nn}^{(l)}) \subseteq \Phi_{nn}$ for $1 \leqslant l \leqslant n-1$ is similar.

Now, it follows from $\Phi_{11}^{*} \cup \Phi_{nn}^{*} = \mathcal{W}_{11} - \bigcup_{2 \leqslant k \leqslant n} \Phi_{11}^{(k)} - \bigcup_{1 \leqslant k \leqslant n-1} \Phi_{nn}^{(k)}$ and the above arguments that $\psi_2(\Phi_{11}^{*} \cup \Phi_{nn}^{*}) = \Phi_{11}^{*} \cup \Phi_{nn}^{*}$.

(S_2) $\Delta = \{(ae_1 + e_2)e_2^T \in \Phi_{22} |\ a \in F\}$ is stable under ψ_2.

By Lemma 2.6, for any $a \in F$, $\psi_2((ae_1 + e_2)e_2^T) \in \Phi_{22}$ or $\psi_2((ae_1 + e_2)e_2^T) \in \Phi_{n-1,n-1}$. The case $n = 3$ is clear. If $n \geqslant 4$, by applying ψ_2 to $[(ae_1+e_2)e_2^T] \sim [e_1 e_{n-1}^T]$ we have $\psi_2([(ae_1 + e_2)e_2^T]) \sim [e_1 e_{n-1}^T]$. It follows that $\psi_2([(ae_1 + e_2)e_2^T]) \notin \Phi_{n-1,n-1}$, and so $\psi_2([(ae_1 + e_2)e_2^T]) \in \Phi_{22}$. Assume that

$$\psi_2\left(\left[(ae_1 + e_2)\, e_2^T\right]\right) = \left[(a_1 e_1 + e_2) \left(e_2 + \sum_{3 \leqslant t \leqslant n} a_t e_i\right)^T\right], \tag{3.2}$$

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where $a_1, a_t \in F$, $3 \leqslant t \leqslant n$. For $3 \leqslant t \leqslant n$, applying ψ_2 to $[(ae_1+e_2)e_2^T] \sim [e_te_n^T]$ yields $[(a_1e_1+e_2)(e_2+\sum_{3\leqslant t\leqslant n} a_te_i)^T] \sim [e_te_n^T]$, which implies that $a_t = 0$ for all $3 \leqslant t \leqslant n$ in (3.2). Hence, $\psi_2([(ae_1+e_2)e_2^T]) = [(a_1e_1+e_2)e_2^T] \in \Delta$.

(S_3) $\psi_2(\Phi_{11}^*) = \Phi_{11}^*$ and $\psi_2(\Phi_{nn}^*) = \Phi_{nn}^*$.

Recall that $\psi_2(\Phi_{11}^* \cup \Phi_{nn}^*) = \Phi_{11}^* \cup \Phi_{nn}^*$. If $\psi_2(\Phi_{11}^*) \neq \Phi_{11}^*$, then there exists $[A] \in \Phi_{11}^*$ such that $\psi_2([A]) \in \Phi_{nn}^*$. Let $A = e_1(e_1 + \sum_{2\leqslant i\leqslant n} a_ie_i)^T$ with $a_i \in F^*$, $2 \leqslant i \leqslant n$, and suppose that

$$\psi_2\left([A]\right) = \left[\left(\sum_{1\leqslant j\leqslant n-1} b_je_i + e_n\right)e_n^T\right],$$

where $b_j \in F^*$, $1 \leqslant j \leqslant n-1$. Applying ψ_2 to $[A] \sim [(-a_2e_1+e_2)e_2^T]$ we know that $[(\sum_{1\leqslant j\leqslant n-1} b_je_i + e_n)e_n^T] \sim [(xe_1+e_2)e_2^T]$ for some $x \in F$, which implies that $b_2 = 0$, a contradiction. Thus, $\psi_2(\Phi_{11}^*) = \Phi_{11}^*$ and $\psi_2(\Phi_{nn}^*) = \Phi_{nn}^*$.

Claim 4. *For any $[\alpha_1\beta_1^T] \in V_n$, let $\psi_2([\alpha_1\beta_1^T]) = [\alpha_2\beta_2^T]$. Then the k-th component of α_1 (resp., β_1) is zero if and only if the k-th component of α_2 (resp., β_2) is zero, $1 \leqslant k \leqslant n$.*

If $[\alpha_1\beta_1^T] \in \Phi_{11}$, by Claim 3 we may assume that $\alpha_1 = e_1$ and $\alpha_2 = e_1$, then $\psi_2([e_1\beta_1^T]) = [e_1\beta_2^T]$. Thus

$$\beta_1e_k^T = 0 \Leftrightarrow \left[e_1\beta_1^T\right] \sim \left[e_ke_n^T\right] \Leftrightarrow \left[e_1\beta_2^T\right] \sim \left[e_ke_n^T\right] \Leftrightarrow \beta_2^Te_k = 0.$$

If $[\alpha_1\beta_1^T] \in \Phi_{nn}$, we may assume that $\beta_1 = e_n$ and $\beta_2 = e_n$, then $\psi_2([\alpha_1e_n^T]) = [\alpha_2e_n^T]$. So

$$e_k^T\alpha_1 = 0 \Leftrightarrow \left[e_1e_k^T\right] \sim \left[\alpha_1e_n^T\right] \Leftrightarrow \left[e_1e_k^T\right] \sim \left[\alpha_2e_n^T\right] \Leftrightarrow e_k^T\alpha_2 = 0.$$

If $[\alpha_1\beta_1^T] \notin \mathcal{W}_{11}$, then $[\alpha_2\beta_2^T] \notin \mathcal{W}_{11}$, thus $e_n^T\alpha_1 = e_n^T\alpha_2 = \beta_1^Te_1 = \beta_2^Te_1 = 0$. Now we can see that

$$\begin{aligned} e_k^T\alpha_1 = 0 \ \left(\text{resp.}, \beta_1^Te_k = 0\right) &\Leftrightarrow \left[e_1e_k^T\right] \sim \left[\alpha_1\beta_1^T\right] \ \left(\text{resp.}, \left[\alpha_1\beta_1^T\right] \sim \left[e_ke_n^T\right]\right) \\ &\Leftrightarrow \left[e_1e_k^T\right] \sim \left[\alpha_2\beta_2^T\right] \ \left(\text{resp.}, \left[\alpha_2\beta_2^T\right] \sim \left[e_ke_n^T\right]\right) \\ &\Leftrightarrow e_k^T\alpha_2 = 0 \ \left(\text{resp.}, \ \beta_2^Te_k = 0\right). \end{aligned}$$

Therefore, the k-th component of α_1 (resp., β_1) is zero if and only if the k-th component of α_2 (resp., β_2) is zero.

For $1 \leqslant i < j \leqslant n$ and $a \in F$, Claim 4 shows that there exists a permutation $\tau_j^{(i)}$ on F satisfying $\tau_j^{(i)}(0) = 0$ such that

$$\psi_2\left(\left[e_i\left(e_i + ae_j\right)^T\right]\right) = \left[e_i\left(e_i + \tau_j^{(i)}(a)e_j\right)^T\right]. \tag{3.3}$$

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For convenience, we assume that $\tau_j^{(i)}(a) = 1$ for $a \in F$ and $i \geqslant j$.

Claim 5. *$\psi_2([(ae_i + e_j)e_j^T]) = [(-\tau_j^{(i)}(-a)e_i + e_j)e_j^T]$ for $a \in F$, $1 \leqslant i < j \leqslant n$.*

For $a \in F$ and $1 \leqslant i < j \leqslant n$, by Claim 4, we may assume that $\psi_2([(ae_i+e_j)e_j^T]) = [(be_i + e_j)e_j^T]$, where $b \in F$. Applying ψ_2 on $[e_i(e_i - ae_j)^T] \sim [(ae_i + e_j)e_j^T]$ we have $[e_i(e_i + \tau_j^{(i)}(-a)e_j)^T] \sim [(be_i + e_j)e_j^T]$, which implies that $b = -\tau_j^{(i)}(-a)$.

Claim 6. *For $1 \leqslant i \leqslant n$, we have*

$$\psi_2\left(\left[\left(\textstyle\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i\right)\left(e_i + \textstyle\sum_{i+1\leqslant t\leqslant n} b_t e_t\right)^T\right]\right)$$

$$= \left[\left(-\textstyle\sum_{1\leqslant s\leqslant i-1} \tau_i^{(s)}\left(-a_s\right) e_s + e_i\right)\left(e_i + \textstyle\sum_{i+1\leqslant t\leqslant n} \tau_t^{(i)}(b_t) e_t\right)^T\right]$$

where $a_s,\ b_t \in F$.

By Claim 4, we may assume that

$$\psi_2\left(\left[\left(\textstyle\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i\right)\left(e_i + \textstyle\sum_{i+1\leqslant t\leqslant n} b_t e_t\right)^T\right]\right)$$

$$= \left[\left(\textstyle\sum_{1\leqslant s\leqslant i-1} c_s e_s + e_i\right)\left(e_i + \textstyle\sum_{i+1\leqslant t\leqslant n} d_t e_t\right)^T\right] \in \Phi_{ii},$$

where $c_s,\ d_t \in F$. For $1 \leqslant k < i < l \leqslant n$, it follows from (3.3) and Claim 5 that $\psi_2([e_k(e_k - a_k e_i)^T]) = [e_k(e_k + \tau_i^{(k)}(-a_k)e_i)^T]$ and $\psi_2([(-b_l e_i + e_l)e_l^T]) = [(-\tau_l^{(i)}(b_l)e_i + e_l)e_l^T]$. Clearly,

$$\left[e_k\left(e_k - a_k e_i\right)^T\right] \sim \left[\left(\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} b_t e_t\right)^T\right], \tag{3.4}$$

and

$$\left[\left(\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} b_t e_t\right)^T\right] \sim \left[\left(-b_l e_i + e_l\right) e_l^T\right]. \tag{3.5}$$

Applying ψ_2 to (3.4) and (3.5) we have

$$\left[e_k\left(e_k + \tau_i^{(k)}\left(-a_k\right) e_i\right)^T\right] \sim \left[\left(\sum_{1\leqslant s\leqslant i-1} c_s e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} d_t e_t\right)^T\right]$$

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and

$$\left[\left(\sum_{1\leqslant s\leqslant i-1} c_s e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} d_t e_t\right)^T\right] \sim \left[\left(-\tau_l^{(i)}\left(b_l\right)e_i + e_l\right)e_l^T\right],$$

which imply that $c_k = -\tau_i^{(k)}(-a_k)$ and $d_l = \tau_l^{(i)}(b_l)$, $1 \leqslant k < i < l \leqslant n$.

Denote $(\tau_2^{(1)}(1))^{-1}\tau_2^{(1)}$ by τ, then $\tau(1) = 1$, and $(\tau_2^{(1)}(1))^{-1}\tau_2^{(1)}(a) = \tau(a)$ for any $a \in F$.

Claim 7.

(i) $\tau_j^{(i)}(ab) = \tau_k^{(i)}(a)\tau_j^{(k)}(b)$ *for* $a,\ b \in F$, $1 \leqslant i < k < j \leqslant n$.

(ii) $(\tau_j^{(i)}(1))^{-1}\tau_j^{(i)}(a) = \tau(a)$ *for* $a \in F$, $1 \leqslant i < j \leqslant n$.

(iii) $\tau(ab) = \tau(a)\tau(b)$ *for* $a,\ b \in F$.

For $a,\ b \in F$ and $1 \leqslant i < k < j \leqslant n$, by Claim 6 we have

$$\psi_2\left(\left[e_i\left(e_i + ae_k + abe_j\right)^T\right]\right) = \left[e_i\left(e_i + \tau_k^{(i)}\left(a\right)e_k + \tau_j^{(i)}(ab)e_j\right)^T\right]$$

and

$$\psi_2\left(\left[\left(-be_k + e_j\right)e_j^T\right]\right) = \left[\left(-\tau_j^{(k)}\left(b\right)e_k + e_j\right)e_j^T\right].$$

Applying ψ_2 to $[e_i(e_i + ae_k + abe_j)^T] \sim [(-be_k + e_j)e_j^T]$ we get

$$\left[e_i\left(e_i + \tau_k^{(i)}\left(a\right)e_k + \tau_j^{(i)}\left(ab\right)e_j\right)^T\right] \sim \left[\left(-\tau_j^{(k)}\left(b\right)e_k + e_j\right)e_j^T\right],$$

which implies that $\tau_j^{(i)}(ab) = \tau_k^{(i)}(a)\tau_j^{(k)}(b)$. This completes the proof of (i).

When $i = 1$ and $j \geqslant 3$, we can see from (i) that

$$\left(\tau_j^{(1)}(1)\right)^{-1}\tau_j^{(1)}(a) = \left(\tau_2^{(1)}(1)\tau_j^{(2)}(1)\right)^{-1}\tau_2^{(1)}(a)\tau_j^{(2)}(1) = \tau(a).$$

When $i \geqslant 2$ and $j \geqslant 3$, we have

$$\tau_j^{(i)}(1) = (\tau_i^{(1)}(1))^{-1}\tau_j^{(1)}(1), \quad \tau_j^{(i)}(a) = (\tau_i^{(1)}(1))^{-1}\tau_j^{(1)}(a),$$

which implies that $(\tau_j^{(i)}(1))^{-1}\tau_j^{(i)}(a) = \tau_i^{(1)}(1)(\tau_j^{(1)}(1))^{-1}(\tau_i^{(1)}(1))^{-1}\tau_j^{(1)}(a) = \tau(a)$. Thus, $(\tau_j^{(i)}(1))^{-1}\tau_j^{(i)}(a) = \tau(a)$ for all $a \in F$ and all $1 \leqslant i < j \leqslant n$. This completes the proof of (ii).

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For $a,\ b \in F$, by (ii) we have $\tau(ab) = (\tau_n^{(1)}(1))^{-1}\tau_n^{(1)}(ab)$. On the other hand, by (i) we have $\tau_n^{(1)}(1) = \tau_2^{(1)}(1)\tau_n^{(2)}(1)$, $\tau_n^{(1)}(ab) = \tau_2^{(1)}(a)\tau_n^{(2)}(b)$. Therefore,

$$\tau(ab) = \left(\tau_2^{(1)}(1)\right)^{-1} \tau_2^{(1)}(a) \left(\tau_n^{(2)}(1)\right)^{-1} \tau_n^{(2)}(b) = \tau(a)\tau(b),$$

which completes the proof of (iii).

By Claim 7 (iii), one can easily see that $(\tau(-1))^2 = 1$ and $\tau(-1) = -1$. Thus, $\tau(-a) = -\tau(a)$ for any $a \in F$.

Claim 8. *There exists an inner automorphism $\overline{\sigma}_D$ with D an invertible diagonal matrix such that*

$$\begin{aligned}&\overline{\sigma}_D \cdot \psi_2\left(\left[\left(\textstyle\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} b_t e_t\right)^T\right]\right)\\ &= \left[\left(\textstyle\sum_{1\leqslant s\leqslant i-1} \tau(a_s) e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} \tau(b_t) e_t\right)^T\right]\end{aligned} \tag{3.6}$$

for all $a_s, b_t \in F$ and all $1 \leqslant i \leqslant n$.

For $2 \leqslant j \leqslant n$ and $a \in F$, by Claim 7 (ii) we have $\tau_j^{(1)}(a) = \tau(a)\tau_j^{(1)}(1)$. For $2 \leqslant s < i < t \leqslant n$ and $a \in F$, by Claim 7 (i) we have

$$\tau_i^{(s)}(a) = \left(\tau_s^{(1)}(1)\right)^{-1} \tau_i^{(1)}(a) = \left(\tau_s^{(1)}(1)\right)^{-1} \tau(a)\,\tau_i^{(1)}(1)$$

and

$$\tau_t^{(i)}(a) = \left(\tau_i^{(1)}(1)\right)^{-1} \tau_t^{(1)}(a) = \left(\tau_i^{(1)}(1)\right)^{-1} \tau(a)\tau_t^{(1)}(1).$$

Now, for $1 \leqslant i < j \leqslant n$, it follows from Claim 6 that

$$\begin{aligned}&\psi_2\left(\left[\left(\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i\right)\left(e_i + \sum_{i+1\leqslant t\leqslant n} b_t e_t\right)^T\right]\right)\\ &= \left[\left(\sum_{1\leqslant s\leqslant i-1} \frac{\tau(a_s)\tau_i^{(1)}(1)}{\tau_s^{(1)}(1)} e_s + e_i\right)\left(e_j + \sum_{i+1\leqslant t\leqslant n} \frac{\tau(b_t)\tau_t^{(1)}(1)}{\tau_i^{(1)}(1)} e_t\right)^T\right].\end{aligned} \tag{3.7}$$

Let $D = \mathrm{diag}(1, (\tau_2^{(1)}(1))^{-1}, \ldots, (\tau_n^{(1)}(1))^{-1})$. Then $D \in \mathcal{U}_n^{-1}(F)$ and the equality in (3.7) can be rewritten as the form in (3.6).

Denote $\psi_3 = \overline{\sigma}_D \cdot \psi_2$.

Claim 9. *τ is an automorphism of the field F.*

By Claim 7 (iii), it suffices to prove that τ is additive. For any $a,\ b \in F$, by Claim 8 we know that $\psi_3([e_1(e_1 + ae_2 + be_3)^T]) = [e_1(e_1 + \tau(a)e_2 + \tau(b)e_3)^T]$ and

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$\psi_3([(-(a+b)e_1+e_2+e_3)e_3^T]) = [(-\tau(a+b)e_1+e_2+e_3)e_3^T]$. Then by applying ψ_3 to $[e_1(e_1+ae_2+be_3)^T] \sim [(-(a+b)e_1+e_2+e_3)e_3^T]$, we get $[e_1(e_1+\tau(a)e_2+\tau(b)e_3)^T] \sim [(-\tau(a+b)e_1+e_2+e_3)e_3^T]$, which implies that $\tau(a+b) = \tau(a)+\tau(b)$.

Claim 9 shows that τ can induce a field automorphism $\overline{\theta}_\tau$ of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. In the following, we denote $\overline{\theta}_\tau^{-1} \cdot \psi_3$ by ψ_4. Then by Claim 8 we have

$$\psi_4\left([A]\right) = [A] \text{ for any } [A] \in \Phi_{ii}, \quad 1 \leqslant i \leqslant n. \tag{3.8}$$

Claim 10. *ψ_4 is the identity automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.*

By (3.8), it suffices to prove that $\psi_4([A]) = [A]$ for any $A \in \Phi_{ij}$, $1 \leqslant i < j \leqslant n$. For $1 \leqslant i < j \leqslant n$ and $A = (\sum_{1\leqslant s\leqslant i-1} a_s e_s + e_i)(e_j + \sum_{j+1\leqslant t\leqslant n} b_t e_t)^T$ with $a_s, b_t \in F$, by Claim 4, we may assume that

$$\psi_4\left([A]\right) = \left[\left(\sum_{1\leqslant s\leqslant i-1} c_s e_s + e_i\right)\left(e_j + \sum_{j+1\leqslant t\leqslant n} d_t e_t\right)^T\right] \in \Phi_{ij},$$

where $c_s, d_t \in F$. For $1 \leqslant k \leqslant i-1$ and $j+1 \leqslant l \leqslant n$ (if exists), by (3.8) we have

$$\psi_4\left(\left[e_k\left(e_k - a_k e_i\right)^T\right]\right) = \left[e_k\left(e_k - a_k e_i\right)^T\right],$$

$$\psi_4\left(\left[\left(-b_l e_j + e_l\right)e_l^T\right]\right) = \left[\left(-b_l e_j + e_l\right)e_l^T\right].$$

Applying ψ_4 to $[e_k(e_k - a_k e_i)^T] \sim [A]$ and $[A] \sim [(-b_l e_j + e_l)e_l^T]$, respectively, we get

$$\left[e_k\left(e_k - a_k e_i\right)^T\right] \sim \left[\left(\sum_{1\leqslant s\leqslant i-1} c_s e_s + e_i\right)\left(e_j + \sum_{j+1\leqslant t\leqslant n} d_t e_t\right)^T\right]$$

and

$$\left[\left(\sum_{1\leqslant s\leqslant i-1} c_s e_s + e_i\right)\left(e_j + \sum_{j+1\leqslant t\leqslant n} d_t e_t\right)^T\right] \sim \left[\left(-b_l e_j + e_l\right)e_l^T\right],$$

which implies that $c_k = a_k$ and $d_l = b_l$, $1 \leqslant k \leqslant i-1$ and $j+1 \leqslant l \leqslant n$. Thus, $\psi_4([A]) = [A]$ for any $[A] \in V_n$.

The above discussions show that $\psi = \overline{\eta}^\delta \cdot \overline{\sigma}_P \cdot \overline{\theta}_\tau$, where $P = D^{-1}Q^{-1}$, which completes the proof. $\square$

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4. Automorphisms of $\Gamma(\mathcal{U}_n^1(F))$. In this section, we first show how to reduce an automorphism of $\Gamma(\mathcal{U}_n^1(F))$ to that of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.

LEMMA 4.1. *Let $A, B \in \mathcal{U}_n^1(F)$. Then $\mathcal{N}(A)\backslash\{B\} = \mathcal{N}(B)\backslash\{A\}$ (in $\Gamma(\mathcal{U}_n^1(F))$) if and only if B is a nonzero multiple of A.*

Proof. The sufficiency is obvious. We only prove the necessity. Suppose that $[A] \in \Phi_{ij}$, $1 \leqslant i \leqslant j \leqslant n$, then by the proof of Lemma 2.2 we conclude that there exists $P \in \mathcal{U}_n^{-1}(F)$ such that $P^{-1}AP = ae_ie_j^T$ with some $a \in F^*$. By $\mathcal{N}(A)\backslash\{B\} = \mathcal{N}(B)\backslash\{A\}$ we know that $\mathcal{N}(e_ie_j^T)\backslash\{C\} = \mathcal{N}(C)\backslash\{e_ie_j^T\}$, where $C = P^{-1}BP$. Suppose that

$$C = \left(\sum_{1\leqslant k\leqslant t} a_k e_k\right)\left(\sum_{t\leqslant l\leqslant n} b_l e_l\right)^T, \quad \text{where} \quad a_k,\ b_l \in F.$$

For $s \neq i, j$, by $e_se_s^T \in \mathcal{N}(e_ie_j^T)\backslash\{C\}$, we have $e_se_s^T \in \mathcal{N}(C)\backslash\{e_ie_j^T\}$, which implies that $C(e_se_s^T) = (e_se_s^T)C$. By the arbitrariness of s, we get $C = ae_ie_i^T + be_ie_j^T + ce_je_j^T$, where $a = a_ib_i$, $b = a_ib_j$, $c = a_jb_j$. If $i = j$, then it is easily seen that C is a nonzero multiple of $e_ie_i^T$. If $i < j$, we claim that $a = 0$ and $c = 0$. Indeed, if $a \neq 0$, Since $C \in \mathcal{U}_n^1(F)$, we have $C = ae_ie_i^T + be_ie_j^T$. It follows that $be_ie_j^T - ae_je_j^T \in \mathcal{N}(C)\backslash\{e_ie_j^T\}$, and so $be_ie_j^T - ae_je_j^T \in \mathcal{N}(e_ie_j^T)\backslash\{C\}$, a contradiction. In a similar way we conclude that $c = 0$. Thus, C is a nonzero multiple of $e_ie_j^T$ and then B is a nonzero multiple of A. □

Let θ be an automorphism of $\Gamma(\mathcal{U}_n^1(F))$. We define $\overline{\theta}$ on $\overline{\Gamma}(\mathcal{U}_n^1(F))$ by

$$\overline{\theta}\left([A]\right) = \left[\theta\left(A\right)\right], \ \forall A \in \mathcal{U}_n^1(F). \tag{4.1}$$

LEMMA 4.2. *Let θ be an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, then $\overline{\theta}$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$.*

Proof. If $[A] = [B] \in V_n$, then B is a nonzero multiple of A. By Lemma 4.1 we have $\mathcal{N}(A)\backslash\{B\} = \mathcal{N}(B)\backslash\{A\}$, which implies that

$$\mathcal{N}\left(\theta\left(A\right)\right) \backslash \left\{\theta\left(B\right)\right\} = \mathcal{N}\left(\theta\left(B\right)\right) \backslash \left\{\theta\left(A\right)\right\}.$$

Hence, $\theta(B)$ is also a nonzero multiple of $\theta(A)$, and then $\overline{\theta}$ is well defined. It's clear that $\overline{\theta}$ is a bijection, and $\overline{\theta}([A]) \sim \overline{\theta}([B])$ if and only if $[A] \sim [B]$. Thus, $\overline{\theta}$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. □

Next, by Theorem 3.1 and Lemma 4.2, we can describe the automorphisms of $\Gamma(\mathcal{U}_n^1(F))$ with $n \geqslant 3$ immediately.

THEOREM 4.3. *Let $n \geqslant 3$. If θ is an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, then θ is of the form*

$$\theta = \eta^\delta \cdot \sigma_P \cdot \theta_\tau \cdot \xi,$$

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where η, σ_P, θ_τ *and* ξ *respectively are an extremal automorphism, an inner automorphism, a field automorphism and a local scalar multiplication of* $\Gamma(\mathcal{U}_n^1(F))$ *defined as in section 1,* $\delta = 0$ *or* 1.

Proof. Let θ be an automorphism of $\Gamma(\mathcal{U}_n^1(F))$, and define $\overline{\theta}$ as in (4.1), then by Lemma 4.2 we see that $\overline{\theta}$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_n^1(F))$. Applying Theorem 3.1, we know that there exists a matrix $P \in \mathcal{U}_n^{-1}(F)$ and an automorphism τ of F such that $\overline{\theta} = \overline{\eta}^\delta \cdot \overline{\sigma}_P \cdot \overline{\theta}_\tau$, where $\delta = 0$ or 1. This shows that $\overline{\theta}_\tau^{-1} \cdot \overline{\sigma}_P^{-1} \cdot \overline{\eta}^\delta \cdot \overline{\theta}$ acts as the identity automorphism on $\overline{\Gamma}(\mathcal{U}_n^1(F))$, or equivalently, $\theta_\tau^{-1} \cdot \sigma_P^{-1} \cdot \eta^\delta \cdot \theta$ sends each rank one upper triangular matrix A to a scalar multiple of A. Thus, $\theta_\tau^{-1} \cdot \sigma_P^{-1} \cdot \eta^\delta \cdot \theta$ is exactly a local scalar multiplication of $\Gamma(\mathcal{U}_n^1(F))$. The proof is complete. ☐

Proof of Theorem 1.2. The necessity follows from Theorem 4.3, and the sufficiency is obvious. ☐

Now, we describe all automorphisms of $\Gamma(\mathcal{U}_n^1(F))$ for $n = 2$. To this end, we need construct two exceptional type of automorphisms.

- For any $a \in F$, we define a permutation ρ_a on $\mathcal{U}_2^1(F)$ as follows: for any $r \in F^*$, $re_1(e_1 + ae_2)^T$ is sent to $r(-ae_1 + e_2)e_2^T$; $r(-ae_1 + e_2)e_2^T$ is sent to $re_1(e_1 + ae_2)^T$; each other vertices in $\mathcal{U}_2^1(F)$ is fixed by ρ_a. Then ρ_a is an automorphism of $\Gamma(\mathcal{U}_2^1(F))$, and $\rho_a^2 = 1$ (in fact $\rho_0 = \eta$). Denote $\rho = \prod\limits_{a \in F} \rho_a^{\delta_a}$, where $\delta_a = 0$ or 1.
- Let π be a permutation of F satisfying $\pi(0) = 0$. Define the map θ_π : $\mathcal{U}_2^1(F) \to \mathcal{U}_2^1(F)$ as follows: for any $a, b \in F$, $r \in F^*$, $re_1(e_1 + ae_2)^T$ is sent to $re_1(e_1 + \pi(a)e_2)^T$; $r(be_1 + e_2)e_2^T$ is sent to $r(-\pi(-b)e_1 + e_2)e_2^T$; each $re_1e_2^T$ is fixed by φ_π. Then φ_π is an automorphism of $\Gamma(\mathcal{U}_2^1(F))$.

Denote by K_n the complete graph on n vertices. Let $\mathcal{W}_0 = [e_{12}]$. For any $a \in F$, we denote

$$\mathcal{W}(a) = \left[e_1\left(e_1 + ae_2\right)^T\right] \cup \left[\left(-ae_1 + e_2\right)e_2^T\right].$$

It's clear that $\mathcal{W}_0$, $\mathcal{W}(a)$ with $a \in F$, are $|F| + 1$ connected components in $\overline{\Gamma}(\mathcal{U}_2^1(F))$, and that $\overline{\Gamma}[\mathcal{W}_0] \cong K_1$, $\overline{\Gamma}[\mathcal{W}(a)] \cong K_2$. Automorphisms of $\Gamma(\mathcal{U}_2^1(F))$ are characterized as follows.

Theorem 4.4. *θ is an automorphism of $\Gamma(\mathcal{U}_2^1(F))$ if and only if*

$$\theta = \sigma_U \cdot \theta_\pi \cdot \rho \cdot \xi,$$

where σ_U (with U a 2×2 unit upper triangular matrix, all of whose diagonal entries are 1, over F) and ξ are respectively an inner automorphism and a local scalar multiplication of $\Gamma(\mathcal{U}_2^1(F))$ defined as in section 1, θ_π (with π a permutation on F fixing 0) and ρ are exceptional type of automorphisms of $\Gamma(\mathcal{U}_2^1(F))$ defined as above.

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Proof. The sufficiency is obvious. For the necessity, assume that θ is an automorphism of $\Gamma(\mathcal{U}_2^1(F))$. Define $\overline{\theta}$ as in (4.1), then by Lemma 4.2 we see that $\overline{\theta}$ is an automorphism of $\overline{\Gamma}(\mathcal{U}_2^1(F))$.

We first consider the action of $\overline{\theta}$ on $[e_1e_1^T]$. If $\overline{\theta}([e_1e_1^T]) = [e_1(e_1 + xe_2)^T]$ for some $x \in F$, then $\overline{\sigma}_U^{-1} \cdot \overline{\theta}([e_1e_1^T]) = [e_1e_1^T]$, where $U = e_1e_1^T + e_2e_2^T + xe_1e_2$ is a unit upper triangular matrix over F. If $\overline{\theta}([e_1e_1^T]) = [(ye_1 + e_2)e_2^T]$ for some $y \in F$, then $\overline{\rho}_0 \cdot \overline{\theta}([e_1e_1^T]) = [e_1(e_1+xe_2)^T]$ with $x = -y$. It follows that $\overline{\sigma}_U^{-1} \cdot \overline{\rho}_0 \cdot \overline{\theta}([e_1e_1^T]) = [e_1e_1^T]$, and so $\overline{\sigma}_U^{-1} \cdot \overline{\rho}_0 \cdot \overline{\theta}([e_2e_2^T]) = [e_2e_2^T]$. Thus, we may assume that $\delta_0 = 0$ or 1 such that

$$\overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}\left(\left[e_ie_i^T\right]\right) = \left[e_ie_i^T\right], \quad i = 1, 2.$$

Next, for $a \in F$, it follows from $\overline{\Gamma}[\mathcal{W}(a)] \cong K_2$ that $\overline{\Gamma}[\overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}(\mathcal{W}(a))] \cong K_2$, which implies that $\overline{\Gamma}[\overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}(\mathcal{W}(a))] = \overline{\Gamma}[\mathcal{W}(b)]$ for some $b \in F$. Then, there exists a permutation π of F such that $\overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}(\mathcal{W}(a)) = \mathcal{W}(\pi(a))$ for all $a \in F$. Obviously, $\pi(0) = 0$. By this π we can induce a automorphism θ_π of $\Gamma(\mathcal{U}_2^1(F))$ such that $\overline{\theta}_\pi^{-1} \cdot \overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}(\mathcal{W}(a)) = \mathcal{W}(a)$, $a \in F$. Now, we conclude that for $a \in F$, either

$$\overline{\theta}_\pi^{-1} \cdot \overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}\left(\left[e_1\left(e_1 + ae_2\right)^T\right]\right) = \left[e_1\left(e_1 + ae_2\right)^T\right]$$

or

$$\overline{\theta}_\pi^{-1} \cdot \overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}\left(\left[e_1\left(e_1 + ae_2\right)^T\right]\right) = \left[\left(-ae_1 + e_2\right)e_2^T\right].$$

For $a \in F^*$, choose $\delta_a = 0$ or 1 such that $\overline{\rho_a}^{\delta_a} \cdot \overline{\theta}_\pi^{-1} \cdot \overline{\sigma}_U^{-1} \cdot \overline{\rho}_0^{\delta_0} \cdot \overline{\theta}([e_1(e_1+ae_2)^T]) = [e_1(e_1+ae_2)^T]$, and denote $\rho = \prod_{a\in F} \rho_a^{\delta_a}$, then $\overline{\rho} \cdot \overline{\theta}_\pi^{-1} \cdot \overline{\sigma}_U^{-1} \cdot \overline{\theta}([e_1(e_1 + ae_2)^T]) = [e_1(e_1 + ae_2)^T]$ for any $a \in F$.

The above discussions show that $\overline{\rho} \cdot \overline{\theta}_\pi^{-1} \cdot \overline{\sigma}_U^{-1} \cdot \overline{\theta}([A]) = [A]$ for any $[A] \in V_2$. In a similar way as in the proof of Theorem 4.3, we conclude that $\rho \cdot \theta_\pi^{-1} \cdot \sigma_U^{-1} \cdot \theta$ is exactly a local scalar multiplication of $\Gamma(\mathcal{U}_2^1(F))$. This completes the proof. ◻

5. Applications. In this section, we denote by Γ_n the graph $\Gamma(\mathcal{U}_n^1(F))$. The set of all automorphisms of Γ_n, denoted by $\texttt{Aut}(\Gamma_n)$, forms a group under composition of transformations. Let $\texttt{Inn}(\Gamma_n)$, $\texttt{Fie}(\Gamma_n)$, $\texttt{Ext}(\Gamma_n)$ and $\texttt{Loc}(\Gamma_n)$, respectively, be the set of all extremal automorphisms, inner automorphisms, field automorphisms and local scalar multiplications of $\Gamma(\mathcal{U}_n^1(F))$ (see the definition in section 1), and denote by $\texttt{Per}_1(\Gamma_2)$ and $\texttt{Per}_2(\Gamma_2)$ the set of all permutations $\rho = \prod_{a\in F} \rho_a^{\delta_a}$ and all permutations θ_π (with π a permutation on F fixing 0) on $\mathcal{U}_2^1(F)$, respectively. Then it's easy to verify that $\texttt{Inn}(\Gamma_n)$, $\texttt{Fie}(\Gamma_n)$, $\texttt{Ext}(\Gamma_n)$ and $\texttt{Loc}(\Gamma_n)$ (resp., $\texttt{Per}_1(\Gamma_2)$ and $\texttt{Per}_2(\Gamma_2)$) are all subgroups of $\texttt{Aut}(\Gamma_n)$ (resp., $\texttt{Aut}(\Gamma_2)$). If G and H are two subgroups of a

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group, we use $G \times H$ and $G \rtimes H$ to denote their direct product, semidirect product with G normal, respectively. Also $\underbrace{G \times \cdots \times G}_{k}$ is denoted by kG.

Now, we consider the orbit partition of the vertex set (see Corollary 5.1) and the order of the group of automorphisms (see Corollary 5.2).

Corollary 5.1. *The orbit partition of* $\mathcal{U}_n^1(F)$ *under the automorphisms is* $\mathcal{U}_n^1(F) = \bigcup_{1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor}^{k \leqslant l \leqslant n-k+1} \mathcal{W}_{kl}$. *The number of orbits is* $\lfloor \frac{n^2}{4} \rfloor$, *unless* $n = 2$ *and in this case, the number of orbits is* 2.

Proof. For $1 \leqslant k \leqslant \lfloor \frac{n+1}{2} \rfloor$ and $k \leqslant l \leqslant n-k+1$, Lemma 2.6 shows that each $\mathcal{W}_{kl}$ is stabilized under any automorphism. It suffices to prove that for any $A \in \mathcal{W}_{kl}$, there exists an automorphism θ such that $\theta(A) \in [e_k e_l^T]$. If $[A] \in \Phi_{kl}$, suppose that $A = r(\sum_{1 \leqslant s \leqslant k-1} a_s e_s + e_k)(e_l + \sum_{l+1 \leqslant t \leqslant n} b_t e_t)^T$, where $r \in F^*$, $a_s \in F$, $b_t \in F$. Set $P = I - \sum_{1 \leqslant s \leqslant k-1} a_s e_s e_k^T + \sum_{l+1 \leqslant t \leqslant n} b_t e_l e_t^T \in \mathcal{U}_n^{-1}(F)$, then $\sigma_P(A) = r e_k e_l^T$. If $[A] \in \Phi_{n-l+1,n-k+1}$, we see that $[\eta(A)] \in \Phi_{kl}$. Then by what we obtained above we get that there exists a matrix $P \in \mathcal{U}_n^{-1}(F)$ such that $\sigma_P \cdot \eta(A) \in [e_k e_l^T]$. The second result is obvious. $\square$

Corollary 5.2. *Let* $|F| = q = p^m$ *with* p *a prime. Then*

$$|\mathtt{Aut}(\Gamma_n)| = 2mq^{\frac{n(n-1)}{2}} (q-1)^{n-1} ((q-1)!)^{|V_n|} \quad for \quad n \geqslant 3, \tag{5.1}$$

and

$$|\mathtt{Aut}(\Gamma_n)| = 2^q q(q-1) ((q-1)!)^{|V_2|+1} \quad for \quad n = 2. \tag{5.2}$$

Proof. If $n \geqslant 3$, then by Theorem 4.3, each automorphism θ can be written as $\theta = \eta^\delta \cdot \sigma_P \cdot \theta_\tau \cdot \xi$, where $\delta = 0$ or 1, $P \in \mathcal{U}_n^{-1}(F)$, $\tau \in \mathtt{Aut}(F)$, ξ is a permutation on $\mathcal{U}_n^1(F)$ such that $\xi([A]) = [A]$ for any $A \in \mathcal{U}_n^1(F)$. If $\eta^{\delta_1} \cdot \sigma_{P_1} \cdot \theta_{\tau_1} \cdot \xi_1 = \eta^{\delta_2} \cdot \sigma_{P_2} \cdot \theta_{\tau_2} \cdot \xi_2$, then $\eta^{\delta_3} \cdot \sigma_{P_0} = \theta_{\tau_0} \cdot \xi_0$, where $\delta_3 = 0$ or 1, $P_0 = P_2^{-1} P_1$, $\tau_0 = \tau_2 \cdot \tau_1^{-1}$, $\xi_0 = \xi_2 \cdot \xi_1^{-1}$. Since $[e_i e_j^T]$ is stable under $\theta_{\tau_0} \cdot \xi_0$, we have $\eta^{\delta_3} \cdot \sigma_{P_0}([e_i e_j^T]) = \eta^{\delta_3} \cdot [P_0^{-1}(e_i e_j^T) P_0] = [e_i e_j^T]$ for all $1 \leqslant i \leqslant j \leqslant n$. This shows that $\delta_3 = 0$, and P_0 is a diagonal matrix. By $\delta_3 = 0$, we get $\delta_1 = \delta_2$, and so $\sigma_{P_0} = \theta_{\tau_0} \cdot \xi_0$. Let $P_0 = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i \in F^*$, then

$$\sigma_{P_0}\left(\left[e_1 \left(e_1 + a e_j\right)^T\right]\right) = \left[e_1 \left(e_1 + a d_1^{-1} d_j e_j\right)^T\right],$$

$$\sigma_{P_0}\left(\left[\left(a e_2 + e_n\right) e_n^T\right]\right) = \left[\left(a d_2^{-1} d_n e_2 + e_n\right) e_n^T\right]$$

for all $a \in F$ and all $2 \leqslant j \leqslant n$. On the other hand,

$$\theta_{\tau_0} \cdot \xi_0 \left(\left[e_1 \left(e_1 + a e_j\right)^T\right]\right) = \left[e_1 \left(e_1 + \tau_0(a) e_j\right)^T\right],$$

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$$\theta_{\tau_0} \cdot \xi_0 \left(\left[(ae_2 + e_n)\, e_n^T \right] \right) = \left[(\tau_0(a)e_2 + e_n)\, e_n^T \right].$$

Consequently, $\tau_0(a) = ad_1^{-1}d_j = ad_2^{-1}d_n$, $a \in F$, $2 \leqslant j \leqslant n$. It follows that P_0 is a nonzero scalar matrix and $\tau_0(a) = a$ for any $a \in F$. Hence, $P_1 = d_1P_2$ and $\tau_1 = \tau_2$, which implies that $\xi_1 = \xi_2$. Now, the above discussion shows that

$$|\mathtt{Aut}(\Gamma_n)| = 2 \cdot \frac{|\mathcal{U}_n^{-1}(F)|}{q-1} \cdot |\mathtt{Aut}(F)| \cdot |\mathtt{Loc}(\Gamma_n)|.$$

Denote by K_n^c the complement of the complete K_n, i.e., the graph consisting of n isolated vertices. Clearly, $\mathtt{Aut}(K_n) \cong \mathtt{Aut}(K_n^c) \cong S_n$, where S_n is the symmetric group of degree n. For any $A \in \mathcal{U}_n^1(F)$, we see that the subgraph induced by $[A]$ in $\Gamma(\mathcal{U}_n^1(F))$ is isomorphic to K_{q-1} or K_{q-1}^c. This shows that $\mathtt{Loc}(\Gamma_n) \cong kS_{q-1}$ with $k = \frac{|\mathcal{U}_n^1(F)|}{q-1} = |V_n|$. It is not difficult to see that $|\mathcal{U}_n^{-1}(F)| = (q-1)^n q^{\frac{n(n-1)}{2}}$, $|\mathtt{Aut}(F)| = m$, $|S_{q-1}| = (q-1)!$. Thus, we get (5.1).

When $n = 2$, it is easily seen that the number of permutations ρ on $\mathcal{U}_n^1(F)$ is 2^q and the number of permutations π on F satisfying $\pi(0) = 0$ is $(q-1)!$. Hence, in a similar way as above, we have (5.2). $\square$

Finally, by Theorem 4.3, Theorem 4.4 and the proof of Corollary 5.2, we have the following result.

Corollary 5.3. *Let $|F| = q$. Then, the following hold:*

(i) When $n \geqslant 3$, $\mathtt{Aut}(\Gamma_n) \cong ((\frac{\mathcal{U}_n^{-1}(F)}{K} \times |V_n|S_{q-1}) \rtimes \mathtt{Aut}(F)) \rtimes S_2$, where $K = \{aI \mid a \in F^*\}$;

(ii) When $n = 2$, $\mathtt{Aut}(\Gamma_n) \cong ((U_2(F) \times |V_2|S_{q-1}) \rtimes S_{q-1}) \rtimes qS_2$, where $U_2(F)$ is the set of all 2×2 unit upper triangular matrices over F.

Proof. If $n \geqslant 3$, then by Lemma 1.1 and Theorem 4.3, we get

$$\mathtt{Aut}(\Gamma_n) = ((\mathtt{Inn}(\Gamma_n) \times \mathtt{Loc}\,(\Gamma_n)) \rtimes \mathtt{Fie}(\Gamma_n)) \rtimes \mathtt{Ext}(\Gamma_n).$$

The proof of Corollary 5.2 shows that $\mathtt{Inn}(\Gamma_n) \cong \frac{\mathcal{U}_n^{-1}(F)}{K}$ with $K = \{aI \mid a \in F^*\}$, $\mathtt{Fie}(\Gamma_n) \cong \mathtt{Aut}(F)$, $\mathtt{Ext}(\Gamma_n) \cong S_2$ and $\mathtt{Loc}(\Gamma_n) \cong |V_n|S_{q-1}$, from which we get (i).

When $n = 2$, by Theorem 4.4 and the proof of Corollary 5.2, it is easily seen that $\mathtt{Inn}(\Gamma_n) \cong U_2(F)$, where $U_2(F)$ is the set of all 2×2 unit upper triangular matrices over F, $\mathtt{Per}_1(\Gamma_2) \cong qS_2$ and $\mathtt{Per}_2(\Gamma_2) \cong S_{q-1}$. Hence, in a similar way as above, we obtain the result of (ii). $\square$

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