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A NOTE ON THE REAL NONNEGATIVE INVERSE EIGENVALUE PROBLEM*

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Abstract. The Real Nonnegative Inverse Eigenvalue Problem (RNIEP) asks when is a list

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

consisting of real numbers the spectrum of an $n \times n$ nonnegative matrix A. In that case, σ is said to be realizable and A is a realizing matrix. In a recent paper dealing with RNIEP, P. Paparella considered cases of realizable spectra where a realizing matrix can be taken to have a special form, more precisely such that the entries of each row are obtained by permuting the entries of the first row. A matrix of this form is called permutative. Paparella raised the question whether any realizable list σ can be realized by a permutative matrix or a direct sum of permutative matrices. In this paper, it is shown that in general the answer is no.

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Key words. Nonnegative matrix, Nonnegative inverse eigenvalue problem, Permutative matrix.

1. Introduction. Let $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a list of n real numbers. We say that σ is *realizable* if there exists an $n \times n$ nonnegative matrix A with spectrum σ, and A is said to be a *realizing* matrix. The problem of characterizing the realizable real lists is known as the *Real Nonnegative Inverse Eigenvalue Problem (RNIEP)*, and has been introduced by Suleimanova [5]. It is solved for $n \leq 4$, where the cases $n = 1, 2$ are trivial, and the cases $n = 3, 4$ were solved by Loewy and London [2]. The RNIEP is open for any $n \geq 5$, an evidence of its difficulty. Well known related problems are the so called Nonnegative Inverse Eigenvalue Problem (NIEP) and Symmetric Nonnegative Inverse Eigenvalue Problem (SNIEP), but they will not be discussed here.

The purpose of this paper is to consider the case where the real list σ can be realized by a special type of a (nonnegative) matrix, defined as follows.

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Definition 1.1. Let A be an $n \times n$ matrix. We call A a *permutative matrix* (a term coined by C.R. Johnson, see a footnote in [3]) if there exist $x = (x_1, x_2, \dots, x_n)^t$ and $n \times n$ permutation matrices $P_2, P_3, \dots, P_n$ such that

$$B = \begin{bmatrix} x^t \\ (P_2 x)^t \\ \vdots \\ (P_n x)^t \end{bmatrix}.$$

Paparella [3] considers realizable lists which can be realized by permutative matrices or their direct sum. He shows that for $n \leq 4$ this is always possible. Moreover, it is shown in [3] that if σ contains one positive number and is realizable then it can be realized by a permutative matrix, thus giving a constructive proof of a well known theorem of Suleimanova [5]. At the end of [3], the author raises the following problem. *Can every realizable (real) list be realized by a permutative matrix or a direct sum of such matrices*?

The problem raised by Paparella is significant in the following sense: If the answer is positive, then for *any* realizable (real) list there is a realizing matrix which has a simple and explicit form involving only n numbers. It is our purpose to answer this question negatively. This will be done in the next section.

Let $s_k(\sigma) = \sum_{i=1}^{k} \lambda_i^k$ be the k-th moment (power sum) of σ, $k = 1, 2, \dots$

2. The main result. In this section, we exhibit a list of five real numbers which is realizable, but cannot be realized by a permutative matrix or a direct sum of such matrices.

Let

$$a = 8/25 + \sqrt{51}/50 \quad \text{and} \quad b = 8/25 - \sqrt{51}/50, \tag{2.1}$$

and let

$$\sigma = (1, a, b, -4/5, -21/25). \tag{2.2}$$

For simplicity, and since σ is fixed in this section, we write s_k for $s_k(\sigma)$. It is straightforward to check that

$$s_1 = 0, \quad s_2 = 3239/1250, \quad s_3 = 0, \quad s_4 = 6107321/3125000, \quad s_5 = 861/3125.$$

Moreover, $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} = 10332/3125$, hence σ is realizable by a result of Laffey and Meehan [1]. In fact, σ can be realized even as the spectrum of a 5×5 symmetric nonnegative matrix. This follows from a recent paper of Spector

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[4], (cf. Theorem 3 there), since s_1 and s_3 are 0, while $a - 21/25 < 0$. It follows from (2.2) that every realizing matrix has to be irreducible.

Theorem 2.1. *The list σ cannot be realized by a permutative matrix or a direct sum of such matrices.*

Proof. We argue by contradiction. Suppose there exists a permutative matrix A with spectrum σ. Then,

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

where x_i, $i = 1, 2, 3, 4, 5$, are nonnegative real numbers, and each row of A consists of a suitable permutation of these numbers. Also, here and throughout, "$\cdot$" denotes an entry which might be 0 or positive. Note that $s_1 = 0$ implies $\mathrm{tr}(A) = 0$ and $s_3 = 0$ implies $\mathrm{tr}(A^3) = 0$. It is convenient to think about these equalities in terms of $G(A)$, the directed graph corresponding to A. Recall that the vertices of $G(A)$ are labeled 1 to 5, and there is a (directed) edge from vertex i to vertex j if and only if $a_{ij} > 0$. Then it follows that $G(A)$ contains no loops or 3-cycles. In particular, all main diagonal entries of A are zero.

It follows that $x_1 = 0$. If no other x_i is equal to zero, then all the main diagonal entries of A are equal to x_1, and hence, all off-diagonal entries of A are positive. Then, clearly, $G(A)$ contains a 3-cycle, which is impossible. So, at least one additional x_i is positive, and since we can apply a permutation similarity, we can assume without loss of generality that $x_2 = 0$. On the other hand, since $A \neq 0$ there exists a positive x_i, so without loss of generality (same argument) we may assume $x_3 > 0$.

We now show that at least one of x_4, x_5 must be positive. If this is not the case, then each row of A contains exactly one positive entry, namely x_3. If there is a column of A containing no x_3, then A is singular, which is not the case. Otherwise, there exists a 5×5 permutation matrix P such that $A = x_3 P$, which is also impossible. Hence, at least one of x_4, x_5 must be positive, which we may assume to be x_4.

Suppose that $x_5 > 0$. Then each row of A contains 3 positive entries. Hence, $a_{34} + a_{35} > 0$. We now distinguish three cases.

(I) Suppose that a_{34} and a_{35} are positive. Then, as $G(A)$ contains no 3-cycles, we must have $a_{41} = a_{51} = 0$. But then a_{42}, a_{43}, a_{45}, a_{52}, a_{53} and a_{54} are all

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positive. So,

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & + & + \\ 0 & + & + & 0 & + \\ 0 & + & + & + & 0 \end{bmatrix},$$

where "$+$" denotes throughout a positive entry. But this is impossible, because if $a_{23} > 0$, then $2 \mapsto 3 \mapsto 4 \mapsto 2$ is a 3-cycle in $G(A)$, which is impossible, while if $a_{24} > 0$, then $2 \mapsto 4 \mapsto 5 \mapsto 2$ is a 3-cycle in $G(A)$, which is also impossible. Hence, this case is not possible.

(II) Suppose that $a_{34} > 0$ and $a_{35} = 0$. Hence, a_{31} and a_{32} are positive. We must have $a_{53} = 0$ or else $1 \mapsto 5 \mapsto 3 \mapsto 1$ is a 3-cycle in $G(A)$, a contradiction. Likewise, we must have $a_{43} = 0$. Hence, a_{41}, a_{42}, a_{45}, a_{51}, a_{52} and a_{54} are all positive, so

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ + & + & 0 & + & 0 \\ + & + & 0 & 0 & + \\ + & + & 0 & + & 0 \end{bmatrix},$$

and this leads to a contracdiction as follows: At least one of a_{24}, a_{25} is positive. If a_{24} is positive, then $2 \mapsto 4 \mapsto 5 \mapsto 2$ is a 3-cycle in $G(A)$ while if a_{25} is positive, then $2 \mapsto 5 \mapsto 4 \mapsto 2$ is a 3-cycle in $G(A)$. Hence, this case is impossible.

(III) Supose that $a_{35} > 0$ and $a_{34} = 0$. Hence, a_{31} and a_{32} are positive, and as in (II), we must have $a_{43} = a_{53} = 0$, implying a_{41}, a_{42}, a_{45}, a_{51}, a_{52} and a_{54} are all positive. Hence,

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ + & + & 0 & 0 & + \\ + & + & 0 & 0 & + \\ + & + & 0 & + & 0 \end{bmatrix},$$

and we get a contradiction as in (II), namely either $a_{24} > 0$, in which case we have the 3-cycle $2 \mapsto 4 \mapsto 5 \mapsto 2$, or $a_{25} > 0$, in which case we have the 3-cycle $2 \mapsto 5 \mapsto 4 \mapsto 2$.

This shows that assuming $x_5 > 0$ leads to a contradiction.

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Suppose now that $x_5 = 0$. Then each row of A contains 2 positive entries, x_3 and x_4. We assume first that $a_{21} > 0$. Then we must have $a_{32} = a_{42} = 0$, otherwise $G(A)$ contains the 3-cycle $2 \mapsto 1 \mapsto 3 \mapsto 2$ or the 3-cycle $2 \mapsto 1 \mapsto 4 \mapsto 2$, a contradiction. Then, as A is nonsingular, we must have $a_{52} > 0$. So A looks like

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ + & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & 0 & \cdot \\ \cdot & + & \cdot & \cdot & 0 \end{bmatrix},$$

leading us to consider two cases.

(IV) Suppose that $a_{31} > 0$. Then, $a_{43} = 0$, or else we have the 3-cycle $1 \mapsto 4 \mapsto 3 \mapsto 1$. We conclude that a_{41} and a_{45} are positive (two positive entries in every row). Not allowing 3-cycles implies $a_{51} = a_{24} = a_{34} = 0$. Hence, $a_{35} > 0$, implying $a_{23} = 0$, and hence $a_{25} > 0$. It follows that each of the second, third and fourth rows has the pattern (+,0,0,0,+), where one of the positive entries is x_3 and the other is x_4. This means that two of these rows are identical, so A is singular, a contradiction. This implies that when $a_{21} > 0$ we must have $a_{31} = 0$.

(V) Suppose that $a_{31} = 0$. Then A looks like

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ + & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & 0 & \cdot \\ \cdot & + & \cdot & \cdot & 0 \end{bmatrix},$$

and we must have (two positive entries in each row) $a_{34}, a_{35} > 0$. Continuing as in previous arguments we conclude $a_{41} = a_{51} = 0$, $a_{43}, a_{45} > 0$, $a_{53} = 0$ and $a_{54} > 0$. This leads to a contradiction, as we get the 3-cycle $3 \mapsto 5 \mapsto 4 \mapsto 3$.

This shows that when $x_5 = 0$ *we must have* $a_{21} = 0$. Then (two positive entries in every row) at least one of a_{23}, a_{24} must be positive, and by symmetry (between the indices 3 and 4) we may assume $a_{23} > 0$. We consider two cases.

(a) Suppose that $a_{24} > 0$. Then $a_{25} = 0$. If $a_{51} = a_{52} = 0$, then a_{53} and a_{54} must be positive, implying that each of of the first, second and fifth rows has the pattern (+,0,0,0,+), where one of the positive entries is x_3 and the other is x_4. Hence, A has two equal rows, so is singular, a contradiction. By the symmetry of the indices 1 and 2 we may therefore assume that $a_{51} > 0$. Since no 3-cycle is allowed we conclude that

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$a_{35} = a_{45} = 0$, leading to a contradiction as the last column of A is zero.

(b) Suppose that $a_{24} = 0$. Then $a_{25} > 0$. We claim that $a_{31} + a_{32} > 0$. Otherwise $a_{34}, a_{35} > 0$, and as no 3-cycles are allowed we must have $a_{41} = a_{51} = 0$, so the first column of A is zero, a contradiction. By symmetry, we may perform a permutation similarity which interchanges 1 and 2 as well as 4 and 5. This allows us to assume that $a_{31} > 0$, so A looks like

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & + & 0 & + \\ + & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix},$$

implying (no 3-cycles) $a_{43} = 0$. We claim that $a_{41} > 0$. Otherwise (two positive entries in each row) $a_{42}, a_{45} > 0$, implying (no 3-cycles) $a_{51} = a_{54} = 0$, and so $a_{52}, a_{53} > 0$. This forces $a_{34} = a_{35} = 0$ and $a_{32} > 0$. Hence, $G(A)$ contains the 3-cycle $2 \mapsto 5 \mapsto 3 \mapsto 2$, a contradiction. Therefore, $a_{41} > 0$, so A looks like

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & + & 0 & + \\ + & \cdot & 0 & \cdot & \cdot \\ + & \cdot & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix},$$

implying $a_{34} = 0$. We claim that $a_{32} > 0$. Otherwise we must have $a_{35} > 0$, and so $a_{51} = a_{52} = 0$, while $a_{53}, a_{54} > 0$. As no column of A can be zero we must have $a_{42} > 0$, but then we have the 3-cycle $2 \mapsto 5 \mapsto 4 \mapsto 2$, a contradiction. Hence, $a_{32} > 0$, so $a_{35} = a_{53} = 0$. If $a_{54} = 0$, then we must have $a_{51}, a_{52} > 0$, implying $a_{45} = 0$. But now we have that $A[345]$, the principal submatrix of A based on indices 3,4,5 is zero, so A is singular, which is impossible. Hence, $a_{54} > 0$, implying $a_{42} = 0$, $a_{45} > 0$, $a_{51} = 0$, $a_{52} > 0$. We finally get to the only $(+, 0)$ pattern that A can take, namely,

$$A = \begin{bmatrix} 0 & 0 & + & + & 0 \\ 0 & 0 & + & 0 & + \\ + & + & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + \\ 0 & + & 0 & + & 0 \end{bmatrix},$$

so the undirected graph corresponding to $G(A)$ is just the 5-cycle, and in $G(A)$ if $i \mapsto j$, then $j \mapsto i$.

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We find it convenient to change the notation, and we replace x_3, x_4 by x, y. Recall that the spectral radius of σ is 1, and on the other hand each row sum of A is $x + y$, hence we must have $x + y = 1$, so $y = 1 - x$. We also find it convenient to perform a permutation similarity of our matrix so that in $G(A)$ we have $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $3 \leftrightarrow 4$, $4 \leftrightarrow 5$, and $5 \leftrightarrow 1$. Hence, the first row of A is $(0, x, 0, 0, y = 1 - x)$, and every other row has each of x, y one time, so altogether we have 16 possible matrices to check. We will show that none will give the desired spectrum. The computations throughout are performed in Maple 18 and use exact arithmetic.

Denote by $q(t)$ the monic polynomial in the indeterminate t whose roots are the elements of σ, that is,

$$q(t) = (t - 1)(t - a)(t - b)(t + 4/5)(t + 21/25),$$

where a and b are defined by (2.1). It follows that

$$q(t) = t^5 - (3239/2500)t^3 + (21919/62500)t - 861/15625.$$

Then, the coefficient of t^3 in $q(t)$ is $-3239/2500$. We call the 16 possible matrices $A_1, A_2, \ldots, A_{16}$. For each of those matrices, we will compute the characteristic polynomial, which we call $q_i(t)$. For each matrix A_i, if its spectrum is σ, the coefficient of t^3 in $q_i(t)$ must be $-3239/2500$. This will be used to determine x. We then show that the computed x does not satisfy other conditions forced by equality of $q(t)$ and $q_i(t)$. To describe uniquely each matrix A_i it suffices to indicate the location of x in each of its rows except the first one (where, as indicated, the x is located in the second place). In the following list, the row vector v_i consists of 4 components, where in the ith component we give the location of x in the $(i + 1)$-th row of A_i.

- $v_1 = (1, 4, 3, 4)$, $v_2 = (1, 4, 3, 1)$, $v_3 = (1, 4, 5, 4)$, $v_4 = (1, 4, 5, 1)$,
- $v_5 = (1, 2, 3, 4)$, $v_6 = (1, 2, 3, 1)$, $v_7 = (1, 2, 5, 4)$, $v_8 = (1, 2, 5, 1)$,
- $v_9 = (3, 4, 3, 4)$, $v_{10} = (3, 4, 3, 1)$, $v_{11} = (3, 4, 5, 4)$, $v_{12} = (3, 4, 5, 1)$,
- $v_{13} = (3, 2, 3, 4)$, $v_{14} = (3, 2, 3, 1)$, $v_{15} = (3, 2, 5, 4)$, $v_{16} = (3, 2, 5, 1)$,

so, for example,

$$A_1 = \begin{bmatrix} 0 & x & 0 & 0 & 1-x \\ x & 0 & 1-x & 0 & 0 \\ 0 & 1-x & 0 & x & 0 \\ 0 & 0 & x & 0 & 1-x \\ 1-x & 0 & 0 & x & 0 \end{bmatrix},$$

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and

$$A_{12} = \begin{bmatrix} 0 & x & 0 & 0 & 1-x \\ 1-x & 0 & x & 0 & 0 \\ 0 & 1-x & 0 & x & 0 \\ 0 & 0 & 1-x & 0 & x \\ x & 0 & 0 & 1-x & 0 \end{bmatrix}.$$

Computing the characteristic polynomials of the matrices A_i, $i = 1, 2, \ldots, 16$, yields four different polynomials, namely:

- $q_1(t) = t^5 - (3x^2 - 3x + 2)t^3 - (-x^4 + 2x^3 - 4x^2 + 3x - 1)t - x^4 + 2x^3 - x^2$,
- $q_4(t) = t^5 - (-x^2 + x + 1)t^3 - (3x^4 - 6x^3 + 5x^2 - 2x)t + 3x^4 - 6x^3 + 4x^2 - x$,
- $q_6(t) = t^5 - (-x^2 + x + 1)t^3 - (-x^4 + 2x^3 - x)t - x^4 + 2x^3 - x^2$,
- $q_{12}(t) = t^5 + 5x(-1+x)t^3 - (-5x^4 + 10x^3 - 5x^2)t - 5x^4 + 10x^3 - 10x^2 + 5x - 1$,

and we also have

- $q_1(t) = q_2(t) = q_3(t) = q_7(t) = q_{15}(t)$,
- $q_4(t) = q_5(t) = q_{10}(t) = q_{11}(t) = q_{16}(t)$,
- $q_6(t) = q_8(t) = q_9(t) = q_{13}(t) = q_{14}(t)$.

Consider first the matrix A_1. If its spectrum is equal to σ, then the coefficients of t^3 in $q(t)$ and $q_1(t)$ must coincide, so x has to satisfy

$$-(3x^2 - 3x + 2) = -3239/2500,$$

which yields the two solutions $x1m, x1p$ given by $x1m = 1/2 - \sqrt{38}/50$, $x1p = 1/2 + \sqrt{38}/50$. Substituting either one of those numbers into A_1 we get, in both cases, the characteristic polynomial

$$t^5 - (3239/2500)t^3 + (2192069/6250000)t - 344569/6250000,$$

which is not equal to $q(t)$, so the spectrum of A_1 cannot be σ.

Consider now A_4. A similar argument leads to the equation

$$-(-x^2 + x + 1) = -3239/2500,$$

which has no real roots. Hence, the spectrum of A_4 cannot be σ. We note that the coefficients of t^3 in $q_4(t)$ and $q_6(t)$ are the same polynomial in x, implying that the spectrum of A_6 cannot be σ as well. Finally, we consider A_{12}. A similar argument leads to the equation

$$-5x + 5x^2 = -3239/2500,$$

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which has no real roots, so the spectrum of A_{12} cannot be σ. As these are the only distinct cases we have shown that σ cannot be realized as the spectrum of a permutative matrix. Since any nonnegative matrix with spectrum σ must be irreducible, this completes the proof. $\square$

REFERENCES

[1] T.J. Laffey and E. Meehan. A characterization of trace zero nonnegative 5×5 matrices. *Linear Algebra and its Applications*, 302/303:295–302, 1999.
[2] R. Loewy and D. London. A note on an inverse problem for nonnegative matrices. *Linear and Multilinear Algebra*, 6:83–90, 1978.
[3] P. Paparella. Realizing Suleimanova-type spectra via permutative matrices. *Electronic Journal of Linear Algebra*, 31:306–312, 2016.
[4] O. Spector. A characterization of trace zero symmetric nonnegative 5×5 matrices. *Linear Algebra and its Applications*, 434:1000–1017, 2011.
[5] H.R. Suleimanova. Stochastic matrices with real characteristic numbers. *Doklady Akademii Nauk SSSR (N.S.)*, 66:343–345, 1949.