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THE ϕ_S POLAR DECOMPOSITION WHEN THE COSQUARE OF S IS NONDEROGATORY*

RALPH JOHN DE LA CRUZ† AND DARYL Q. GRANARIO‡

Abstract. For $S \in GL_n$, define $\phi_S : M_n \to M_n$ by $\phi_S(A) = S^{-1}A^TS$. A matrix $A \in M_n$ is ϕ_S *orthogonal* if $\phi_S(A) = A^{-1}$; A is ϕ_S *symmetric* if $\phi_S(A) = A$; A has a ϕ_S *polar decomposition* if $A = ZY$ for some ϕ_S orthogonal Z and ϕ_S symmetric Y. If A has a ϕ_S polar decomposition, then A commutes with the cosquare $S^{-T}S$. Conditions under which the converse implication holds for the case where $S^{-T}S$ is nonderogatory, are obtained.

Key words. ϕ_S Orthogonal matrices, ϕ_S Symmetric matrices, ϕ_S Polar decomposition, Nonderogatory.

AMS subject classifications. 15A21, 15A23.

1. Introduction. Denote by M_n the set of all n-by-n complex matrices and by GL_n the set of all n-by-n nonsingular complex matrices. If $S \in GL_n$, define the map $\phi_S : M_n \to M_n$ by $\phi_S(A) = S^{-1}A^TS$. We say that $A \in M_n$ is ϕ_S *orthogonal* if $\phi_S(A) = A^{-1}$; A is ϕ_S *symmetric* if $\phi_S(A) = A$; A is ϕ_S *skew symmetric* if $\phi_S(A) = -A$; and A has a ϕ_S *polar decomposition* if we can write A as a product $A = ZY$, where Z is ϕ_S orthogonal and Y is ϕ_S symmetric.

Every matrix $A \in M_n$ has a classical *polar decomposition*, that is, $A = QR$, where Q is unitary and R is positive semidefinite. The *algebraic polar decomposition* or the *orthogonal-symmetric polar decomposition* is the ϕ_S polar decomposition when $S = I$. Kaplansky [9] showed that a matrix $A \in M_n$ has an algebraic polar decomposition if and only if AA^T is similar to A^TA. In particular, every nonsingular matrix has an algebraic polar decomposition. Horn and Merino [7, Theorem 2.3] showed that a matrix $A \in M_n$ has a *circular polar decomposition*, that is $A = QR$ for some real matrix Q ($\overline{Q} = Q$) and coninvolutory R ($\overline{R} = R^{-1}$), if and only if A and $\overline{A}$ have the same range. In particular, every nonsingular matrix has a circular polar decomposition. Let $S \in GL_n$. A matrix $A \in M_n$ has a ψ_S *polar decomposition* if $A = QR$, where Q is ψ_S *orthogonal* ($S^{-1}\overline{Q}^{-1}S = Q^{-1}$ if Q is nonsingular or

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†Institute of Mathematics, University of the Philippines, Diliman, Quezon City 1101, Philippines (rjdelacruz@math.upd.edu.ph).

‡Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA (dqg0001@auburn.edu).

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equivalently $S^{-1}\overline{Q}S = Q$ if Q is singular) and R is ψ_S *symmetric* ($S^{-1}\overline{R}^{-1}S = R$). When $S = I$, the ψ_S polar decomposition is the circular polar decomposition. If $S^{-T}S$ is normal, Granario, Merino, and Paras [4, Corollary 16] showed that a matrix $A \in M_n$ has a ψ_S polar decomposition if and only if A commutes with $\overline{S}S$, rank A and rank$(\overline{S}S - \lambda I)A$ have the same parity for every negative eigenvalue λ of $\overline{S}S$, and the ranges of SA and $\overline{A}$ are the same. In particular, a nonsingular matrix A has a ψ_S polar decomposition if and only if A commutes with $\overline{S}S$.

Let $S \in GL_n$. If Q is ϕ_S orthogonal and R is ϕ_S symmetric, then

$$\phi_S(\phi_S(Q)) = \phi_S(Q^{-1}) = \phi_S(Q)^{-1} = Q \tag{1.1}$$

and

$$\phi_S(\phi_S(R)) = \phi_S(R) = R. \tag{1.2}$$

One checks that $\phi_S(\phi_S(A)) = A$ if and only if A commutes with $S^{-T}S$. Thus, every ϕ_S orthogonal matrix and ϕ_S symmetric matrix commutes with $S^{-T}S$, which implies that

$$\text{if } A \text{ has a } \phi_S \text{ polar decomposition, then } A \text{ commutes with } S^{-T}S. \tag{1.3}$$

If A is nonsingular, we show that the converse of (1.3) is true (see Theorem 2.1). Under certain assumptions on S, necessary and sufficient conditions for a (not necessarily nonsingular) matrix $A \in M_n$ to have a ϕ_S polar decomposition are given in the following.

1. If $S \in GL_n$ is symmetric, then A has a ϕ_S polar decomposition if and only if $A\phi_S(A)$ is similar to $\phi_S(A)A$ [6, Theorem 28].
2. If $S \in GL_n$ is skew symmetric, then A has a ϕ_S polar decomposition if and only if $A\phi_S(A)$ is similar to $\phi_S(A)A$ and rank$([A\phi_S(A)]^k A)$ is even for each nonnegative integer k [1, Corollary 10].
3. If $S \in GL_n$ is a real involution, then A has a ϕ_S polar decomposition if and only if A commutes with $S^{-T}S$, $A = X^{-1}\phi_S(A)Y$ and $\phi_S(A) = Y^{-1}AX$ for some $X, Y \in GL_n$ satisfying $\phi_S(\phi_S(X)) = X$ and $\phi_S(\phi_S(Y)) = Y$ [2, Theorem 11].
4. If $S \in GL_n$ is a real skew involution, then A has a ϕ_S polar decomposition if and only if A commutes with $S^{-T}S$, $A = X^{-1}\phi_S(A)Y$ and $\phi_S(A) = Y^{-1}AX$ for some $X, Y \in GL_n$ satisfying $\phi_S(\phi_S(X)) = X$ and $\phi_S(\phi_S(Y)) = Y$, and rank$([A\phi_S(A)]^k A)$ is even for each nonnegative integer k [2, Theorem 12].
5. If $S \in GL_n$ and $S^{-T}S$ is normal, then A has a ϕ_S polar decomposition if and only if A commutes with $S^{-T}S$, $A = X^{-1}\phi_S(A)Y$ and $\phi_S(A) = Y^{-1}AX$ for some $X, Y \in GL_n$ satisfying $\phi_S(\phi_S(X)) = X$ and $\phi_S(\phi_S(Y)) = Y$, and rank$[(S^{-T}S - I)(A\phi_S(A))^k A]$ is even for each nonnegative integer k [3, Theorem 9].

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A matrix $A \in M_n$ is *nonderogatory* if for every eigenvalue λ of A, there is only one Jordan block corresponding to λ in the Jordan canonical form of A. If $S \in GL_n$ and S is symmetric or skew symmetric, then $S^{-T}S = \pm I$ is far from nonderogatory. In this paper we study the ϕ_S polar decomposition when $S^{-T}S$ is nonderogatory. We use the following notation for the statement of our main theorem.

Definition 1.1. Let $S \in GL_n$ and let $\mu_1, \mu_1^{-1}, \ldots, \mu_k, \mu_k^{-1}$ be the distinct eigenvalues of $S^{-T}S$ that are not 1 and -1. We define

$$\mathcal{S}(S, \pm) := \left(\prod_{i=1}^{k} (S^{-T}S - \mu_i I)^n (S^{-T}S - \mu_i^{-1} I)^n \right) (S^{-T}S \pm I)^n.$$

The following theorem is the main result of this paper.

Theorem 1.2. *Let $S \in GL_n$ and suppose that $S^{-T}S$ is nonderogatory. Then A has a ϕ_S polar decomposition if and only if*

1. *A commutes with $S^{-T}S$,*
2. *$A\phi_S(A)$ is similar to $\phi_S(A)A$ via a matrix that commutes with $S^{-T}S$,*
3. $\operatorname{rank}(\mathcal{S}(S,+)A)$ *is zero or odd, and*
4. $\operatorname{rank}(\mathcal{S}(S,-)A)$ *is even.*

In Section 2, we give some preliminary results. In particular, we give properties of the operator ϕ_S and we give a canonical form of matrices in GL_n under congruence. In Section 3, we prove Theorem 1.2.

2. Preliminaries. We denote by $\sigma(A)$ the spectrum of the matrix $A \in M_n$. Let $f : M_n \to M_n$ be a linear operator such that $\sigma(f(A)) = \sigma(A)$ and $f(AB) = f(B)f(A)$ for all $A, B \in M_n$. By [5, Theorem 4.5.7], there exists $S \in GL_n$ such that $f(A) = \phi_S(A)$. Conversely, the operator ϕ_S satisfies the two conditions $\sigma(\phi_S(A)) = \sigma(A)$ and $\phi_S(AB) = \phi_S(B)\phi_S(A)$ for all $A, B \in M_n$. If, in addition, we have $\phi_S(\phi_S(A)) = A$ for all $A \in M_n$, we can choose S to be symmetric or skew-symmetric [6, Lemma 15]. If $A \in M_n$ and p is an element of $\mathbb{C}[x]$, the set of polynomials with complex coefficients, then $p(\phi_S(A)) = \phi_S(p(A))$. This implies that the set of ϕ_S symmetric and the set of ϕ_S skew symmetric matrices are subspaces of M_n.

Define the set

$$\mathcal{C}(S^{-T}S) = \{A \in M_n : A(S^{-T}S) = (S^{-T}S)A\},$$

the *centralizer* of the *cosquare* $S^{-T}S$. Then $\phi_S(\phi_S(A)) = A$ if and only if $A \in \mathcal{C}(S^{-T}S)$. One also checks that the cosquare of S is ϕ_S orthogonal.

Let $S \in GL_n$ and let $A \in \mathcal{C}(S^{-T}S)$ be nonsingular. The matrix $\phi_S(A)A$ is ϕ_S symmetric. Since A is nonsingular, it follows that $\phi_S(A)A$ is nonsingular and has a

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square root R such that $R = f(\phi_S(A)A)$ for some $f \in \mathbb{C}[x]$. Observe that

$$\phi_S(R) = \phi_S(f(\phi_S(A)A)) = f(\phi_S(\phi_S(A)A)) = f(\phi_S(A)A) = R,$$

that is, R is ϕ_S symmetric. Now, let $Q \equiv AR^{-1}$. Note that

$$Q\phi_S(Q) = AR^{-1}(R^{-1}\phi_S(A)) = A(R^2)^{-1}\phi_S(A) = A(\phi_S(A)A)^{-1}\phi_S(A) = I,$$

that is, Q is ϕ_S orthogonal. Finally, since $A = QR$, the matrix A has a ϕ_S polar decomposition.

Theorem 2.1. *Let $S, A \in GL_n$. Then A has a ϕ_S polar decomposition if and only if A commutes with $S^{-T}S$.*

Let $A, B \in M_n$. We say that A is *congruent* to B if there exists $X \in GL_n$ such that $A = X^TBX$. Given congruent matrices $S, S_0 \in GL_n$, we have the following properties.

Proposition 2.2. *Let $X, S, S_0 \in GL_n$, and suppose that $S = X^TS_0X$. Let $A \in M_n$ and set $A_0 = XAX^{-1}$. Then the following hold:*

1. *$A \in \mathcal{C}(S^{-T}S)$ if and only if $A_0 \in \mathcal{C}(S_0^{-T}S_0)$.*
2. *A is ϕ_S orthogonal if and only if A_0 is ϕ_{S_0} orthogonal.*
3. *A is ϕ_S skew symmetric if and only if A_0 is ϕ_{S_0} skew symmetric.*
4. *A is ϕ_S symmetric if and only if A_0 is ϕ_{S_0} symmetric.*
5. *$X\mathcal{S}(S,\pm)AX^{-1} = \mathcal{S}(S_0,\pm)A_0$.*

Under the assumptions of Proposition 2.2, a matrix A has a ϕ_S polar decomposition if and only if XAX^{-1} has a ϕ_{S_0} polar decomposition. Whenever it is convenient, we may replace a nonsingular matrix S by a matrix that is congruent to it. The following theorem gives a canonical form of nonsingular matrices under congruence. For $\lambda \in \mathbb{C}$, we denote by $J_k(\lambda)$ the k-by-k upper triangular Jordan block corresponding to λ.

Theorem 2.3. *[8, Theorem 1.1] Let $A \in M_n$. Then there exist $X \in GL_n$ and nonnegative integers n_i, m_j, p_r, and μ_r such that*

$$X^TAX = \left(\bigoplus_{i=1}^{\alpha} J_{n_i}(0)\right) \oplus \left(\bigoplus_{j=1}^{\beta} \Gamma_{m_j}\right) \oplus \left(\bigoplus_{r=1}^{\gamma} H_{2p_r}(\mu_r)\right), \tag{2.1}$$

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where each Γ_{m_j} is of the form

$$\Gamma_{m_j} = \begin{bmatrix} 0 & & & & (-1)^{m_j+1} \\ & & & \cdot^{\cdot^{\cdot}} & (-1)^{m_j} \\ & & 1 & \cdot^{\cdot^{\cdot}} & \\ & -1 & -1 & & \\ 1 & 1 & & & 0 \end{bmatrix} \quad \textit{for } m_j > 1 \textit{ and } \Gamma_1 = [1],$$

each $H_{2p_r}(\mu_r)$ is of the form

$$H_{2p_r}(\mu) = \begin{bmatrix} 0 & I_{p_r} \\ J_{p_r}(\mu) & 0 \end{bmatrix},\ 0 \neq \mu \neq (-1)^{p_r+1},$$

and μ is determined up to replacement by μ^{-1}. Moreover, the direct sum in (2.1) is determined uniquely up to permutation of the direct summands.

Let $A, B \in GL_n$. If $A = X^T B X$ for some $X \in GL_n$, then

$$A^{-T}A = (X^T B X)^{-T}(X^T B X) = X^{-1}B^{-T}BX.$$

That is, the cosquares of A and B are similar. The converse is also true [8, Lemma 2.1].

Let $S \in GL_n$ and let $S^{-T}S$ be nonderogatory. Then $S^{-T}S$ is similar to

$$J = \bigoplus_{j=1}^{k} \left(J_{m_j}(\mu_j) \oplus J_{m_j}(\mu_j^{-1})\right) \oplus J_{2a+1}(1) \oplus J_{2b}(-1), \tag{2.2}$$

where $\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{C}\backslash\{-1, 0, 1\}$ are such that $\mu_j \neq \mu_r, \mu_r^{-1}$ if $j \neq r$. By the uniqueness assertion of Theorem 2.3, S is congruent to

$$S_0 = \left(\bigoplus_{j=1}^{k} H_{2m_j}(\mu_j)\right) \oplus \Gamma_{2a+1} \oplus \Gamma_{2b}, \tag{2.3}$$

where the cosquare of S_0 is

$$\bigoplus_{j=1}^{k} \left(J_{m_j}(\mu_j) \oplus J_{m_j}^{-T}(\mu_j)\right) \oplus G_{2a+1} \oplus G_{2b}, \tag{2.4}$$

and G_k is similar to $J_k((-1)^{k+1})$.

Set $\mathcal{P}(A) \equiv \{p(A) \,|\, p \in \mathbb{C}[x]\}$. It is known that if A is nonderogatory, then $\mathcal{C}(A) = \mathcal{P}(A)$ [5, Corollary 4.4.18].

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Let $S \in M_n$ be a direct sum of the form (2.3), and let $A \in \mathcal{C}(S^{-T}S)$. Since $S^{-T}S$ is nonderogatory, it follows that $\mathcal{C}(S^{-T}S) = \mathcal{P}(S^{-T}S)$. Hence, if $A \in \mathcal{C}(S^{-T}S)$, then $A = f(S^{-T}S)$ for some $f \in \mathbb{C}[x]$. Now,

$$\begin{aligned} A &= f(S^{-T}S) \\ &= \bigoplus_{j=1}^{k} \Big(f(J_{m_j}(\mu_j)) \oplus f(J_{m_j}^{-T}(\mu_j)) \Big) \oplus f(G_{2a+1}) \oplus f(G_{2b}), \end{aligned}$$

which is block diagonal conformal to S. Moreover,

$$\phi_S(A) = \phi_S(f(S^{-T}S)) = f(\phi_S(S^{-T}S))) = f(S^{-1}S^T) = f((S^{-T}S)^{-1}).$$

For $j \in \{1, 2, \ldots, k\}$, let $A_j = f(J_{m_j}(\mu_j)) \oplus f(J_{m_j}(\mu_j)^{-T})$, let $A_- = f(G_{2a+1})$ and $A_+ = f(G_{2b})$ so that

$$A = \bigoplus_{j=1}^{k} A_j \oplus A_- \oplus A_+. \tag{2.5}$$

In particular, every ϕ_S orthogonal or ϕ_S symmetric matrix is a matrix of the form (2.5). Hence, if A is a matrix of the form (2.5) and A has a ϕ_S polar decomposition, say $A = QR$, where Q is ϕ_S orthogonal and R is ϕ_S symmetric, then

$$Q = \textstyle\bigoplus_{j=1}^{k} Q_j \oplus Q_- \oplus Q_+ \quad \text{and} \quad R = \bigoplus_{j=1}^{k} R_j \oplus R_- \oplus R_+$$

are partitioned conformal to A. Since each Q_* and R_* are respectively ϕ_{S_*} orthogonal and ϕ_{S_*} symmetric, where $* \in \{1, 2, \ldots, k\} \cup \{+, -\}$, it follows that A_* has a ϕ_{S_*} polar decomposition. The converse is also true.

LEMMA 2.4. *Let S and A be the direct sum of the form (2.3) and (2.5), respectively. Then A has a ϕ_S polar decomposition if and only if A_j has a ϕ_{S_j} polar decomposition for all $j \in \{1, 2, \ldots, k\} \cup \{-, +\}$.*

3. Proof of Theorem 1.2. Let $S \in GL_n$ and suppose that $S^{-T}S$ is nonderogatory. Suppose that A commutes with $S^{-T}S$. Then there exists $X \in GL_n$ such that $S_0 = X^{-T}SX^{-1}$ is the direct sum in (2.3). Assume that $A_0 = XAX^{-1}$ is the direct sum in (2.5), that is, $A_0 \in \mathcal{C}(S_0^{-T}S_0)$. Using (2.4), one computes

$$\mathcal{S}(S_0, +) = 0_{n-(2a+1+2b)} \oplus (G_{2a+1} + I_{2a+1})^n \oplus 0_{2b} \tag{3.1}$$

and

$$\mathcal{S}(S_0, -) = 0_{2n-2b} \oplus (G_{2b} - I_{2b})^n. \tag{3.2}$$

Since $\sigma(G_{2a+1}) = \{1\}$ and $\sigma(G_{2b}) = \{-1\}$, we have

$$\operatorname{rank}(\mathcal{S}(S_0, +)A_0) = \operatorname{rank} A_+ \quad \text{and} \quad \operatorname{rank}(\mathcal{S}(S_0, -)A_0) = \operatorname{rank} A_-. \tag{3.3}$$

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Now A_0 has a ϕ_{S_0} polar decomposition if and only if A_j has a $\phi_{H_{2m_j}(\mu_j)}$ polar decomposition for all $j \in \{1, 2, \ldots, k\}$, A_+ has a $\phi_{\Gamma_{2a+1}}$ polar decomposition, and A_- has a $\phi_{\Gamma_{2b}}$ polar decomposition, due to Lemma 2.4. This is equivalent to the following set of conditions (see Theorem 3.4 and Theorem 3.7, which we prove in the succeeding subsections):

(a) For all $j \in \{1, 2, \ldots, k\}$, $A_j \phi_{H_{2m_j}(\mu_j)}(A_j)$ is similar to $\phi_{H_{2m_j}(\mu_j)}(A_j)A_j$ via a matrix in $\mathcal{C}(H_{2m_j}(\mu_j)^{-T} H_{2m_j}(\mu_j))$.
(b) $\operatorname{rank} A_+$ is zero or is odd and $\operatorname{rank} A_-$ is even.

Since $\mathcal{C}(G_n)$ is a commutative algebra, we have $A_- \phi_{\Gamma_{2b}}(A_-) = \phi_{\Gamma_{2b}}(A_-)A_-$ and $A_+ \phi_{\Gamma_{2a+1}}(A_+) = \phi_{\Gamma_{2a+1}}(A_+)A_+$. Therefore, condition (a) above is equivalent to

(a^*) $A_0 \phi_{S_0}(A_0)$ is similar to $\phi_{S_0}(A_0)A_0$ via a matrix in $\mathcal{C}(S_0^{-T} S_0)$.

By (3.3), condition (b) is equivalent to

(b^*) $\operatorname{rank}(\mathcal{S}(S_0, +)A_0)$ is zero or odd and $\operatorname{rank}(\mathcal{S}(S_0, -)A_0)$ is even.

One checks that (a^*) and (b^*) are satisfied and $A_0 \in \mathcal{C}(S_0^{-T} S_0)$ if and only if S and A satisfy the conditions in Theorem 1.2 (see Proposition 2.2). This proves Theorem 1.2, subject to verification of our claims about (a) ($S = H_{2n}(\mu)$) and (b) ($S = \Gamma_n$) which we address in the succeeding subsections.

3.1. The case $S = \Gamma_n$. Let $S = \Gamma_n$. If $n = 1$, then every matrix has a ϕ_S polar decomposition since every 1-by-1 matrix is ϕ_S symmetric. Let $n > 1$. We can write $G_n = S^{-T} S$ as a polynomial in $J_n(0)$:

$$G_n = (-1)^n I_n + 2(-1)^n \sum_{k=1}^{n-1} J_n(0)^k.$$

Therefore, for all $A \in \mathcal{C}(G_n)$, we have $A = f(J_n(0))$ for some $f \in \mathbb{C}[x]$ with $\deg f < n$ if $A \neq 0$.

LEMMA 3.1. *Let $n > 1$. Then $\phi_{\Gamma_n}(J_n(0)) = -J_n(0)$.*

It follows from Lemma 3.1 that $A \in M_n$ is ϕ_S symmetric if and only if f is an even polynomial. Now, if A and B are ϕ_S symmetric, then there exist even polynomials p and q such that $p(J_n(0)) = A$ and $q(J_n(0)) = B$. Hence, $C = AB = r(J_n(0))$ where $r = pq$ is even, that is, C is also ϕ_S symmetric.

LEMMA 3.2. *Let $A, B \in \mathcal{C}(G_n)$.*

(a) If A and B are ϕ_S symmetric, then AB is ϕ_S symmetric.
(b) If A and B have ϕ_S polar decompositions, then AB has a ϕ_S polar decomposition.

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Proof. The first statement is done. The second statement follows from the first and the fact that $\mathcal{C}(G_n)$ is a commuting family. $\square$

Let

$$A = \sum_{i=0}^{m} a_{2i}(J_n(0))^{2i} \tag{3.4}$$

be ϕ_S symmetric. Let $2j$ be the minimum integer in (3.4) such that $a_{2j} \neq 0$. Then $\operatorname{rank} A = \operatorname{rank}(J_n(0))^{2j} = n - 2j$. Thus, the rank of A has the same parity as n.

LEMMA 3.3. *Let A be nonzero and ϕ_{Γ_n} symmetric. Then* $\operatorname{rank} A$ *has the same parity as n. In particular, if a matrix has a ϕ_{Γ_n} polar decomposition, then its rank has the same parity as n.*

Let $A \in \mathcal{C}(G_n)$ be nonzero. If A is nonsingular, then A has a ϕ_S polar decomposition. Assume that $A = f(J_n(0))$ for some $f \in \mathbb{C}[x]$ with $\deg f < n$. If A is singular, then $A = J_n(0)^s B$ for some $B \in GL_n \cap \mathcal{C}(G_n)$ and $0 < s \leq \deg f$. Now if $\operatorname{rank} A$ has the same parity as n, then $\operatorname{rank} J_n(0)^s = n - s$ has the same parity as n; this happens only if s is even. Thus, $J_n(0)^s$ is ϕ_S symmetric, and since B has a ϕ_S polar decomposition, A has a ϕ_S polar decomposition.

THEOREM 3.4. *Let $S \in GL_n$ and let S be congruent to Γ_n. Let $A \in M_n$ be nonzero. Then A has a ϕ_S polar decomposition if and only if $A \in \mathcal{C}(S^{-T}S)$ and* $\operatorname{rank} A$ *has the same parity as n.*

3.2. The case $S = H_{2n}(\mu)$. Let $S = H_{2n}(\mu)$, where $\mu \in \mathbb{C}\backslash\{-1, 0, 1\}$. Let $A \in \mathcal{C}(S^{-T}S)$ be given. Then $A = B \oplus C$ where $B, C \in M_n$ and B, C^T are upper triangular Toeplitz. Conversely, if $A = B \oplus C$ for some $B, C \in M_n$ such that B, C^T are upper triangular Toeplitz, then $A \in \mathcal{C}(S^{-T}S)$. Moreover, $\phi_S(A) = C^T \oplus B^T$ so A is ϕ_S symmetric if and only if $C = B^T$; A is ϕ_S orthogonal if and only if B is nonsingular and $C = B^{-T}$.

Let $A, B, C, D \in M_n$. We say that the pair (A, B) is *contragrediently equivalent* to the pair (C, D) if there exist $X, Y \in GL_n$ such that $A = X^{-1}CY$ and $B = Y^{-1}DX$. In this case, we write $(A, B) \sim (C, D)$. The following theorem gives equivalent conditions for a matrix to have a ϕ_S polar decomposition if $S = H_{2n}(\mu)$. Similar results were proved by Horn and Merino [6, Theorem 28] if $S \in GL_n$ is symmetric, and by Granario, Merino, and Paras [3, Theorem 8] if $S \in GL_n$ and $S^{-T}S$ is normal. Our proof is parallel to those in [6] and [3].

THEOREM 3.5. *Let $S = H_{2n}(\mu)$, where $\mu \in \mathbb{C}\backslash\{-1, 0, 1\}$. Let $B, C \in M_n$. Suppose that $A = B \oplus C \in \mathcal{C}(S^{-T}S)$. The following are equivalent:*

1. $(B, C^T) \sim (C^T, B)$.

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2. *$(A, \phi_S(A)) \sim (\phi_S(A), A)$ and the equivalence can be achieved using ϕ_S orthogonal matrices.*
3. *$(A, \phi_S(A)) \sim (\phi_S(A), A)$ and the equivalence can be achieved using matrices in $\mathcal{C}(S^{-T}S)$.*
4. *There are ϕ_S orthogonal matrices Q_1 and Q_2 such that $A = Q_1\phi_S(A)Q_2$.*
5. *There is a $U \in M_n$ such that $(B, C^T) \sim (U, U)$.*
6. *There are ϕ_S orthogonal matrices Q_3, Q_4 and a ϕ_S symmetric matrix V such that $A = Q_3VQ_4$.*
7. *There is a ϕ_S orthogonal matrix Q and a ϕ_S symmetric matrix L such that $A = QL$.*
8. *There is a ϕ_S orthogonal matrix Q such that $A = Q\phi_S(A)Q$.*

Proof. Let $A = B \oplus C$, where $B, C^T \in M_n$ are upper triangular Toeplitz. Suppose that there exist a ϕ_S orthogonal $Z = Z_1 \oplus Z_1^{-T}$ and a ϕ_S symmetric $Y = Y_1 \oplus Y_1^T$ such that $A = ZY$. Then $B = Z_1Y_1$ and $C^T = Y_1Z_1^{-1}$, that is, $(B, C^T) \sim (Y_1, Y_1)$. Note that $(B, C^T) \sim (Y_1, Y_1)$ implies $(B, C^T) \sim (C^T, B)$ and in turn, $(B, C^T) \sim (C^T, B)$ implies that $(A, \phi_S(A)) \sim (\phi_S(A), A)$ and the equivalence is achieved using ϕ_S orthogonal matrices.

If $(A, \phi_S(A)) \sim (\phi_S(A), A)$ and the equivalence is achieved using ϕ_S orthogonal matrices, then $(A, \phi_S(A)) \sim (\phi_S(A), A)$ and the equivalence is achieved using matrices in $\mathcal{C}(S^{-T}S)$. The converse is also true [3, Lemma 4].

Suppose that $(A, \phi_S(A)) \sim (\phi_S(A), A)$ and the equivalence is achieved using ϕ_S orthogonal matrices. By [3, Lemma 4], there are ϕ_S orthogonal matrices

$$Q_1 = X_1 \oplus X_1^{-T} \quad \text{and} \quad Q_2 = X_2 \oplus X_2^{-T} \tag{3.5}$$

such that $A = Q_1\phi_S(A)Q_2$. Set

$$E = AQ_1^{-1} = Q_1\phi_S(A)Q_2Q_1^{-1} \quad \text{and} \quad Q = Q_2Q_1^{-1}. \tag{3.6}$$

Then $E = \phi_S(E)Q$. Notice that

$$EQ^{-1} = \phi_S(E) = \phi_S(\phi_S(E)Q) = Q^{-1}E. \tag{3.7}$$

Hence, E commutes with Q^{-1}. Consequently, E commutes with Q. Also, $\phi_S(E)$ commutes with Q. Let R be a square root of Q that is polynomial in Q. Then $R \in \mathcal{C}(S^{-T}S)$ and R commutes with both E and $\phi_S(E)$. Now,

$$(ER^{-1})^2 = E^2(R^2)^{-1} = E^2Q^{-1} = E\phi_S(E) = A\phi_S(A), \tag{3.8}$$

that is, $A\phi_S(A)$ has a square root. Observe that $Q = X_2X_1^{-1} \oplus (X_2X_1^{-1})^{-T}$ and

$$R = f(Q) = f(X_2X_1^{-1}) \oplus f((X_2X_1^{-1})^{-T}) \tag{3.9}$$

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for some polynomial $f(t)$. Now,

$$E = AQ_1^{-1} = BX_1^{-1} \oplus CX_1^T, \tag{3.10}$$

so that

$$\phi_S(E) = X_1C^T \oplus X_1^{-T}B^T. \tag{3.11}$$

Let

$$U = (f(X_2X_1^{-1}))^{-1}(BX_1^{-1}) \ \text{ and } \ P = X_1C^Tf(X_2X_1^{-1}). \tag{3.12}$$

Note that U is the $(1,1)$ block of $R^{-1}E$, while P is the $(1,1)$ block of $\phi_S(E)R$. Since $EQ^{-1} = \phi_S(E)$, we have $ER^{-1} = \phi_S(E)R$. Moreover, because R commutes with E and $\phi_S(E)$, we have $U = P$. Therefore,

$$B = f(X_2X_1^{-1})UX_1 \ \text{ and } \ C = X_1^{-1}U(f(X_2X_1^{-1}))^{-1}, \tag{3.13}$$

so that $(B, C^T) \sim (U, U)$.

Suppose that $(B, C^T) \sim (U, U)$. Then there are $X_3, X_4 \in GL_n$ such that $B = X_3^{-1}UX_4$ and $C^T = X_4^{-1}UX_3$. Let $V = U \oplus U^T$, $Q_3 = X_3^{-1} \oplus X_3^T$, and $Q_4 = X_4 \oplus X_4^{-T}$. Then $A = Q_3VQ_4$, where V is ϕ_S symmetric and Q_3, Q_4 are ϕ_S orthogonal.

Suppose that there is a ϕ_S symmetric V and ϕ_S orthogonal matrices Q_3, Q_4 such that $A = Q_3VQ_4$. Set $Q = Q_3Q_4$ and $L = Q_4^{-1}VQ_4$. Then Q is ϕ_S orthogonal, L is ϕ_S symmetric, and $A = QL$.

Suppose that $A = QL$ for some ϕ_S orthogonal Q and ϕ_S symmetric L. Then $Q^{-1}A = L = \phi_S(L) = \phi_S(Q^{-1}A) = \phi_S(A)Q$ so that $A = Q\phi_S(A)Q$.

Suppose that there is a ϕ_S orthogonal $Q = X \oplus X^{-T}$ such that $A = Q\phi_S(A)Q$. Then $B = XC^TX^{-1}$ and $C = X^{-T}B^TX^T$, that is $C^T = XBX^{-1}$. Hence, $(B, C^T) \sim (C^T, B)$. ∎

The following theorem is from [6, Corollary 11].

Theorem 3.6. *Let $A, B, C, D \in M_n$. Then $(A, B) \sim (C, D)$ if and only if*

1. *AB is similar to CD,*
2. *$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$, and*
3. *$\operatorname{rank}((AB)^kA) = \operatorname{rank}((CD)^kC)$ for all $k \in \mathbb{N} \cup \{0\}$.*

Let $A = B \oplus C \in \mathcal{C}(S^{-T}S)$ and suppose that $A\phi_S(A)$ is similar to $\phi_S(A)A$ via a matrix in $\mathcal{C}(S^{-T}S)$. One computes that BC^T is similar to C^TB. By Theorem 3.6, $(B, C^T) \sim (C^T, B)$, and so by Theorem 3.5, A has a ϕ_S polar decomposition.

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Conversely, if A has a ϕ_S polar decomposition, then $A\phi_S(A)$ is similar to $\phi_S(A)A$ via a ϕ_S orthogonal matrix, and hence via a matrix in $\mathcal{C}(S^{-T}S)$.

THEOREM 3.7. *Let $S = H_{2n}(\mu)$, where $\mu \in \mathbb{C}\backslash\{-1, 0, 1\}$ and $A \in M_{2n}$. Then A has a ϕ_S polar decomposition if and only if A commutes with $S^{-T}S$ and $A\phi_S(A)$ is similar to $\phi_S(A)A$ via a matrix that commutes with $S^{-T}S$.*

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