

CONTROLLABILITY AND NONSINGULAR SOLUTIONS OF SYLVESTER EQUATIONS*

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Abstract. The singularity problem of the solutions of some particular Sylvester equations is studied. As a consequence of this study, a good choice of a Sylvester equation which is associated to a linear continuous time system can be made such that its solution is nonsingular. This solution is then used to solve an eigenstructure assignment problem for this system. From a practical point view, this study can also be applied to automatic control when the system is subject to input constraints.

Key words. Stabilizability, Controllability indices, Sylvester equation, Nilpotent matrices, Eigenstructure assignment.

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1. Introduction. Sylvester equations play a central role in many areas of applied mathematics especially in systems and control theory. In [1, 7, 13, 17] among other references, the solution of a Sylvester equation has been used to deal with the problem of (partial) eigenstructure assignment for linear continuous time systems especially when this solution is nonsingular. The existence of the feedback matrix allowing this (partial) eigenstructure assignment is based on the non-singularity of the solution of the Sylvester equation.

In order to clarify the objective which first motivates the present work and also for a use in the sequel, we shall give an outline of the eigenstructure assignment method described in [1] or in its generalization in [13] which fails when the solution of some Sylvester equation is singular. The method presented in this paper can then be used to overcome this failure. Let

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1.1)$$

be a linear continuous time system. The matrices A and B are real and constant: $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $1 \leq m \leq n$. The vector $x(t) \in \mathbb{R}^n$ represents the state vector of the system and $u(t) \in \mathbb{R}^m$ is the control vector.

We suppose that the spectrum of the matrix A has $n - m$ desirable or stable

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eigenvalues and m undesirable or unstable eigenvalues. As a consequence of this assumption, we get a decomposition of the state space $\mathbb{R}^n$ in two complementary subspaces $\mathbb{F}_s$ and $\mathbb{F}_u$ called, respectively, stable and unstable.

In this method, we prescribe a matrix $H \in \mathbb{R}^{m \times m}$ with stable eigenvalues and we look for a full rank feedback matrix $F \in \mathbb{R}^{m \times n}$ such that $FA + FBF = HF$. If such a matrix exists, then the eigenvalues of $A + BF$ are those of H together with the stable eigenvalues of A . According to [13], one first computes $V \in \mathbb{R}^{m \times n}$ so that its rows span the orthogonal complement $\mathbb{F}_s^\perp$ of the stable subspace $\mathbb{F}_s$ and $\Lambda = VAV^\top (VV^\top)^{-1}$. Then the feedback matrix F exists if and only if the solution X of the Sylvester equation

$$\Lambda X - XH = C \quad (1.2)$$

is nonsingular, where $C = -VB$. The feedback matrix F is then given by the formula $F = X^{-1}V$. The method can then be used to solve a pole placement problem if there is a way to prescribe a suitable matrix H so that the solution of the Sylvester equation (1.2) is nonsingular. The singularity problem of the solutions of Sylvester equations has been studied in [9, 12, 15] and necessary conditions related to the controllability of the pair (Λ, C) and the observability of the pair (C, H) have been provided. It has also been shown [15] that the non-singularity of the solution of the Sylvester equation (1.2) is a generic property when Λ and H are fixed and C is generic.

It is true that there was a stream of literature that employed the Sylvester equation for the solution of pole placement problems. The just described method is particularly important in automatic control [1, 3, 4, 5] when the system (1.1) is subject to input constraints

$$-u_{\min} \leq u(t) \leq u_{\max}, \quad (1.3)$$

where $u_{\min}$ and $u_{\max}$ denote column matrices with positive components $u_{\min,i}$ and $u_{\max,i}$ for $i = 1, \dots, m$. Constraints (1.3) mean that $-u_{\min,i} \leq u_i(t) \leq u_{\max,i}$, where $u_i(t)$ are the components of $u(t)$ for $i = 1, \dots, m$. The linear feedback controller $u(t) = Fx(t)$ asymptotically stabilizes (1.1). In order to respect constraints (1.3), the matrix H should be suitably chosen [5] so that $\tilde{H}U \leq 0$, where

$$U = \begin{pmatrix} u_{\max} \\ u_{\min} \end{pmatrix} \in \mathbb{R}^{2m}, \quad \tilde{H} = \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix} \in \mathbb{R}^{2m \times 2m},$$

$$[H_1]_{i,j} = \begin{cases} H_{i,i} & \text{if } i = j \\ \max(H_{i,j}, 0) & \text{if } i \neq j \end{cases} \quad \text{and} \quad [H_2]_{i,j} = \begin{cases} 0 & \text{if } i = j \\ \max(-H_{i,j}, 0) & \text{if } i \neq j. \end{cases}$$

In this paper, we solve the singularity problem of the solution of the Sylvester equation (1.2) in the particular case where the matrix H has only one eigenvalue

μ which is not an eigenvalue of Λ , where $\Lambda, C \in \mathbb{K}^{m \times m}$, $\mu \in \mathbb{K}$ and $\mathbb{K}$ is a given field. More precisely, we show how to find the matrix $H = \mu \mathbb{I}_m + N$, where $\mathbb{I}_m$ is the identity matrix and N is a nilpotent matrix, such that the solution of the Sylvester equation (1.2) is nonsingular. It is true that the choice of the matrix H with only one eigenvalue is restrictive, but it allows us to solve the constrained problem because of the simple formula that links the matrices $\tilde{H}$ and $\tilde{N}$. The constrained problem will also be solved through the prescription of a suitable matrix H or N such that the solution of the Sylvester equation (1.2) is nonsingular and $\tilde{H}U \leq 0$. We also show that the non-singularity of the solution of the Sylvester equation (1.2) is a generic property when Λ and C are fixed and $N = H - \mu \mathbb{I}_m$ is generic in the cone $\mathcal{N}$ of nilpotent matrices. In the particular case of $m = 2$ and in order to illustrate the genericity property, we show that $\mathcal{N}$ can be identified to $\mathbb{R}^2$ and, under the controllability of the pair (Λ, C) , the set $\mathcal{N}_0$ of nilpotent matrices for which the solution of the Sylvester equation (1.2) is singular is either empty or exactly a conic of $\mathbb{R}^2$.

2. A nonsingular solution of a Sylvester equation. Throughout this section, we will consider the two matrices $\Lambda, C \in \mathbb{K}^{m \times m}$ and a scalar μ which is not an eigenvalue of the matrix Λ and we will study the singularity problem of the solution of the Sylvester equation

$$\Lambda X - X(\mu \mathbb{I}_m + N) = C, \quad (2.1)$$

where $N \in \mathcal{N}$ is a nilpotent matrix.

A general Sylvester equation $FX - XG = H$, for appropriate matrices F, G and H , has a unique solution X whenever the spectra $\sigma(F)$ and $\sigma(G)$ of F and G are disjoint. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, there are many ways to express the solution X . For example,

$$X = \sum_{j=0}^{\infty} F^{-1-j} H G^j, \quad (2.2)$$

where $\sigma(G) \subset \{z : |z| < r\}$ and $\sigma(F) \subset \{z : |z| > r\}$ for some $r > 0$ [2].

If G is nilpotent, we do not have to worry about the convergence of the above series or specify the field $\mathbb{K}$.

PROPOSITION 2.1. *The unique solution of the Sylvester equation (2.1) is*

$$X = \sum_{j=0}^{m-1} (\Lambda - \mu \mathbb{I}_m)^{-1-j} C N^j. \quad (2.3)$$

Proof. As the spectra of the matrices Λ and $\mu \mathbb{I}_m + N$ are disjoint, the Sylvester equation (2.1) has a unique solution. We just check that the matrix X given by (2.3)

is a solution of (2.1):

$$\begin{aligned}\Lambda X - X(\mu \mathbb{I}_m + N) &= (\Lambda - \mu \mathbb{I}_m)X - XN \\ &= \sum_{j=0}^{m-1} (\Lambda - \mu \mathbb{I}_m)^{-j} C N^j - \sum_{j=0}^{m-1} (\Lambda - \mu \mathbb{I}_m)^{-1-j} C N^{j+1} \\ &= C - (\Lambda - \mu \mathbb{I}_m)^{-m} C N^m \\ &= C.\end{aligned}$$

The last equality follows from the fact that $N \in \mathbb{K}^{m \times m}$ is nilpotent. $\square$

In the following proposition, we give a lower bound of the rank of nilpotent matrices N for which the solution of the Sylvester equation (2.1) is nonsingular.

PROPOSITION 2.2. *If the solution of the Sylvester equation (2.1) is nonsingular, then $m - \text{rank}(C) \leq \text{rank}(N)$.*

Proof. Let y be in $\mathbb{R}^m$ and let X be the Sylvester equation (2.1). If y belongs to $\ker \begin{pmatrix} C \\ N \end{pmatrix} = \ker C \cap \ker N$, then $Xy = 0$. As X is nonsingular, we get $y = 0$ and then $\text{rank} \begin{pmatrix} C \\ N \end{pmatrix} = m$. The desired inequality follows from the fact that $\begin{pmatrix} C \\ N \end{pmatrix} \leq \text{rank}(C) + \text{rank}(N)$. $\square$

Even though the controllability of the pair (Λ, C) is only a necessary condition for the solution X to be nonsingular [12], it will help us find a nilpotent matrix N with a minimum rank such that the solution of the Sylvester equation (2.1) is nonsingular. Recall first that the controllability matrix of the pair (Λ, C) is the $m \times m^2$ matrix $\mathcal{C} = [C, \Lambda C, \dots, \Lambda^{m-1} C]$.

If the pair (Λ, C) is controllable, then the rank of $\mathcal{C}$ is m . Let C_i be the i th column of C and r be its rank. By eliminating the columns depending linearly on the previous ones from the left in the matrix $\mathcal{C}$ we get, after reordering the vectors, the nonsingular $m \times m$ matrix

$$\mathcal{L} = [C_{\sigma(1)}, \dots, \Lambda^{\alpha_1-1} C_{\sigma(1)}, \dots, C_{\sigma(r)}, \dots, \Lambda^{\alpha_r-1} C_{\sigma(r)}]$$

where σ is a permutation of $\{1, \dots, m\}$. The integers of the list $\{\alpha_1, \dots, \alpha_r\}$ are called the controllability indices of the pair (Λ, C) and the integer α , which is the maximum of the controllability indices, is called the controllability index of the pair (Λ, C) and is the smallest integer such that the rank of the matrix $[C, \Lambda C, \dots, \Lambda^{\alpha-1} C]$ is m . These controllability indices are unique up to permutations of the elements. The permutation σ can be chosen such that $\alpha_1 \geq \dots \geq \alpha_r$. In this case, we have [6, 16] the dual relations:

$$\alpha_j = \text{card}(\{i : \alpha'_i \geq j\}) \quad \text{and} \quad \alpha'_j = \text{card}(\{i : \alpha_i \geq j\}). \quad (2.4)$$

where $\alpha'_1 = \dim S_1$, $\alpha'_j = \dim S_j - \dim S_{j-1}$ for $j \geq 2$ and where S_j is the linear subspace spanned by the vectors

$$C_1, \dots, C_m, \dots, \Lambda^{j-1}C_1, \dots, \dots, \Lambda^{j-1}C_m.$$

The subspace S_j is also the span of the Krylov subspaces $\mathcal{K}_j(\Lambda, C_i)$ for $i = 1, \dots, m$, where $\mathcal{K}_j(\Lambda, C_i) = \text{span}(C_i, \dots, \Lambda^{j-1}C_i)$.

The following proposition establishes the link between the controllability indices of the pairs (Λ, C) and $(M^{-1}, M^{m-1}C)$, where $M = \Lambda - \mu\mathbb{I}_m$.

PROPOSITION 2.3. *If the pair (Λ, C) is controllable, then the pair $(M^{-1}, M^{m-1}C)$ is controllable, where $M = \Lambda - \mu\mathbb{I}_m$. Moreover, the two pairs have the same controllability indices.*

Proof. From the well-known fact that Krylov subspaces are translation invariant; $\mathcal{K}_j(\Lambda, C_i) = \mathcal{K}_j(\Lambda - \mu\mathbb{I}_m, C_i)$, one can see that the subspace S_j introduced above is also spanned by the vectors

$$C_1, \dots, C_m, \dots, M^{j-1}C_1, \dots, \dots, M^{j-1}C_m.$$

The dual relations (2.4) show that the pairs (Λ, C) and (M, C) have the same controllability indices and, in particular, the pair (M, C) is controllable. Besides, the controllability matrix of the pair $(M^{-1}, M^{m-1}C)$ is

$$C' = [M^{m-1}C, M^{m-2}C, \dots, C]$$

which is of rank m because the pair (M, C) is controllable. The pair $(M^{-1}, M^{m-1}C)$ is then controllable. Let us denote the controllability indices of the pair $(M^{-1}, M^{m-1}C)$ by β_i . Let also T_j be the subspace spanned by the following vectors

$$M^{m-1}C_1, \dots, M^{m-1}C_m, \dots, M^{m-j}C_1, \dots, M^{m-j}C_m,$$

$\beta'_1 = \dim T_1$ and $\beta'_j = \dim T_j - \dim T_{j-1}$. One can see that $T_j = M^{m-j}S_j$ and, since M is nonsingular, $\beta'_j = \alpha'_j$. The dual relations (2.4) show again that $\alpha_j = \beta_j$. Therefore, the controllability indices $\{\alpha_1, \dots, \alpha_r\}$ of the pair (Λ, C) are also the controllability indices of the pair $(M^{-1}, M^{m-1}C)$. $\square$

The just introduced controllability indices are used to establish the following result which is the key to find the nilpotent matrix N discussed previously.

THEOREM 2.4. *Let $M \in \mathbb{K}^{m \times m}$ be a nonsingular matrix such that the pair (M, C) is controllable. Then, there is a nilpotent matrix $N \in \mathbb{K}^{m \times m}$ such that the following matrix*

$$K = \sum_{j=0}^{m-1} M^{m-1-j} C N^j$$

is nonsingular. Moreover, N can be chosen so that its rank is $m - r$ and its nilpotency index is α , where r is the rank of C and α is the controllability index of the pair (M, C) .

Proof. Let $\{\alpha_1, \dots, \alpha_r\}$ be the controllability indices of the pair (M, C) . We know that they are also the controllability indices of the pair $(M^{-1}, M^{m-1}C)$. One can extract columns from the controllability matrix C' of the pair $(M^{-1}, M^{m-1}C)$ so that the following matrix

$$\mathcal{L} = [M^{m-1}C_{\sigma(1)}, \dots, M^{m-\alpha_1}C_{\sigma(1)}, \dots, M^{m-1}C_{\sigma(r)}, \dots, M^{m-\alpha_r}C_{\sigma(r)}]$$

is nonsingular. For $1 \leq i \leq r$, let us take $e_i \in \mathbb{K}^m$ such that $C_i = Ce_i$. We then choose the following matrix

$$Y = [y_{\alpha_1,1}, \dots, y_{1,1}, \dots, y_{\alpha_r,r}, \dots, y_{1,r}]$$

such that $y_{\alpha_j,j} = e_{\sigma(j)}$ for $1 \leq j \leq r$ and $\mathcal{B} = \{y_{i,j} ; 1 \leq j \leq r, 1 \leq i \leq \alpha_j - 1\}$ is a basis for $\ker C$. This is possible because $\mathcal{B}$ contains $m - r = \dim \ker C$ vectors. From the way the matrix Y is defined, it is nonsingular. In fact, let

$$z = (z_{\alpha_1,1}, \dots, z_{1,1}, \dots, z_{\alpha_r,r}, \dots, z_{1,r})^\top \in \mathbb{K}^m$$

such that $Yz = 0$, then $CYz = 0$. We have

$$CY = [e_{\sigma(1)}, 0, \dots, 0, e_{\sigma(2)}, 0, \dots, 0, e_{\sigma(r)}, 0, \dots, 0]$$

and, from $CYz = 0$, we get $z_{\alpha_j,j} = 0$ since $e_{\sigma(1)}, \dots, e_{\sigma(r)}$ are linearly independent. Then Yz is a linear combination of the vectors in the basis $\mathcal{B}$ and therefore $z_{i,j} = 0$ for $1 \leq j \leq r$ and $i \leq \alpha_j - 1$. Now, we define the matrix N such that:

$$\begin{cases} Ny_{\alpha_1,1} = 0, \dots, Ny_{\alpha_r,r} = 0 \\ Ny_{i,j} = \gamma_{i,j}y_{i+1,j} \text{ for } 1 \leq j \leq r, 1 \leq i \leq \alpha_j - 1, \end{cases} \quad (2.5)$$

where $\gamma_{i,j}$ is a nonzero element in $\mathbb{K}$. More generally, for $1 \leq k \leq r$ and $1 \leq i \leq \alpha_k$, we have

$$N^j y_{i,k} = \begin{cases} 0 & \text{if } i + j > \alpha_k, \\ \gamma_{i,k} \cdots \gamma_{i+j-1,k} y_{i+j,k} & \text{if } i + j \leq \alpha_k \end{cases}$$

for every $j \geq 0$, where we let $\gamma_{0,k} = 1$. In particular, we have $N^{\alpha_k} y_{i,k} = 0$ and $N^{\alpha_k-1} y_{1,k} = \gamma_{1,k} \cdots \gamma_{\alpha_k-1,k} y_{\alpha_k,k} \neq 0$. This shows that the matrix N is nilpotent and that its nilpotency index is equal to $\alpha = \max(\alpha_1, \dots, \alpha_r)$. It is also easy to see that the rank of NY is $m - r$ which shows that the rank of N is $m - r$. From another

hand, we have

$$\begin{aligned}
 Ky_{i,k} &= \sum_{j=0}^{m-1} M^{m-1-j} CN^j y_{i,k} \\
 &= \sum_{j=0}^{\alpha_k-i} M^{m-1-j} CN^j y_{i,k} + \sum_{j=\alpha_k-i+1}^{m-1} M^{m-1-j} CN^j y_{i,k} \\
 &= \gamma_{i,k} \cdots \gamma_{\alpha_k-1,k} M^{m-1-\alpha_k+i} C y_{\alpha_k,k} \\
 &\quad + \sum_{j=0}^{\alpha_k-i-1} \gamma_{i,k} \cdots \gamma_{i+j-1,k} M^{m-1-j} C y_{i+j,k} \\
 &= \gamma_{i,k} \cdots \gamma_{\alpha_k-1,k} M^{m-1-\alpha_k+i} C_{\sigma(k)}.
 \end{aligned}$$

The last equality follows from the fact that $y_{i+j,k} \in \ker C$ for $i+j < \alpha_k$. This means that $KY = \mathcal{L}D_\gamma$, where D_γ is the diagonal matrix whose diagonal elements are the nonzero scalars $\gamma_{i,k} \cdots \gamma_{\alpha_k-1,k}$. This proves that the matrix K is nonsingular. $\square$

We point out again that the nilpotent matrix N obtained in Theorem 2.4 is of minimal rank. We now show that it is also of minimal nilpotency index.

PROPOSITION 2.5. *Let $M \in \mathbb{K}^{m \times m}$ be a nonsingular matrix such that the pair (M, C) is controllable and let $N \in \mathbb{K}^{m \times m}$ be a nilpotent matrix such that the matrix*

$$K = \sum_{j=0}^{m-1} M^{m-1-j} CN^j$$

is nonsingular. Then, the nilpotency index of N is at least α , where α is the controllability index of the pair (M, C) .

Proof. Suppose that K is nonsingular and let $\beta \geq 1$ be the nilpotency index of N . Then, the columns of $K = \sum_{j=0}^{\beta-1} M^{m-1-j} CN^j$ belong to the subspace S which is spanned by the columns of the matrix $\mathcal{C}_\beta = [M^{m-1}C, M^{m-2}C, \dots, M^{m-\beta}C]$. Since the matrix K is nonsingular, we have $S = \mathbb{K}^m$ and the rank of $\mathcal{C}_\beta$ is m . This shows that $\beta \geq \alpha$. $\square$

The goal of this section is reached by the following theorem.

THEOREM 2.6. *If the pair (Λ, C) is controllable, then there is a nilpotent matrix N such that the unique solution of the Sylvester equation (2.1) is nonsingular. Moreover, the rank of N is $m - r$ and its nilpotency index is the controllability index of the pair (Λ, C) , where r is the rank of C .*

Proof. In the proof of Proposition 2.3, we have seen that the subspace S_j is also spanned by the vectors

$$C_1, \dots, C_m, \dots, (\Lambda - \mu \mathbb{I}_m)^{j-1} C_1, \dots, (\Lambda - \mu \mathbb{I}_m)^{j-1} C_m.$$

This shows that the pairs (Λ, C) and $(\Lambda - \mu \mathbb{I}_m, C)$ have the same controllability index and, in particular, the $(\Lambda - \mu \mathbb{I}_m, C)$ is controllable.

From the fact that the matrix $\Lambda - \mu \mathbb{I}_m$ is nonsingular and the the fact that the pair $(\Lambda - \mu \mathbb{I}_m, C)$ is controllable, Theorem 2.4 ensures that there is a nilpotent matrix N of rank $m - r$ and nilpotency index which is equal to the controllability index of the pair $(\Lambda - \mu \mathbb{I}_m, C)$ such that the matrix

$$K = \sum_{j=0}^{m-1} (\Lambda - \mu \mathbb{I}_m)^{m-1-j} C N^j$$

is nonsingular. Finally, Proposition 2.1 shows that $X = (\Lambda - \mu \mathbb{I}_m)^{-m} K$, which is nonsingular, is the unique solution of the Sylvester equation (2.1). $\square$

3. Application toward an eigenstructure assignment. In this section, we will be interested in the linear continuous time system (1.1) with the same assumptions and notations used in the introduction section. We first make precise some of these notations. Let the $n - m$ desirable or stable eigenvalues of A be denoted by $\lambda_{m+1}, \dots, \lambda_n$ and the m undesirable or unstable eigenvalues be denoted $\lambda_1, \dots, \lambda_m$. That is $\text{Re}(\lambda_i) \geq 0$ for $1 \leq i \leq m$ and $\text{Re}(\lambda_i) < 0$ for $m + 1 \leq i \leq n$, where $\text{Re}(\lambda)$ stands for the real part of some complex number λ . To these eigenvalues partitioning, we associate the polynomials

$$P(x) = (x - \lambda_1) \cdots (x - \lambda_m) \quad \text{and} \quad Q(x) = (x - \lambda_{m+1}) \cdots (x - \lambda_n)$$

that are factors of the characteristic polynomial of the matrix A and the two subspaces $\mathbb{F}_u = \ker P(A)$ and $\mathbb{F}_s = \ker Q(A)$ of $\mathbb{R}^n$. The matrices Λ and V satisfy the following properties [13]:

- $VA = \Lambda V$,
- the eigenvalues of H are the unstable eigenvalues of A , and
- $\ker V = \mathbb{F}_s$.

If μ is a negative real number and $H = \mu \mathbb{I}_m + N$ for some nilpotent matrix $N \in \mathbb{R}^{m \times m}$ such that the unique solution X of the Sylvester equation (2.1) is nonsingular, then the eigenvalues of $A + BF$ are $\mu, \lambda_{m+1}, \dots, \lambda_n$, where $F = X^{-1}V$ [13]. The linear feedback controller $u(t) = Fx(t)$ asymptotically stabilizes then the system (1.1). To effectively prescribe the suitable nilpotent matrix N , we shall suppose that the pair $(\Lambda, -VB)$ is controllable in order to use Theorem 2.6. The pair $(\Lambda, -VB)$ represents the unstable part of the system (1.1) and its controllability is a consequence of the stabilizability of the pair (A, B) . So, under the assumption of the stabilizability of the original pair (A, B) which is a natural assumption for pole placement methods, Theorem 2.6 can be used to solve the singularity problem encountered in the methods

described in [1, 13] at least in the particular case of only one eigenvalue. This method has the advantage to deal with the problem of the constraints on the inputs. In fact, let us suppose that the system (1.1) is subject to input constraints (1.3). Then our aim is to asymptotically stabilize this system through a linear feedback controller $u(t) = Fx(t)$ without violating constraints (1.3). If we keep the notations used in the proof of Theorem 2.4, the equalities (2.5) can be stated as $NY = YJ(\gamma)$, where

$$J(\gamma) = \begin{pmatrix} J_{1,\gamma} & 0 & \dots & 0 \\ 0 & J_{2,\gamma} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{r,\gamma} \end{pmatrix} \quad \text{and} \quad J_{i,\gamma} = \begin{pmatrix} 0 & \gamma_{i,\alpha_i-1} & 0 & \dots & 0 \\ 0 & \ddots & \gamma_{i,\alpha_i-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & \gamma_{i,1} \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

It can be seen that we have $\tilde{H} = \mu \mathbb{I}_{2m} + \tilde{N}$, where $\tilde{H}$ or $\tilde{N}$ is the matrix defined in the introduction section. The inequality $\tilde{H}U \leq 0$ becomes then $\tilde{N}U \leq -\mu U$. As the coefficients of the matrix $\tilde{N}$ are obtained from those of $N = YJ(\gamma)Y^{-1}$ using the continuous functions min and max, the matrix $\tilde{N}$ depends continuously on the coefficients $\gamma_{i,\ell}$. If we choose the coefficients $\gamma_{i,\ell}$ small enough, the equality $\tilde{N}U \leq -\mu U$ will be satisfied since $-\mu$ and the components of U are positive. The constrained problem can then be solved. This will be illustrated through the following example.

EXAMPLE 3.1. Let the system (1.1) be given with

$$A = \begin{pmatrix} -4 & -9 & -9 \\ 3 & 14 & 15 \\ 1 & -6 & -7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 2 \\ 0 & -2 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = -1$ and the stable subspace $\mathbb{F}_s = \ker(A + \mathbb{I}_3)$ is spanned the vector $(0, -1, 1)^\top$. If we take $V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, then the columns of $V^\top$ span $\mathbb{F}_s^\perp$ and $\Lambda = VAV^\top (VV^\top)^{-1} = \begin{pmatrix} 0 & -1 \\ 4 & 4 \end{pmatrix}$. We also have $C = -VB = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$. According to (2.5), the nilpotent matrix N is defined by $Ne_1 = 0$ and $Ny = \gamma e_1$, where $e_1 = (1, 0)^\top$, γ is a nonzero real number and $y = (-1, 1)^\top \in \ker C$. This means that $N = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$. If $\mu = -1$, then the solution of the Sylvester equation $\Lambda X - X(-\mathbb{I}_2 + N) = C$ is

$$X = (\Lambda + \mathbb{I}_2)^{-1}C + (\Lambda + \mathbb{I}_2)^{-2}CN = \frac{1}{27} \begin{pmatrix} -12 & -5\gamma - 12 \\ 15 & 7\gamma + 15 \end{pmatrix}$$

which is nonsingular since its determinant is $-\frac{\gamma}{81}$. One can check that the only eigenvalue of the matrix

$$A + BF = \frac{1}{\gamma} \begin{pmatrix} -46\gamma & -81\gamma & -81\gamma \\ 3\gamma - 90 & 14\gamma - 162 & 15\gamma - 162 \\ 22\gamma + 90 & 30\gamma + 162 & 29\gamma + 162 \end{pmatrix}$$

is $\lambda = -1$ (multiplicity 3), where

$$F = X^{-1}V = \frac{1}{\gamma} \begin{pmatrix} -21\gamma - 45 & -36\gamma - 81 & -36\gamma - 81 \\ 45 & 81 & 81 \end{pmatrix}.$$

Then the linear controller $u(t) = Fx(t)$ which depends on the nonzero scalar γ asymptotically stabilizes the system (1.1). Let us now suppose that the input vector $u(t) = (u_1(t), u_2(t))^T$ is constrained to evolve in the polyhedral set

$$D = \left\{ u = (u_1, u_2)^T \in \mathbb{R}^2 : -1 \leq u_1 \leq 2 \text{ and } -2 \leq u_2 \leq 3 \right\}.$$

That is $u_{\min} = (1, 2)^T$ and $u_{\max} = (2, 3)^T$. If $x(0) = x_0$ is an initial state such that $u(0) = Fx_0$ belongs to D , then $u(t) = Fx(t)$ should remain in D for all future time $t \geq 0$, where $x(t)$ is the solution of the system (1.1) with $x(0) = x_0$. This will be satisfied if $\tilde{H}U \leq 0$. We have $U = (2, 3, 1, 2)^T$ and as $N = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$, we also have

$$\tilde{H} = \begin{pmatrix} -1 & \max(\gamma, 0) & 0 & \min(-\gamma, 0) \\ 0 & -1 & 0 & 0 \\ 0 & \min(-\gamma, 0) & -1 & \max(\gamma, 0) \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then

$$\tilde{H}U \leq 0 \iff \begin{cases} 3 \max(\gamma, 0) + 2 \min(-\gamma, 0) \leq 2 \\ 3 \min(-\gamma, 0) + 2 \max(\gamma, 0) \leq 1 \end{cases}$$

which is satisfied for $|\gamma| \leq \frac{1}{3}$.

4. Genericity of the solution. The system (1.1) or equivalently the pair (A, B) can be considered as a point of the space $\mathbb{R}^{n \times (m+n)}$. If some property of the system (1.1) is preserved in an open and dense set of $\mathbb{R}^{n \times (m+n)}$, we say that this property is generic. A lot of classical properties of linear systems can be studied in terms of genericity [10]. The properties of the controllability or the stabilizability of the system (1.1) are known to be generic, see [18] in which the notion of genericity is discussed in $\mathbb{K}^\ell$, where $\mathbb{K}$ is a given field and ℓ is an appropriate integer. Also, the property of

the solution of the Sylvester equation (1.2) of being nonsingular is generic [15] with Λ , H fixed and C generic in $\mathbb{K}^{m \times m}$.

In this section, we will be interested in the genericity of the property of the solution of the Sylvester equation (2.1) of being nonsingular with Λ , C fixed and N generic in the set $\mathcal{N}$ of all nilpotent matrices of $\mathbb{K}^{m \times m}$. The space $\mathbb{K}^{m \times m}$ will then be seen as an affine space of the coordinate ring

$$\mathbb{K}[x_{1,1}, \dots, x_{1,m}, \dots, x_{m,1}, \dots, x_{m,m}] = \mathbb{K}[x_{i,j}],$$

where the variable $x_{i,j}$ stands for the (i, j) row-column of matrices in $\mathbb{K}^{m \times m}$ and the set $\mathcal{N}$ will be seen as the set of the commons zeros in $\mathbb{K}^{m \times m}$ of the m polynomials

$$\text{tr}(N), \dots, \text{tr}(N^m), \quad (4.1)$$

where $\text{tr}(M)$ is the trace of some square matrix M in $\mathbb{K}^{m \times m}$. In order to give a precise definition of the a generic property in $\mathcal{N}$, some notions from the algebraic geometry theory and one can see [8] for more details. This paragraph is divided into two subparagraphs. In the first subparagraph, we will recall the necessary notions from the algebraic geometry theory and then show that, under the controllability condition of the pair (Λ, C) , the non-singularity of the solution of the Sylvester equation (2.1) is generic which means that for “almost every” nilpotent matrix N , the matrix

$$X(N) = \sum_{j=0}^{m-1} (\Lambda - \mu \mathbb{I}_m)^{-1-j} C N^j$$

is nonsingular. The second subparagraph will be devoted to the particular case of $m = 2$. In this particular case, it is possible to parametrize nilpotent matrices which helps us to give explicit and precise characterizations of generic nilpotent matrices.

4.1. An overview of the general case. The set of the commons zeros of a family of polynomials in $\mathbb{K}[x_{i,j}]$ is called an affine variety or a Zariski closed set of $\mathbb{K}^{m \times m}$ according to the Zariski topology. An affine variety is irreducible if it cannot be the union of proper subvarieties. Any affine variety is a union of finitely many irreducible varieties. The dimension of an irreducible variety is given by the length of the longest decreasing chain of irreducible subvarieties. The dimension of an affine variety is the largest dimension of its irreducible components.

DEFINITION 4.1. Let Z be an affine variety. A subset $S \subset Z$ is called generic if it contains a Zariski dense open subset of Z . A property is generic if the set of points on which it holds is a generic set.

We shall point out that the affine variety Z inherits its Zariski topology from $\mathbb{K}^{m \times m}$ and the subset S is Zariski dense in Z if its closure is Z . Furthermore, if Z is irreducible and $S \subset Z$ is Zariski open and nonempty, then S is Zariski dense [11].

When $\mathbb{K} = \mathbb{C}$ is the complex field [14], the set $\mathcal{N}$ is an affine irreducible variety of dimension $m^2 - m$. Let $\mathcal{N}_0$ be the subset of $\mathcal{N}$ of the matrices N such that $X(N)$ is singular. In other words, $\mathcal{N}_0$ is the affine variety of the common zeros of the polynomials

$$\mathrm{tr}(N), \dots, \mathrm{tr}(N^m), \det(X(N))$$

which is a subvariety of $\mathcal{N}$.

PROPOSITION 4.2. *If the pair (Λ, C) is controllable and $\mathbb{K} = \mathbb{C}$, then, in the affine variety $\mathcal{N}$, the property “ $X(N)$ is nonsingular” is generic. In other words, the solution of the Sylvester equation (2.1) is nonsingular for almost every choice of a nilpotent matrix N .*

Proof. Let $\mathcal{N}'$ be the subset of $\mathcal{N}$ of matrices N on which the property holds. That is $\mathcal{N}' = \mathcal{N} \setminus \mathcal{N}_0$ and is then Zariski open. According to Theorem 2.6, the subset $\mathcal{N}'$ is nonempty. As mentioned before, the affine variety $\mathcal{N}$ is irreducible. The subset $\mathcal{N}'$ is then generic. $\square$

4.2. Particular case of $m = 2$. In this section, we will discuss the case when $m = 2$. In the particular case of $m = 2$, the solution of the Sylvester equation (2.1) is nonsingular if and only if the matrix

$$K(N) = (\Lambda - \mu \mathbb{I}_2)C + CN$$

is nonsingular. The following result shows that the map $N \mapsto K(N)$ is the restriction to $\mathcal{N}$ of an affine map.

PROPOSITION 4.3. *Suppose that the field $\mathbb{K}$ is infinite, then for $N \in \mathcal{N}$, one has*

$$\det(K(N)) = \mathrm{tr}(YN) + \det(C) \det(\Lambda - \mu \mathbb{I}_2), \quad (4.2)$$

where $Y = (C - \mathrm{tr}(C)\mathbb{I}_2)(\Lambda - \mu \mathbb{I}_2)C$.

Proof. If $C = \mathbb{I}_2$, one should show that

$$\det[\Lambda - \mu \mathbb{I}_2 + N] = \det(\Lambda - \mu \mathbb{I}_2) - \mathrm{tr}(\Lambda N). \quad (4.3)$$

If $\Lambda - \mu \mathbb{I}_2 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $N = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, then direct computation shows that both the two quantities present in equality (4.3) are equal to

$$b_1b_4 - a_1b_1 - a_4b_4 + a_4a_1 - a_2a_3 - a_2b_3 - a_3b_2 - b_2b_3.$$

If now C is nonsingular, then

$$\det[(\Lambda - \mu \mathbb{I}_2)C + CN] = \det(C) \det[C^{-1}(\Lambda - \mu \mathbb{I}_2)C + N].$$

Using the previous case, linearity of the trace and the fact that

$$\det(C)C^{-1} = \operatorname{tr}(C)\mathbb{I}_2 - C,$$

which is a consequence of Cayley-Hamilton theorem, one gets

$$\begin{aligned}\det[K(N)] &= \det(C) (\det(\Lambda - \mu\mathbb{I}_2) - \operatorname{tr}(C^{-1}(\Lambda - \mu\mathbb{I}_2)CN)) \\ &= \det(C) \det(\Lambda - \mu\mathbb{I}_2) - \operatorname{tr}((\operatorname{tr}(C)\mathbb{I}_2 - C)(\Lambda - \mu\mathbb{I}_2)CN) \\ &= \det(C) \det(\Lambda - \mu\mathbb{I}_2) + \operatorname{tr}(YN).\end{aligned}$$

Then the two polynomial functions

$$C \mapsto \det[(\Lambda - \mu\mathbb{I}_2)C + CN] \quad \text{and} \quad C \mapsto \operatorname{tr}(YN) + \det(C) \det(\Lambda - \mu\mathbb{I}_2)$$

are equal on $\mathbb{GL}_2(\mathbb{K})$, the set of all nonsingular matrices in $\mathbb{K}^{2 \times 2}$. This set is dense in $\mathbb{K}^{2 \times 2}$ according to the Zariski topology or the usual topology when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. These two polynomial functions are then equal everywhere. $\square$

REMARK 4.4. Let the matrices $Y = (C - \operatorname{tr}(C)\mathbb{I}_2)(\Lambda - \mu\mathbb{I}_2)C$ and the nilpotent matrix N be written as

$$Y = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x & z \\ y & -x \end{pmatrix}$$

where $x^2 + yz = 0$. The equality (4.2) becomes

$$\det(K(N)) = (a - d)x + cy + bz + \det(C)\Delta_\Lambda(\mu), \quad (4.4)$$

where Δ_Λ is the characteristic polynomial of Λ . The affine variety $\mathcal{N}$ can be seen as the cone of $\mathbb{K}^3$ defined by $x^2 + yz = 0$ and its subset $\mathcal{N}_0$ is the intersection of $\mathcal{N}$ and the “plane” $\mathcal{P}$ defined by

$$(a - d)x + cy + bz + \det(C)\Delta_\Lambda(\mu) = 0. \quad (4.5)$$

This shows that $\mathcal{N}_0$ is a conic, may be degenerate or even empty, of $\mathcal{P}$ and that its dimension is at most one. This justifies why for almost every $N \in \mathcal{N}$, one has $N \notin \mathcal{N}_0$. We should then provide conditions under which $\mathcal{P}$ is in fact a plane. That is, at least one of the scalars $a - d$, b or c is nonzero.

In the following proposition, we make precise the statements in Remark 4.4. Recall that a matrix is said to be scalar if it can be written as $\lambda\mathbb{I}_2$ for some scalar λ .

PROPOSITION 4.5. *If the pair (Λ, C) is controllable, then*

1. *$a = d$ and $b = c = 0$ if and only if the matrix Λ is scalar;*
2. *if the matrix Λ is scalar, then the matrix C is nonsingular and $\mathcal{P}$ is empty.*

Proof. Let $\mathcal{C} = [C, (\Lambda - \mu\mathbb{I}_2)C]$ be the controllability matrix of the pair $(\Lambda - \mu\mathbb{I}_2, C)$. The rank of $\mathcal{C}$ is two. Suppose that $a = d$ and $b = c = 0$. The matrix Y is then $Y = a\mathbb{I}_2$. Suppose for a moment that $a = 0$. If C is nonsingular, then $C = \text{tr}(C)\mathbb{I}_2$ which is impossible. The matrix C is then singular. Since $C^2 - \text{tr}(C)C = 0$, we get

$$(C - \text{tr}(C)\mathbb{I}_2)\mathcal{C} = [C^2 - \text{tr}(C)C, Y] = [0, 0]$$

and then again $C = \text{tr}(C)\mathbb{I}_2$ because $\mathcal{C}$ is of full rank. As C is singular, we get $C = 0$ and this is not true since the pair (Λ, C) is controllable. So a is nonzero. Now from the fact that a is nonzero and $a\mathbb{I}_2 = Y = (C - \text{tr}(C)\mathbb{I}_2)(\Lambda - \mu\mathbb{I}_2)C$, the matrix C is nonsingular and

$$\begin{aligned} a\mathbb{I}_2 &= (C - \text{tr}(C)\mathbb{I}_2)(\Lambda - \mu\mathbb{I}_2)C \\ &= -\det(C)C^{-1}(\Lambda - \mu\mathbb{I}_2)C \end{aligned}$$

and then $\Lambda = \left(\mu - \frac{a}{\det(C)}\right)\mathbb{I}_2$ is a scalar matrix. Suppose now that $\Lambda = \lambda\mathbb{I}_2$ is a scalar matrix, where $\lambda \in \mathbb{K} \setminus \{\mu\}$. Then,

$$\begin{aligned} Y &= (C - \text{tr}(C)\mathbb{I}_2)(\Lambda - \mu\mathbb{I}_2)C \\ &= (\lambda - \mu)(C^2 - \text{tr}(C)C) \\ &= -\det(C)(\lambda - \mu)\mathbb{I}_2. \end{aligned}$$

In particular, $a = d = -\det(C)(\lambda - \mu)$ and $b = c = 0$. Since the rank of

$$\mathcal{C} = [C, (\Lambda - \mu\mathbb{I}_2)C] = [C, (\lambda - \mu)C]$$

is two, the matrix C is nonsingular. The set $\mathcal{P}$ is defined by:

$$\det(C)\Delta_\Lambda(\mu) = 0$$

which is impossible since C is nonsingular and μ is not an eigenvalue of Λ . $\square$

Proposition 4.5 shows that if the pair (Λ, C) is controllable and Λ is not a scalar matrix, then the set $\mathcal{P}$ is a plane. The subset $\mathcal{N}_0$ is then a conic of $\mathcal{P}$.

The nature of $\mathcal{N}_0$ depends on the matrix Y and information on its coefficients can help to figure out the nature of $\mathcal{N}_0$. The following result establishes the relation between the coefficients of Y , $\det(C)$ and the eigenvalues λ_1 and λ_2 of Λ .

PROPOSITION 4.6. *We have*

$$(a - d)^2 + (b + c)^2 - (b - c)^2 = \det(C)^2(\lambda_1 - \lambda_2)^2. \quad (4.6)$$

Proof. From the equality $C^2 - \text{tr}(C)C = -\det(C)\mathbb{I}_2$, we get

$$\begin{aligned}\text{tr}(Y) &= \text{tr}[(C - \text{tr}(C)\mathbb{I}_2)(\Lambda - \mu\mathbb{I}_2)C] \\ &= \text{tr}[(C^2 - \text{tr}(C)C)(\Lambda - \mu\mathbb{I}_2)] \\ &= -\det(C)\text{tr}(\Lambda - \mu\mathbb{I}_2).\end{aligned}$$

For the same reason, we also have $\det(Y) = \det(C)^2\Delta_\Lambda(\mu)$. Then,

$$\begin{aligned}(a-d)^2 + (b+c)^2 - (b-c)^2 &= (a+d)^2 + 4(ad-bc) \\ &= \text{tr}(Y)^2 - 4\det(Y) \\ &= \det(C)^2 [\text{tr}(\Lambda - \mu\mathbb{I}_2)^2 - 4\det(\Lambda - \mu\mathbb{I}_2)] \\ &= \det(C)^2 [(\lambda_1 + \lambda_2 - 2\mu)^2 - 4(\lambda_1 - \mu)(\lambda_2 - \mu)] \\ &= \det(C)^2(\lambda_1 - \lambda_2)^2\end{aligned}$$

which is the desired equality. $\square$

In the sequel, we will be interested in particular case when $\mathbb{K} = \mathbb{R}$ and discuss the nature of subset $\mathcal{N}_0$. We first need to parametrize the cone $\mathcal{N}$.

Using the linear transformation $u = x$, $v = \frac{\sqrt{2}}{2}(y+z)$ and $w = \frac{\sqrt{2}}{2}(y-z)$, the equation $x^2 + yz = 0$ becomes

$$u^2 + \frac{1}{2}v^2 - \frac{1}{2}w^2 = 0. \quad (4.7)$$

The cone $\mathcal{N}$ can then be parametrized as follows

$$\begin{cases} u &= \alpha \cos \beta, \\ v &= \sqrt{2}\alpha \sin \beta, \\ w &= \sqrt{2}\alpha, \end{cases} \quad \text{or} \quad \begin{cases} x &= \alpha \cos \beta, \\ y &= \alpha(1 + \sin \beta), \\ z &= \alpha(-1 + \sin \beta) \end{cases}$$

where α and β are arbitrary real numbers. In other words, one can identify $\mathcal{N}$ with $\mathbb{R}^2$. That is, one can identify any $(x, y) = \alpha(\cos \beta, \sin \beta) \in \mathbb{R}^2$ to the matrix

$$N = \alpha \begin{pmatrix} \cos \beta & -1 + \sin \beta \\ 1 + \sin \beta & -\cos \beta \end{pmatrix}.$$

Also the equation (4.5) allows us to identify the subset $\mathcal{N}_0$ to the curve of the $\mathbb{R}^2$ which is defined by the polar equation

$$\alpha [(a-d)\cos \beta + (c+b)\sin \beta + c-b] + \det(C)\Delta_\Lambda(\mu) = 0. \quad (4.8)$$

PROPOSITION 4.7. *Suppose that the pair (Λ, C) is controllable and Λ is not a scalar matrix.*

1. *If $\lambda_1 = \lambda_2$ or C is singular, then $b \neq c$ and $\mathcal{N}_0$ is a line of $\mathbb{R}^2$.*

2. If λ_1 and λ_2 are different real numbers and C is nonsingular, then
 - (a) $\mathcal{N}_0$ is a line of $\mathbb{R}^2$ if Y is symmetric;
 - (b) $\mathcal{N}_0$ is a hyperbola of $\mathbb{R}^2$ if Y is non-symmetric.
3. If λ_1 and λ_2 are not real numbers and C is nonsingular, then
 - (a) $\mathcal{N}_0$ is a circle of $\mathbb{R}^2$ if Y is a matrix of a similitude of $\mathbb{R}^2$;
 - (b) $\mathcal{N}_0$ is an ellipse of $\mathbb{R}^2$ if Y is not a matrix of a similitude of $\mathbb{R}^2$.

Proof. Let $\Delta = \sqrt{(a-d)^2 + (b+c)^2}$.

1. Suppose that $\lambda_1 = \lambda_2$ or C is singular. From (4.6), we have $\Delta = |b-c|$. If $b = c$, then $a = d$ and $b = c = 0$ which is not possible because the pair (Λ, C) is controllable and Λ is not a scalar matrix. The equality $\Delta = |b-c|$ shows that there is $\gamma \in \mathbb{R}$ such that $\cos(\gamma) = \frac{a-d}{b-c}$ and $\sin(\gamma) = \frac{b+c}{b-c}$. The equation (4.8) becomes $\alpha [\cos(\beta - \gamma) - 1] = 0$. This shows that $\mathcal{N}_0$ can be identified to the line that passes through the origin and makes the angle γ with the x axis.
2. Suppose that λ_1 and λ_2 are different real numbers and C is nonsingular. From (4.6), we have $\Delta > |b-c|$. The equation (4.8) can be written as

$$\alpha [\Delta \cos(\beta - \gamma) + c - b] = -\det(C)\Delta_\Lambda(\mu) \quad (4.9)$$

where $\gamma \in \mathbb{R}$ is such that $\cos(\gamma) = \frac{a-d}{\Delta}$ and $\sin(\gamma) = \frac{b+c}{\Delta}$.

- (a) If the matrix Y is symmetric, that is $b = c$, then $\alpha = \frac{-\det(C)\Delta_\Lambda(\mu)}{\Delta \cos(\beta - \gamma)}$ which represents the polar equation of a line in $\mathbb{R}^2$.
- (b) If the matrix Y is non-symmetric, that is $b \neq c$, then

$$\alpha = \frac{-\frac{1}{c-b} \det(C)\Delta_\Lambda(\mu)}{1 + \frac{\Delta}{c-b} \cos(\beta - \gamma)}$$

which represents the polar equation of a hyperbola in $\mathbb{R}^2$ since $\left| \frac{\Delta}{c-b} \right| > 1$.

3. Suppose that λ_1 and λ_2 are not real numbers and C is nonsingular. Since the matrix Λ is real, its eigenvalues λ_1 and λ_2 are complex conjugate. Then, $(\lambda_1 - \lambda_2)^2 = -4\text{Im}(\lambda_1)^2 < 0$, where $\text{Im}(\lambda_1)$ is the imaginary part of λ_1 . Using again (4.6), we get $\Delta < |b-c|$.
 - (a) If Y is the matrix of a similitude of $\mathbb{R}^2$, that is $a = d$ and $b = -c$, then by the equation (4.8), we have $-2b\alpha + \det(C)\Delta_\Lambda(\mu) = 0$ which represents the circle of radius $\left| \frac{\det(C)\Delta_\Lambda(\mu)}{2b} \right|$ with center at the origin. Note that $b \neq 0$ otherwise the matrix Y is zero.
 - (b) If Y is not the matrix of a similitude of $\mathbb{R}^2$, then as (2b) we have

$$\alpha = \frac{-\frac{1}{c-b} \det(C)\Delta_\Lambda(\mu)}{1 + \frac{\Delta}{c-b} \cos(\beta - \gamma)}$$

which represents the polar equation of an ellipse in $\mathbb{R}^2$ since $\left| \frac{\Delta}{c-b} \right| < 1$.
 Again, note that $b \neq c$ since $\Delta < |b - c|$. $\square$

In the following proposition, we give more information on the nature of $\mathcal{N}_0$ when the matrix C is singular.

PROPOSITION 4.8. *Suppose that the pair (Λ, C) is controllable, Λ is not a scalar matrix and C is singular. Let θ be the angle between the vector $(1, 0)$ and a nonzero row vector of the matrix C . Then, $\mathcal{N}_0$ is the line passing through the origin and making the angle 2θ with the y axis.*

Proof. Using (4.6) and the fact that C is singular, we get $(a - d)^2 + (b + c)^2 = (b - c)^2$. If $w = (u, v)$ is a nonzero row vector which is collinear to the row vectors of C , then there are real numbers x and y that are not all zero such that $C = \begin{pmatrix} xu & xv \\ yu & yv \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} u & v \end{pmatrix}$. Since $C - \text{tr}(C)\mathbb{I}_2 = \begin{pmatrix} -v \\ u \end{pmatrix} \begin{pmatrix} y & -x \end{pmatrix}$, we get

$$Y = r \begin{pmatrix} -v \\ u \end{pmatrix} \begin{pmatrix} u & v \end{pmatrix} = r \begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix}$$

where $r = (y \ -x) (\Lambda - \mu\mathbb{I}_2) \begin{pmatrix} x \\ y \end{pmatrix}$. In particular, we have $\begin{cases} a - d = -2uvr \\ b - c = (u^2 + v^2)r \\ b + c = (u^2 - v^2)r. \end{cases}$

From the proof of Proposition 4.7, we know that $b \neq c$ and then $\frac{a-d}{c-b} = \frac{2uv}{u^2+v^2}$ and $\frac{b+c}{b-c} = \frac{u^2-v^2}{u^2+v^2}$. If we write the nonzero complex number $u + iv$ as $u + iv = \rho e^{i\theta}$, we get $\frac{u+iv}{u-iv} = e^{2i\theta}$. This shows that $\cos(2\theta) = \frac{u^2-v^2}{u^2+v^2} = \frac{b+c}{b-c}$ and $\sin(2\theta) = \frac{2uv}{u^2+v^2} = \frac{a-d}{c-b}$. The equation (4.8) becomes now $\alpha [\sin(\beta - 2\theta) - 1] = 0$. This shows that $\mathcal{N}_0$ is the line passing through the origin and making the angle 2θ with the y axis. $\square$

REMARK 4.9. Let us consider the same conditions of Proposition 4.8.

1. If both the two row vectors of C are nonzero, the value of θ depends on the choice of the row vector, but the value of the angle 2θ is the same modulo 2π .
2. Proposition 4.8 can be explained as follows: the solution of the Sylvester equation $\Lambda X - X(\mu\mathbb{I}_2 + N) = C$ is singular if and on only if there is $\alpha \in \mathbb{R}$ such that

$$N = \alpha \begin{pmatrix} -\sin(2\theta) & -1 + \cos(2\theta) \\ 1 + \cos(2\theta) & \sin(2\theta) \end{pmatrix}.$$

3. Proposition 4.8 also shows that $\mathcal{N}_0$ is independent from Λ and μ when the matrix C is singular.

In Example 3.1, we used Theorem 2.6 to find the nilpotent matrix $N = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$ such that the solution $X(N)$ of the Sylvester equation $\Lambda X - X(\mu \mathbb{I}_2 + N) = C$ is nonsingular. In the following example, we will use Proposition 4.8 to find all nilpotent matrices N such that $X(N)$ is nonsingular; that is to describe the elements of $\mathcal{N} \setminus \mathcal{N}_0$.

EXAMPLE 4.10. Let $\Lambda = \begin{pmatrix} 0 & -1 \\ 4 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ as in Example 3.1. According to this Proposition 4.8 and to the parametrization of $\mathcal{N}$, we have

$$\mathcal{N} \setminus \mathcal{N}_0 = \left\{ \alpha \begin{pmatrix} \cos \beta & -1 + \sin \beta \\ 1 + \sin \beta & -\cos \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}, \alpha \neq 0, \beta \not\equiv 2\theta + \frac{\pi}{2} \pmod{2\pi} \right\}$$

where $\theta = \frac{\pi}{4}$ is the angle between vector $(1, 0)$ and the vector $(1, 1)$. In other words

$$\mathcal{N}_0 = \left\{ \alpha \begin{pmatrix} -\sin(2\theta) & -1 + \cos(2\theta) \\ 1 + \cos(2\theta) & \sin(2\theta) \end{pmatrix} = \alpha \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

This means that, for any nilpotent matrix N which is not of the form $\alpha \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, the solution $X(N)$ of the Sylvester equation $\Lambda X - X(\mu \mathbb{I}_2 + N) = C$ is nonsingular.

5. Conclusion. We studied the singularity problem of the solutions of some particular Sylvester equations. This study was motivated by the eigenstructure assignment method presented in [1, 13] and in particular when this method fails. The singularity problem is solved if the unstable eigenvalues are to be assigned to only one stable eigenvalue using some properties of nilpotent matrices. The problem is still unsolved if the unstable eigenvalues are to be assigned to some given stable spectrum. An advantage of this study is its suitability to deal with the problem of the input constraints.

REFERENCES

- [1] A. Baddou, H. Maarouf, and A. Benzaouia. Partial eigenstructure assignment problem and its application to the constrained linear problem. *International Journal of Systems Science*, 5:908–915, 2013.
- [2] R. Bhatia and P. Rosenthal. How and why to solve the operator equation $AX - XB = Y$. *Bulletin of the London Mathematical Society*, 29:1–21, 1997.
- [3] A. Benzaouia. The resolution of the equation $XA + XBX = HX$ and the pole assignment problem. *IEEE Transactions on Automatic Control*, 40:2091–2095, 1994.
- [4] A. Benzaouia and C. Burgat. Regulator problem for linear discrete-time systems with non-symmetrical constrained control. *International Journal of Control*, 48:2441–2451, 1988.
- [5] A. Benzaouia and A. Hmamed. Regulator problem for linear continuous-time systems with nonsymmetrical constrained control. *IEEE Transactions on Automatic Control*, 38:1556–1560, 1993.

- [6] P. Brunovsky. A classification of linear controllable systems. *Kybernetika*, 3:173–187, 1970.
- [7] J.W. Choi. Left eigenstructure via Sylvester equation. *KSM International Journal*, 12:1034–1040, 1998.
- [8] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer-Verlag, Berlin, 1992.
- [9] K. Datta. The matrix equation $XA - BX = R$ and its applications. *Linear Algebra and its Applications*, 109:91–105, 1988.
- [10] J-M. Dion, C. Commault, J. Van Der Woude. Generic properties and control of linear structured systems: A survey. *Automatica*, 39(7):1125–1144, 2003.
- [11] E.L. Allgower and G. Kurt (editors). *Computational Solution of Nonlinear Systems of Equations*. Lectures in Applied Mathematics, Vol. 26, American Mathematical Society, 1990.
- [12] J.Z. Hearon. Nonsingular solutions of $TA - BT = C$. *Linear Algebra and its Applications*, 16:57–63, 1977.
- [13] H. Maarouf and A. Baddou. Eigenstructure Assignment Method and Its Applications to the Constrained Problem. *World Journal of Engineering and Technology*, 2:159–170, 2014.
- [14] M. Skrzyński. On basic geometric properties of the cones of nilpotent matrices. *Universitatis Iagellonicae Acta Mathematica*, 33:219–228, 1996.
- [15] E. Souza and S.P. Bhattacharyya. Controllability, observability and the solution of $AX - XB = C$. *Linear Algebra and its Applications*, 39:167–188, 1981.
- [16] C. Sueur and G. Dauphin-Tanguy. Controllability indices for structured systems. *Linear Algebra and its Applications*, 250:275–287, 1997.
- [17] A. Varga. Sylvester equation based state feedback parametrization. *Computer-Aided Control System Design CACSD*. IEEE International Symposium, 13–18, 2000.
- [18] W.M. Wonham. *Linear Multivariable Control – A Geometric Approach*. Springer-Verlag, Berlin, 1979.