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POTENTIALLY NILPOTENT TRIDIAGONAL SIGN PATTERNS OF ORDER 4*

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Abstract. An $n \times n$ sign pattern $\mathcal{A}$ is said to be potentially nilpotent (PN) if there exists some nilpotent real matrix B with sign pattern $\mathcal{A}$. In [M. Arav, F. Hall, Z. Li, K. Kaphle, and N. Manzagol. Spectrally arbitrary tree sign patterns of order 4. *Electronic Journal of Linear Algebra*, 20:180–197, 2010.], the authors gave some open questions, and one of them is the following: *For the class of* 4×4 *tridiagonal sign patterns, is PN (together with positive and negative diagonal entries) equivalent to being SAP?* In this paper, a positive answer for this question is given.

Key words. Tree sign pattern, Potentially nilpotent pattern, Spectrally arbitrary pattern, Inertially arbitrary pattern.

AMS subject classifications. 15A18, 05C50.

1. Introduction. Our goal is to answer a question raised in [1]. We start with some definitions, terminologies, and some backgrounds of the problem.

A *sign pattern* (*matrix*) is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, $\mathrm{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (resp., negative) entry of B by $+$ (resp., $-$). For a sign pattern $\mathcal{A}$ of order n, the sign pattern class of $\mathcal{A}$, denoted $Q(\mathcal{A})$, is defined as $Q(\mathcal{A}) = \{B = [b_{ij}] \in M_n(\mathbb{R}) \mid \mathrm{sgn}(B) = \mathcal{A}\}$.

The *inertia* of a square real matrix B is the ordered triple $i(B) = (i_+(B), i_-(B), i_0(B))$, in which $i_+(B)$, $i_-(B)$ and $i_0(B)$ are the numbers of eigenvalues (counting multiplicities) of B with positive, negative and zero real parts, respectively.

Let $\mathcal{A}$ be a sign pattern of order $n \geq 2$. If for any given real monic polynomial $f(\lambda)$ of degree n, there is a real matrix $B \in Q(\mathcal{A})$ having characteristic polynomial $f(\lambda)$, then $\mathcal{A}$ is a *spectrally arbitrary sign pattern* (SAP); if for every ordered triple (n_+, n_-, n_0) of nonnegative integers with $n_+ + n_- + n_0 = n$, there exists a real matrix $B \in Q(\mathcal{A})$ such that $i(B) = (n_+, n_-, n_0)$, then $\mathcal{A}$ is an *inertially arbitrary pattern* (IAP); if there is some matrix $B \in Q(\mathcal{A})$ being nilpotent, then $\mathcal{A}$ is *potentially*

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nilpotent (PN); if there is some matrix $B \in Q(\mathcal{A})$ being stable, then $\mathcal{A}$ is *potentially stable* (PS); if every matrix $B \in Q(\mathcal{A})$ being nonsingular, then $\mathcal{A}$ is *sign nonsingular* (SNS).

A sign pattern $\mathcal{A}$ is a minimal spectrally arbitrary pattern if $\mathcal{A}$ is a SAP, but is not a SAP if one or more nonzero entries is replaced by zero.

It is easy to see that the class of $n \times n$ SAPs (IAPs, PN patterns) is closed under negation, transposition, permutation similarity, and signature similarity. We say that two sign patterns are *equivalent* if one can be obtained from the other by using a sequence of such operations.

Recent work (for example, see [1–7] and their references) have examined PN patterns and their relationships with SAPs, IAPs. The following basic relationships on SAPs, IAPs and PN patterns are well known:

- $\mathcal{A}$ is a SAP $\Rightarrow$ $\mathcal{A}$ is an IAP.
- $\mathcal{A}$ is a SAP $\Rightarrow$ $\mathcal{A}$ is PN.
- The converse of each of the above implications does not hold in general.

In [7], it is shown that all potentially nilpotent full sign patterns are spectrally arbitrary. For tree sign patterns, some notable results have been obtained. For example, in [6], it is shown that for an $n \times n$ ($n \geq 2$) star sign pattern $\mathcal{A}$, the following are equivalent: (1) $\mathcal{A}$ is spectrally arbitrary; (2) $\mathcal{A}$ is inertially arbitrary; (3) $\mathcal{A}$ is potentially stable and potentially nilpotent. In [1, 5], it is also shown that for a 4×4 tree sign pattern $\mathcal{A}$, the above three results are equivalent. In [1], the authors gave some open questions and one of them is "For the class of 4×4 tridiagonal sign patterns, is PN (together with positive and negative diagonal entries) equivalent to being SAP?"

In this paper, we prove that for a 4×4 irreducible tridiagonal sign pattern $\mathcal{A}$, PN is equivalent to being SAP.

2. Preliminaries.

Up to equivalence, a 4×4 irreducible tridiagonal sign pattern has the following form

$$\begin{bmatrix} *_0 & + & 0 & 0 \\ * & *_0 & + & 0 \\ 0 & * & *_0 & + \\ 0 & 0 & * & *_0 \end{bmatrix}, \tag{2.1}$$

where $* \in \{+, -\}$, and $*_0 \in \{+, -, 0\}$.

In this section, we determine all 4×4 irreducible potentially nilpotent tridiagonal

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sign patterns. We utilise the following lemmas.

LEMMA 2.1 ([1]). *An $n \times n$ complex matrix B is nilpotent if and only if* $\text{tr}(B) = 0, \text{tr}(B^2) = 0, \text{tr}(B^3) = 0, \ldots, \text{tr}(B^n) = 0$. *The result remains valid when the last condition* $\text{tr}(B^n) = 0$ *is replaced by* $\det(B) = 0$.

LEMMA 2.2 ([1]). *A 4×4 irreducible tridiagonal sign pattern is a SAP if and only if it is a superpattern of a sign pattern equivalent to one of the following minimal irreducible tridiagonal SAPs:*

$$\mathcal{H}_1 = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{H}_2 = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}, \ \mathcal{H}_3 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\mathcal{H}_4 = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{H}_5 = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{H}_6 = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix},$$

$$\mathcal{H}_7 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{H}_8 = \begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}.$$

LEMMA 2.3. *Let $\mathcal{A}$ be a 4×4 tridiagonal sign pattern having the form (2.1). If $\mathcal{A}$ has at most one nonzero diagonal entry, then $\mathcal{A}$ is not potentially nilpotent.*

Proof. If all diagonal entries of $\mathcal{A}$ are zero, then $\mathcal{A}$ is SNS, and so $\mathcal{A}$ is not PN. If $\mathcal{A}$ has exactly one nonzero diagonal entry, then $\text{tr}(\mathcal{A}) \neq 0$, and $\mathcal{A}$ is not PN. ◻

Some calculations in the following proofs are accomplished using Matlab.

LEMMA 2.4. *Let $\mathcal{A}$ be a 4×4 tridiagonal sign pattern having the form (2.1). If $\mathcal{A}$ has exactly two nonzero diagonal entries, then $\mathcal{A}$ is potentially nilpotent if and only if $\mathcal{A}$ is equivalent to one of the following two sign patterns:*

$$\mathcal{A}_1 = \begin{bmatrix} + & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}.$$

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Proof. For sufficiency, noticing that $\mathcal{A}_1$ is equivalent to $\mathcal{H}_2$, and $\mathcal{A}_2$ is equivalent to $\mathcal{H}_1$, we see that $\mathcal{A}_1$ and $\mathcal{A}_2$ are SAPs and therefore they are PN (or see [4], for example).

For necessity, let $\mathcal{A}$ have exactly two nonzero diagonal entries and be potentially nilpotent. By Lemma 2.1, $\mathcal{A}$ has one positive diagonal entry and one negative diagonal entry. Up to equivalence, we consider the following three cases.

Case 1. The pattern

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ * & - & + & 0 \\ 0 & * & 0 & + \\ 0 & 0 & * & 0 \end{bmatrix},$$

where $* \in \{+, -\}$.

For any $B \in Q(\mathcal{A})$, we may assume that

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ a & -b & 1 & 0 \\ 0 & c & 0 & 1 \\ 0 & 0 & d & 0 \end{bmatrix},$$

where $a, b, c, d \neq 0$, and $b > 0$.

Suppose B is nilpotent. By Lemma 2.1,

$$\begin{aligned} &\mathrm{tr}(B) = 1 - b = 0, \\ &\mathrm{tr}(B^2) = 1 + b^2 + 2a + 2c + 2d = 0, \\ &\mathrm{tr}(B^3) = 1 + 3a - 3ab - 3bc - b^3 = 0, \\ &\det(B) = d(a + b) = 0. \end{aligned}$$

From the first equation, we have $b = 1$. Substituting $b = 1$ in the third equation, we obtain $c = 0$, contradicting the assumption $B \in Q(\mathcal{A})$.

Case 2. The pattern

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ * & 0 & + & 0 \\ 0 & * & - & + \\ 0 & 0 & * & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & + & 0 & 0 \\ * & + & + & 0 \\ 0 & * & - & + \\ 0 & 0 & * & 0 \end{bmatrix},$$

where $* \in \{+, -\}$. Note that $\mathcal{A}$ is SNS. So $\mathcal{A}$ is not PN.

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Case 3. The pattern

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ * & 0 & + & 0 \\ 0 & * & 0 & + \\ 0 & 0 & * & - \end{bmatrix},$$

where $* \in \{+, -\}$.

For any $B \in Q(\mathcal{A})$, we may assume that

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & -d \end{bmatrix},$$

where $a, b, c, d \neq 0$ and $d > 0$.

If B is nilpotent, then by Lemma 2.1,

$$\begin{aligned} &\operatorname{tr}(B) = 1 - d = 0, \\ &\operatorname{tr}(B^2) = 1 + 2a + d^2 + 2b + 2c = 0, \\ &\operatorname{tr}(B^3) = 1 + 3a - d^3 - 3cd = 0, \\ &\det(B) = bd + ac = 0. \end{aligned}$$

From the first and third equations, we have $d = 1$ and $c = a$. Substitution in the second and fourth equations obtains

$$2a + 1 + b = 0, \quad b + a^2 = 0.$$

Taking a, b as unknowns and solving the system of equations, we have

$$a = (1 \pm \sqrt{2}), \quad b = -(3 \pm 2\sqrt{2}).$$

These two solutions for a, b, c, d correspond to two forms for $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$.

The lemma now follows. □

Lemma 2.5. *Let $\mathcal{A}$ be a 4×4 tridiagonal sign pattern having the form (2.1). If $\mathcal{A}$ has exactly three nonzero diagonal entries, then $\mathcal{A}$ is potentially nilpotent if and only if it is equivalent to one of the following ten sign patterns:*

$$\mathcal{A}_3 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & 0 \end{bmatrix},\ \mathcal{A}_4 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix},\ \mathcal{A}_5 = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

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$$\mathcal{A}_6 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & + \end{bmatrix}, \mathcal{A}_7 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \mathcal{A}_8 = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\mathcal{A}_9 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & + \end{bmatrix}, \mathcal{A}_{10} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \mathcal{A}_{11} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\mathcal{A}_{12} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}.$$

Proof. Check the following table, where $\mathcal{A}_7, \mathcal{A}_8$ corresponding $\mathcal{H}_1$ means that $\mathcal{A}_7, \mathcal{A}_8$ are equivalent to some superpatterns of $\mathcal{H}_1$, and the others are similar. Then the sufficiency is clear by Lemma 2.2.

$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	$\mathcal{H}_6$	$\mathcal{H}_7$	$\mathcal{H}_8$
$\mathcal{A}_7, \mathcal{A}_8$	$\mathcal{A}_{11}, \mathcal{A}_{12}$	$\mathcal{A}_{10}$	$\mathcal{A}_6$	$\mathcal{A}_9$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$

For necessity, let $\mathcal{A}$ have exactly three nonzero diagonal entries and be potentially nilpotent. By Lemma 2.1, $\mathcal{A}$ has at least one positive diagonal entry and one negative diagonal entry. Up to equivalence, we consider the following two cases.

Case 1. The pattern

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ * & * & + & 0 \\ 0 & * & * & + \\ 0 & 0 & * & 0 \end{bmatrix},$$

where $* \in \{+, -\}$, and $\mathcal{A}$ has at least one negative diagonal entry.

For any $B \in Q(\mathcal{A})$, we may assume that

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ 0 & c & d & 1 \\ 0 & 0 & e & 0 \end{bmatrix},$$

where $a, b, c, d, e \neq 0$ and at least one of b and d is negative.

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If B is nilpotent, then by Lemma 2.1,

$$\begin{aligned}
&\operatorname{tr}(B) = 1 + b + d = 0,\\
&\operatorname{tr}(B^2) = 1 + 2a + b^2 + 2c + d^2 + 2e = 0,\\
&\operatorname{tr}(B^3) = 1 + 3a + 3ab + b^3 + 3bc + 3cd + d^3 + 3de = 0,\\
&\det(B) = (a - b)e = 0.
\end{aligned}$$

From the first and fourth equations, we have

$$d = -1 - b, \quad a = b.$$

Substitution in the second and third equations obtains

$$1 + 2b + b^2 + c + e = 0, \quad c + e + be = 0.$$

So $b \neq -1$, and

$$c = -\frac{(b+1)^3}{b}, \quad e = \frac{(b+1)^2}{b}.$$

Therefore, the signs of a, c, d, e are determined by the value of b according to the following table.

b	a	c	d	e
$b < -1$	$-$	$-$	$+$	$-$
$-1 < b < 0$	$-$	$+$	$-$	$-$
$0 < b$	$+$	$-$	$-$	$+$

The three possibilities for the value of b correspond to $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$.

Case 2. The pattern

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ * & * & + & 0 \\ 0 & * & 0 & + \\ 0 & 0 & * & * \end{bmatrix},$$

where $* \in \{+, -\}$, and $\mathcal{A}$ has at least one negative diagonal entry.

We determine the form of $\mathcal{A}$ according to the signs of the subdiagonal. We only need to verify that if there exists the form which is not equivalent to one of the $\mathcal{A}_3$ through $\mathcal{A}_{12}$, then it is not PN. Up to equivalence, we consider the following cases:

Subcase 2.1. The signs of the subdiagonal are $(-, -, -)$.

In this case, there are three sign patterns $\mathcal{A}_6, \mathcal{A}_7$ and $\mathcal{A}_8$.

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Subcase 2.2. The signs of the subdiagonal are $(+,-,-)$.

In this case, there are the following forms

$$\mathcal{X}_1 = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{X}_2 = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{X}_3 = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}.$$

Note that sign patterns $\mathcal{X}_1$ and $\mathcal{X}_3$ are equivalent to the 11th and 10th patterns of Theorem 3.6 in [1], respectively, and sign pattern $\mathcal{X}_2$ is SNS. So they are not PN.

Subcase 2.3. The signs of the subdiagonal are $(-,+,-)$.

In this case, there are sign patterns $\mathcal{A}_9$, $\mathcal{A}_{10}$ and

$$\mathcal{X}_4 = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}.$$

Sign pattern $\mathcal{X}_4$ is SNS, and so it is not PN.

Subcase 2.4. The signs of the subdiagonal are $(-,-,+)$.

In this case, there are the following forms

$$\mathcal{X}_5 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}, \ \mathcal{X}_6 = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{X}_7 = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}.$$

Note that sign pattern $\mathcal{X}_5$ is equivalent to the 9th pattern of Theorem 3.6 in [1], and sign pattern $\mathcal{X}_7$ is SNS. So $\mathcal{X}_5$ and $\mathcal{X}_7$ are not PN.

For $\mathcal{X}_6$, taking any $B \in Q(\mathcal{X}_6)$, we may assume that

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -a & -b & 1 & 0 \\ 0 & -c & 0 & 1 \\ 0 & 0 & d & -e \end{bmatrix},$$

where a, b, c, d, e are all positive.

Suppose B is nilpotent. By Lemma 2.1,

$$\mathrm{tr}(B) = 1 - b - e = 0,$$

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$$\begin{aligned}
\operatorname{tr}(B^2) &= 1 - 2a + b^2 - 2c + e^2 + 2d = 0,\\
\operatorname{tr}(B^3) &= 1 - 3a + 3ab - b^3 + 3bc - e^3 - 3ed = 0,\\
\det(B) &= -ce - ad + bd = 0.
\end{aligned}$$

From the fourth equation, we obtain $b > a$. So $1 - 2a + b^2 > 0$. Thus, $c > d$ by the second equation. From the fourth equation again, we obtain $b > e$. From the second equation again, we obtain

$$c = d - a + \frac{1 + b^2 + e^2}{2}.$$

Then

$$\begin{aligned}
\operatorname{tr}(B^3) &= 1 - 3a + 3ab - b^3 + 3bc - e^3 - 3ed\\
&= 1 - 3a + \frac{3b}{2} + \frac{b^3}{2} + 3bd - 3de + \frac{3be^2}{2} - e^3\\
&> 1 - 3a + \frac{3a}{2} + \frac{a^3}{2} + 3ed - 3de + \frac{3e^3}{2} - e^3\\
&> 1 - \frac{3a}{2} + \frac{a^3}{2} \ge 0.
\end{aligned}$$

So B is not nilpotent, and $\mathcal{X}_6$ is not PN.

Subcase 2.5. The signs of the subdiagonal are $(-, +, +)$.

In this case, there are the following forms

$$\mathcal{X}_8 = \begin{bmatrix} + & + & 0 & 0\\ - & - & + & 0\\ 0 & + & 0 & +\\ 0 & 0 & + & + \end{bmatrix},\ \mathcal{X}_9 = \begin{bmatrix} + & + & 0 & 0\\ - & - & + & 0\\ 0 & + & 0 & +\\ 0 & 0 & + & - \end{bmatrix},\ \mathcal{X}_{10} = \begin{bmatrix} + & + & 0 & 0\\ - & + & + & 0\\ 0 & + & 0 & +\\ 0 & 0 & + & - \end{bmatrix}.$$

It is shown that $(B^4)_{44} > 0$ for any $B \in Q(\mathcal{X}_8)$ or $B \in Q(\mathcal{X}_{10})$. Sign pattern $\mathcal{X}_9$ is the 8th pattern of Theorem 3.6 in [1]. So $\mathcal{X}_8$ through $\mathcal{X}_{10}$ are not PN.

Subcase 2.6. The signs of the subdiagonal are $(+, -, +)$.

In this case, there are sign patterns $\mathcal{X}_{11}$, $\mathcal{A}_{11}$, and $\mathcal{A}_{12}$, where

$$\mathcal{X}_{11} = \begin{bmatrix} + & + & 0 & 0\\ + & - & + & 0\\ 0 & - & 0 & +\\ 0 & 0 & + & + \end{bmatrix}.$$

Sign pattern $\mathcal{X}_{11}$ is SNS, and so it is not PN.

Subcase 2.7. The signs of the subdiagonal are $(+, +, -)$.

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In this case, there are the following forms

$$\mathcal{X}_{12} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{X}_{13} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{X}_{14} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}.$$

Sign pattern $\mathcal{X}_{12}$ is SNS. For $\mathcal{X}_{13}$, taking any $B \in Q(\mathcal{X}_{13})$ with $\mathrm{tr}(B) = 0$, then $(B^4)_{11} > 0$. For $\mathcal{X}_{14}$, taking any $B \in Q(\mathcal{X}_{14})$, then $(B^4)_{11} > 0$. So $\mathcal{X}_{12}$ through $\mathcal{X}_{14}$ are not PN.

Subcase 2.8. The signs of the subdiagonal are $(+, +, +)$.

In this case, there are the following forms

$$\mathcal{X}_{15} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & + \end{bmatrix}, \ \mathcal{X}_{16} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{X}_{17} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & - \end{bmatrix}.$$

Note that for any $B \in Q(\mathcal{X}_i)$, $i = 15, 16, 17$, all diagonal entries of B^2 are positive, and so $\mathrm{tr}(B^2) > 0$. Then $\mathcal{X}_{16}$ through $\mathcal{X}_{17}$ are not PN.

Above discussions show that sign patterns $\mathcal{A}_3$ through $\mathcal{A}_{12}$ match the conditions. The lemma now follows. □

LEMMA 2.6. *Let $\mathcal{A}$ be a 4×4 tridiagonal sign pattern having the form (2.1). If all diagonal entries of $\mathcal{A}$ are nonzero, then $\mathcal{A}$ is potentially nilpotent if and only if it is equivalent to one of the following thirteen sign patterns:*

$$\mathcal{A}_{13} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{A}_{14} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{A}_{15} = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\mathcal{A}_{16} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{A}_{17} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{A}_{18} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\mathcal{A}_{19} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{A}_{20} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{A}_{21} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & + \end{bmatrix},$$

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$$\mathcal{A}_{22}=\begin{bmatrix} + & + & 0 & 0\\ + & - & + & 0\\ 0 & - & + & +\\ 0 & 0 & + & - \end{bmatrix},\ \mathcal{A}_{23}=\begin{bmatrix} + & + & 0 & 0\\ + & + & + & 0\\ 0 & - & - & +\\ 0 & 0 & + & - \end{bmatrix},\ \mathcal{A}_{24}=\begin{bmatrix} + & + & 0 & 0\\ + & + & + & 0\\ 0 & - & + & +\\ 0 & 0 & + & - \end{bmatrix},$$

$$\mathcal{A}_{25}=\begin{bmatrix} + & + & 0 & 0\\ + & + & + & 0\\ 0 & - & - & +\\ 0 & 0 & + & + \end{bmatrix}.$$

Proof. Check the following table, where $\mathcal{A}_{13}, \mathcal{A}_{14}, \mathcal{A}_{15}$ corresponding $\mathcal{H}_1$ means that $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ are equivalent to some superpatterns of $\mathcal{H}_1$, and the others are similar. Then the sufficiency is clear by Lemma 2.2.

$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{H}_5$	$\mathcal{H}_8$
$\mathcal{A}_{13}, \mathcal{A}_{14}, \mathcal{A}_{15}$	$\mathcal{A}_{22}, \mathcal{A}_{23}, \mathcal{A}_{24}$	$\mathcal{A}_{18}, \mathcal{A}_{19}$	$\mathcal{A}_{16}, \mathcal{A}_{17}$	$\mathcal{A}_{20}, \mathcal{A}_{21}$	$\mathcal{A}_{25}$

For necessity, let $\mathcal{A}$ have four nonzero diagonal entries and be potentially nilpotent. By Lemma 2.1, $\mathcal{A}$ has at least one positive diagonal entry and one negative diagonal entry. We determine the form of $\mathcal{A}$ according to the signs of the subdiagonal. We only need to verify that if there exists the form which is not equivalent to one of the $\mathcal{A}_{13}$ through $\mathcal{A}_{25}$, then it is not PN. Up to equivalence, we consider the following cases:

Case 1. The signs of the subdiagonal are $(-,-,-)$.

Denote

$$\mathcal{Y}_1=\begin{bmatrix} * & + & 0 & 0\\ - & * & + & 0\\ 0 & - & * & +\\ 0 & 0 & - & * \end{bmatrix}.$$

If $(\mathcal{Y}_1)_{11}$ and $(\mathcal{Y}_1)_{44}$ have different signs, up to equivalence, letting $(\mathcal{Y}_1)_{11}=+$, then the corresponding sign patterns are $\mathcal{A}_{13}$, $\mathcal{A}_{14}$ and $\mathcal{A}_{15}$.

If $(\mathcal{Y}_1)_{11}$ and $(\mathcal{Y}_1)_{44}$ have the same sign, up to equivalence, letting $(\mathcal{Y}_1)_{11} = (\mathcal{Y}_1)_{44} = +$, then at least one of $(\mathcal{Y}_1)_{22}$ and $(\mathcal{Y}_1)_{33}$ is negative. The corresponding sign patterns are $\mathcal{A}_{16}$ and $\mathcal{A}_{17}$.

Case 2. The signs of the subdiagonal are $(-,+,-)$.

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Denote

$$\mathcal{Y}_2 = \begin{bmatrix} * & + & 0 & 0 \\ - & * & + & 0 \\ 0 & + & * & + \\ 0 & 0 & - & * \end{bmatrix}.$$

If $(\mathcal{Y}_2)_{11}$ and $(\mathcal{Y}_2)_{44}$ have different signs, up to equivalence, letting $(\mathcal{Y}_2)_{11} = +$, then the corresponding sign patterns are $\mathcal{A}_{18}$, $\mathcal{A}_{19}$ and $\mathcal{Y}_{2-1}$, where

$$\mathcal{Y}_{2-1} = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix}.$$

Note that $\mathcal{Y}_{2-1}$ is SNS. So it is not PN.

If $(\mathcal{Y}_2)_{11}$ and $(\mathcal{Y}_2)_{44}$ have the same sign, up to equivalence, letting $(\mathcal{Y}_2)_{11} = (\mathcal{Y}_2)_{44} = +$, then at least one of $(\mathcal{Y}_2)_{22}$ and $(\mathcal{Y}_2)_{33}$ is negative. The corresponding sign patterns are $\mathcal{A}_{20}$ and $\mathcal{A}_{21}$.

Case 3. The signs of the subdiagonal are $(+, -, -)$.

Denote

$$\mathcal{Y}_3 = \begin{bmatrix} * & + & 0 & 0 \\ + & * & + & 0 \\ 0 & - & * & + \\ 0 & 0 & - & * \end{bmatrix}.$$

If $(\mathcal{Y}_3)_{11}$ and $(\mathcal{Y}_3)_{44}$ have different signs, up to equivalence, letting $(\mathcal{Y}_3)_{11} = +$, then the corresponding patterns are as follows.

$$\mathcal{Y}_{3-1} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{Y}_{3-2} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\mathcal{Y}_{3-3} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{Y}_{3-4} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}.$$

Note that sign pattern $\mathcal{Y}_{3-1}$ is SNS, and sign patterns $\mathcal{Y}_{3-2}$, $\mathcal{Y}_{3-3}$ and $\mathcal{Y}_{3-4}$ are equivalent to the 8th, 3rd and 6th patterns of Theorem 3.7 in [1], respectively. So $\mathcal{Y}_{3-1}$ through $\mathcal{Y}_{3-4}$ are not PN.

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If $(\mathcal{Y}_3)_{11}$ and $(\mathcal{Y}_3)_{44}$ have the same sign, up to equivalence, letting $(\mathcal{Y}_3)_{11} = (\mathcal{Y}_3)_{44} = +$, then at least one of $(\mathcal{Y}_3)_{22}$ and $(\mathcal{Y}_3)_{33}$ is negative. The corresponding patterns are as follows.

$$\mathcal{Y}_{3-5} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \mathcal{Y}_{3-6} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & + \end{bmatrix}, \mathcal{Y}_{3-7} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}.$$

Note that sign patterns $\mathcal{Y}_{3-5}$, $\mathcal{Y}_{3-6}$ and $\mathcal{Y}_{3-7}$ are equivalent to the 7th, 4th and 5th patterns of Theorem 3.7 in [1], respectively. So $\mathcal{Y}_{3-5}$ through $\mathcal{Y}_{3-7}$ are not PN.

Case 4. The signs of the subdiagonal are $(+, -, +)$.

Denote

$$\mathcal{Y}_4 = \begin{bmatrix} * & + & 0 & 0 \\ + & * & + & 0 \\ 0 & - & * & + \\ 0 & 0 & + & * \end{bmatrix}.$$

If $(\mathcal{Y}_4)_{11}$ and $(\mathcal{Y}_4)_{44}$ have different signs, up to equivalence, letting $(\mathcal{Y}_4)_{11} = +$, then the corresponding sign patterns are $\mathcal{A}_{22}$, $\mathcal{A}_{23}$ and $\mathcal{A}_{24}$.

If $(\mathcal{Y}_4)_{11}$ and $(\mathcal{Y}_4)_{44}$ have the same sign, up to equivalence, letting $(\mathcal{Y}_4)_{11} = (\mathcal{Y}_4)_{44} = +$, then at least one of $(\mathcal{Y}_4)_{22}$ and $(\mathcal{Y}_4)_{33}$ is negative. The corresponding sign patterns are $\mathcal{A}_{25}$ and $\mathcal{Y}_{4-1}$, where

$$\mathcal{Y}_{4-1} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & + \end{bmatrix}.$$

Note that $\mathcal{Y}_{4-1}$ is SNS. So it is not PN.

Case 5. The signs of the subdiagonal are $(-, +, +)$.

Denote

$$\mathcal{Y}_5 = \begin{bmatrix} * & + & 0 & 0 \\ - & * & + & 0 \\ 0 & + & * & + \\ 0 & 0 & + & * \end{bmatrix}.$$

If $(\mathcal{Y}_5)_{11}$ and $(\mathcal{Y}_5)_{44}$ have different signs, up to equivalence, letting $(\mathcal{Y}_5)_{11} = +$,

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then the corresponding patterns are as follows.

$$\mathcal{Y}_{5-1} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{Y}_{5-2} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\mathcal{Y}_{5-3} = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{Y}_{5-4} = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & - \end{bmatrix}.$$

Note that $\mathcal{Y}_{5-1}$ and $\mathcal{Y}_{5-2}$ are the 1st and 2nd patterns of Theorem 3.7 in [1], respectively. For $\mathcal{Y}_{5-3}$, taking any $B \in Q(\mathcal{Y}_{5-3})$, it is shown that $(B^4)_{44} > 0$. For $\mathcal{Y}_{5-4}$, taking any $B \in Q(\mathcal{Y}_{5-4})$ with $\mathrm{tr}(B) = 0$, it is shown that $(B^4)_{44} > 0$. So $\mathcal{Y}_{5-1}$ through $\mathcal{Y}_{5-4}$ are not PN.

If $(\mathcal{Y}_5)_{11}$ and $(\mathcal{Y}_5)_{44}$ have the same sign, up to equivalence, letting $(\mathcal{Y}_5)_{11} = (\mathcal{Y}_5)_{44} = +$, then at least one of $(\mathcal{Y}_5)_{22}$ and $(\mathcal{Y}_5)_{33}$ is negative. The corresponding patterns are as follows.

$$\mathcal{Y}_{5-5} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & + \end{bmatrix}, \mathcal{Y}_{5-6} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix}, \mathcal{Y}_{5-7} = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & + \end{bmatrix}.$$

Note that $\mathcal{Y}_{5-5}$ is equivalent to the superpattern of $\mathcal{A}_{4,9}$ in [1, page 194], sign patterns $\mathcal{Y}_{5-6}$ is equivalent to the 9th sign pattern of Theorem 3.7 in [1], and sign patterns $\mathcal{Y}_{5-7}$ is SNS. So $\mathcal{Y}_{5-5}$ through $\mathcal{Y}_{5-7}$ are not PN.

Case 6. The signs of the subdiagonal are $(+, +, +)$.

Denote

$$\mathcal{Y}_6 = \begin{bmatrix} * & + & 0 & 0 \\ + & * & + & 0 \\ 0 & + & * & + \\ 0 & 0 & + & * \end{bmatrix}.$$

Note that for any $B \in Q(\mathcal{Y}_6)$, all diagonal entries of B^2 is positive. Then $\mathcal{Y}_6$ is not PN.

Above discussions show that sign patterns $\mathcal{A}_{13}$ through $\mathcal{A}_{25}$ match the conditions. The lemma now follows. $\square$

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Combining Lemmas 2.3–2.6, we obtain the following theorem.

THEOREM 2.7. *A* 4×4 *irreducible tridiagonal sign pattern is potentially nilpotent if and only if it is equivalent to one of the twenty-five sign patterns* $\mathcal{A}_1$ *through* $\mathcal{A}_{25}$ *defined in Lemmas 2.4–2.6.*

3. Main result. THEOREM 3.1. *Let* $\mathcal{A}$ *be a* 4×4 *irreducible tridiagonal sign pattern. Then* $\mathcal{A}$ *is a SAP if and only if* $\mathcal{A}$ *is potentially nilpotent.*

Proof. The necessity is clear. For sufficiency, let $\mathcal{A}$ be a potentially nilpotent 4×4 irreducible tridiagonal sign pattern. Then $\mathcal{A}$ is equivalent to one of $\mathcal{A}_1$ through $\mathcal{A}_{25}$ by Theorem 2.7. Noting that each one of $\mathcal{A}_1$ through $\mathcal{A}_{25}$ is the superpattern of some one of $\mathcal{H}_1$ through $\mathcal{H}_8$, $\mathcal{A}$ is a SAP by Lemma 2.2. ∎

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