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# GROUP INVERSE EXTENSIONS OF CERTAIN
# $M$-MATRIX PROPERTIES[*]

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**Abstract.** In this article, generalizations of certain $M$-matrix properties are proved for the group generalized inverse. The proofs use the notion of proper splittings of one type or the other. In deriving certain results, a recently introduced notion of a $B_{\#}$-splitting is used. Applications in obtaining comparison results for the spectral radii of matrices are presented.

**Key words.** $M$-Matrix, Group inverse, Proper splitting, Pseudo regular splitting, Weak pseudo regular splitting, $B_{\#}$-Splitting.

**AMS subject classifications.** 47B37, 15A09.

**1. Introduction and motivation.** Let $\mathbb{R}^{n \times n}$ denote the space of all real matrices with $n$ rows and $n$ columns. A matrix $A \in \mathbb{R}^{n \times n}$ is called a *Z-matrix* if the off-diagonal entries of $A$ are nonpositive. A $Z$-matrix $A$ can be written as $A = sI - B$, where $s \geq 0$ and $B \geq 0$. A $Z$-matrix $A$ is called an *M-matrix* if $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of $B$, viz., the maximum of the moduli of the eigenvalues of $B$. It is well known that if $s > \rho(B)$ in the representation described above, then $M$ is invertible and $M^{-1} \geq 0$. We recall that if $C$ is a matrix, by $C \geq 0$ we mean that all the entries of $C$ are nonnegative. In fact, there are many interesting characterizations of invertible $M$-matrices. The book by Berman and Plemmons [6] records more than fifty equivalent conditions. For our purpose, we recall the following result:

THEOREM 1.1. *Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix with the representation $A = sI - B$. Then the following statements are equivalent:*

(a) *$A$ is invertible and $A^{-1} \geq 0$.*

(b) *There exists $x$ such that all the entries of $x$ and $Ax$ are positive.*

(c) *$A$ is an M-matrix with $s > \rho(B)$.*

Let us consider square matrices satisfying condition $(a)$ of the above theorem.

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Such matrices are referred to as *inverse positive* matrices and are related to a notion called monotonicity. A square real matrix $A$ is called *monotone* if $Ax \geq 0$ implies $x \geq 0$. Here, for $y \in \mathbb{R}^n = \mathbb{R}^{n \times 1}$, we use $y \geq 0$ if all the entries of $y$ are nonnegative. The concept of monotonicity was first proposed by Collatz (see [7], for instance), in connection with the application of finite difference methods for solving elliptic partial differential equations. He showed that a matrix is monotone if and only if it is invertible and the inverse is entrywise nonnegative. Hence, monotonicity is equivalent to inverse positivity. Thus, another statement which is equivalent to the three statements of Theorem 1.1 is: $Ax \geq 0$ implies $x \geq 0$.

The notion of monotonicity has been extended in a great variety of ways. Since these generalizations are too many to be included, we only present a brief review, here. Traditionally, splittings of matrices have been used in studying these extensions. For $A \in \mathbb{R}^{n \times n}$, a decomposition $A = U - V$, where $U$ is nonsingular, is referred to as a *splitting* of $A$. With such a splitting, one associates an iterative sequence $x^{k+1} = Hx^k + c$, where $H = U^{-1}V$ is called the iteration matrix and $c = U^{-1}b$, for a nonnegative integer $k$ and given an initial vector $x^0$. It is well known that this sequence converges to the unique solution of the system $Ax = b$ (irrespective of the choice of the initial vector $x^0$) if and only if $\rho(H) < 1$. It is well known that standard iterative methods arise from different choices of $U$ and $V$. For more details one could refer to the books [6] and [16]. Next, we turn our attention to two important types of splittings. A splitting $A = U - V$ where ($U$ is invertible) $U^{-1} \geq 0$ and $V \geq 0$ is called a *regular splitting*. This was proposed by Varga [16], among others and it was shown that $A$ is inverse positive if and only if for any regular splitting $A = U - V$, one has $\rho(U^{-1}V) < 1$. A splitting $A = U - V$ where ($U$ is invertible) $U^{-1} \geq 0$ and $U^{-1}V \geq 0$ is called a *weak regular splitting*. This was proposed by Ortega and Rheinboldt [13]. (Clearly, any regular splitting is a weak regular splitting.) They showed that $A$ is inverse positive if and only if for any weak regular splitting $A = U - V$, one has $\rho(U^{-1}V) < 1$. This establishes a connection between inverse positivity and convergence of an iterative scheme. It is pertinent to point out the fact that if $A$ is an invertible $M$-matrix with the usual representation $A = sI - B$, then $\rho\left(\dfrac{1}{s}B\right) < 1$. This observation immediately implies that the two types of splittings discussed here are genuine generalizations of representations of $M$-matrices. Let us also record the following: A splitting $A = U - V$ where ($U$ is invertible) $U \geq 0$, $U^{-1} \geq 0$ and $V \geq 0$ is called a *completely regular splitting* [1]. This notion was used to prove the following result. This result gives a sufficient condition for a matrix to be inverse positive.

THEOREM 1.2. (Proposition 11, [1]) *If $A = U - V$ is a completely regular splitting, and if $U^{-1}V$ or $VU^{-1}$ has an eigenvector $x > 0$ corresponding to an eigenvalue $\lambda < 1$, then $A^{-1} \geq 0$.*

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Let us review some of the important extensions of monotonicity. Mangasarian [10] called a rectangular matrix $A$ to be monotone if $Ax \geq 0 \Rightarrow x \geq 0$. He showed, using the duality theorem of linear programming, that $A$ is monotone if and only if $A$ has a nonnegative left inverse. Berman and Plemmons generalized the concept of monotonicity in several ways, in a series of papers, where they studied their relationships with nonnegativity of generalized inverses. The book by Berman and Plemmons [6] documents these results. Several applications are also studied there. In order to briefly review these extensions, we need the notion of generalized inverses.

The *Moore-Penrose (generalized) inverse* of a matrix $A \in \mathbb{R}^{m \times n}$, is the unique matrix $X \in \mathbb{R}^{n \times m}$ that satisfies the equations: $A = AXA, X = XAX, (AX)^T = AX$ and $(XA)^T = XA$. It is well known that the Moore-Penrose inverse exists for any matrix; it is denoted by $A^\dagger$. The *group (generalized) inverse* of a matrix $A \in \mathbb{R}^{n \times n}$ (if it exists), denoted by $A^\#$ is the unique matrix $X$ satisfying $A = AXA, X = XAX$ and $AX = XA$. A necessary and sufficient condition for $A^\#$ to exist is the condition $rank(A) = rank(A^2)$. Of course, if $A$ is nonsingular then $A^\# = A^\dagger = A^{-1}$. For more details, we refer to [2].

Let us recall the following result that collects two characterizations for the nonnegativity of the two generalized inverses, viz., the Moore-Penrose inverse and the group inverse. These were proved in [3, Theorem 2] and [4, Thoerem 1], respectively. $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$.

THEOREM 1.3. *Let $A \in \mathbb{R}^{n \times n}$. Then the following hold:*

(a) $A^\dagger \geq 0$ *if and only if*

$$Ax \in \mathbb{R}^n_+ + N(A^T) \ \text{ and } \ x \in R(A^T) \Rightarrow x \geq 0.$$

(b) $A^\#$ *exists and $A^\# \geq 0$ if and only if*

$$Ax \in \mathbb{R}^n_+ + N(A) \ \text{ and } \ x \in R(A) \Rightarrow x \geq 0.$$

It is helpful to observe that $A^\dagger \geq 0$ and $A^\# \geq 0$ are extensions of $A^{-1} \geq 0$ to singular matrices, whereas the second parts of statements $(a)$ and $(b)$ above are generalizations of the implication $Ax \geq 0 \Rightarrow x \geq 0$.

The notion of proper splitting of matrices has proved to be an important tool in the study of nonnegativity of generalized inverses. Let us recall this briefly. For $A \in \mathbb{R}^{m \times n}$, the set of all $m \times n$ matrices of reals, we denote the range space, the null space and the transpose of $A$ by $R(A), N(A)$ and $A^T$, respectively. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a *proper splitting* if $R(A) = R(U)$ and $N(A) = N(U)$. This notion was introduced by Berman and Plemmons [5]. Analogous to the invertible case, with such a splitting, one associates an iterative sequence $x^{k+1} = Hx^k + c$, where (this time) $H = U^\dagger V$ is (again) called the iteration matrix and $c = U^\dagger b$, for

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a nonnegative integer $k$. Once again, it is well known that this sequence converges to the unique solution of the system $Ax = b$ (irrespective of the choice of the initial vector $x^0$) if and only if $\rho(H) < 1$. For details, refer to [6].

The authors in [5] showed that if $A = U - V$ is a proper splitting with $U^\dagger \geq 0$ and $U^\dagger V \geq 0$ then $\rho(U^\dagger V) < 1$ if and only if $A^\dagger \geq 0$. Note that the type of splitting given above is a verbatim extension of what we referred to earlier as a weak regular splitting. We do not prefer to give a name to this type of a splitting. However, since our concern is nonnegativity of the group inverse, we propose the following: A proper splitting $A = U - V$ will be referred to as a *pseudo regular splitting* if $U^\#$ exists, $U^\# \geq 0$ and $V \geq 0$. A proper splitting $A = U - V$ is called a *weak pseudo regular splitting* if $U^\#$ exists, $U^\# \geq 0$ and $U^\# V \geq 0$. Let us observe that if $A = U - V$ is a proper splitting then $A^\#$ exists if and only if $U^\#$ exists. The following result, characterizing the nonnegativity of the group inverse of $A$ if it has a weak pseudo regular splitting can be considered the group inverse analogue of the result of Berman and Plemmons, mentioned previously.

THEOREM 1.4. (Theorem 3.5, [17], paraphrased) *Let $A \in \mathbb{R}^{n \times n}$ with index 1. Let $A = U - V$ be a weak pseudo regular splitting. Then the following statements are equivalent:*

(i) $A^\# \geq 0$.

(ii) $A^\# V \geq 0$.

(iii) $\rho(U^\# V) = \dfrac{\rho(A^\# V)}{1 + \rho(A^\# V)} < 1$.

Let us revert back to the case of the inverse positive matrices to provide a motivation to the results of this article. Peris [14] studied a certain extension of the notion of $Z$-matrices by proposing what are called $B$-splittings. We will not get into the specific details here, but only mention that inverse positivity was characterized in terms of the existence of $B$-splittings. In this regard, mention must be made of the work of Barker [1] who, considered regular splittings and completely regular splittings of a matrix and considered several extensions of the properties of $M$-matrices. One notable contribution in this work is the use of cones in place of the nonnegative orthant of the real Euclidean space. Irreducibility and imprimitivity of matrices were also studied in that work. Also, rather recently, the authors of [8] studied comparison results for certain nonnegative splittings and studied their relationships with inverse positive matrices. The work of Peris mentioned above, has been extended to the case of the Moore-Penrose inverse by Mishra and Sivakumar [12]. In this paper, certain extensions of some of the results of [1] and [8] on group inverses are proved. Also, a generalization of a nice result of Fan [9] which concerns the $M$-matrix property of an invertible matrix $A$ of the type $I - A^{-1}$, is proposed. In the next section, some

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notations, definitions and results are introduced. In Section 3, the main results are proved.

**2. Notation, definitions and preliminaries.** Let $L, M$ be complementary subspaces of $\mathbb{R}^n$, i.e., $L + M = \mathbb{R}^n$ and $L \cap M = \{0\}$. Then $P_{L,M}$ denotes the (not necessarily orthogonal) projection of $\mathbb{R}^n$ onto $L$ along $M$. So, we have $P_{L,M}^2 = P_{L,M}, R(P_{L,M}) = L$ and $N(P_{L,M}) = M$. If in addition $L \perp M$, then $P_{L,M}$ will be denoted by $P_L$. In such a case, we also have $P_L^T = P_L$.

The notions of the Moore-Penrose inverse and the group inverse were discussed in the introduction, where one necessary and sufficient condition for the existence of the group inverse was stated. Another characterization for the group inverse of $A$ to exist is that the subspaces $R(A)$ and $N(A)$ are complementary. (It is now easy to deduce that if $A = U - V$ is a proper splitting, then $A^{\#}$ exists if and only if $U^{\#}$ exists). Next, we collect some well known properties of $A^{\#}$ which will be used: $R(A) = R(A^{\#})$; $N(A) = N(A^{\#})$; $AA^{\#} = P_{R(A),N(A)} = A^{\#}A$ . In particular, if $x \in R(A)$ then $x = A^{\#}Ax$. For the proofs of these and other results, we refer to [2].

The following is a fundamental result concerning systems of linear equations. This will be rather frequently used in deriving some of our results. We refer the reader to [2] for its proof.

LEMMA 2.1. *Let $A \in \mathbb{R}^{n \times n}$ with index 1 and $b \in \mathbb{R}^n$. Then the system of linear equations $Ax = b$ has a solution if and only if $AA^{\#}b = b$. In such a case, the general solution is given by the formula $x = A^{\#}b + z$ for some $z \in N(A)$.*

We frequently use the following result in proving the main results of this paper.

THEOREM 2.2. *(Theorem 3.4.1, [12]) Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{n \times n}$. Suppose that $A^{\#}$ exists. Then the following hold:*

*(a) $U^{\#}$ exists.*

*(b) $AA^{\#} = UU^{\#}$ and $A^{\#}A = U^{\#}U$.*

*(c) $A = U(I - U^{\#}V)$.*

*(d) $I - U^{\#}V$ is invertible.*

*(e) $A^{\#} = (I - U^{\#}V)^{-1}U^{\#}$.*

As mentioned in the introduction, a matrix $A$ is called *nonnegative* if all the entries of $A$ are nonnegative; this is denoted by $A \geq 0$. $A$ is called *positive* if all the entries of $A$ are positive; this is denoted by $A > 0$. For $A, B \in \mathbb{R}^{m \times n}$, the notation $A \leq B$ means that $B - A \geq 0$. A vector $x \in \mathbb{R}^n$ is called *nonnegative* and is denoted by $x \geq 0$ if all its coordinates are nonnegative; $x$ is called *positive* if all its coordinates

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are positive and this will be denoted by $x > 0$. Let $int(\mathbb{R}^n_+)$ denote the set of all interior points of $\mathbb{R}^n_+$. In view of this, if $x$ is positive, sometimes, we also denote that by $x \in int(\mathbb{R}^n_+)$.

Next, we present some results connecting nonnegativity of a matrix and its spectral radius.

THEOREM 2.3. (Theorem 3.16, [16]) *Let $B \in \mathbb{R}^{n \times n}$ and $B \geq 0$. Then $\rho(B) < 1$ if and only if $(I - B)^{-1}$ exists and $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \geq 0$.*

The next result will be used in the proofs of Theorem 4.3 and Theorem 4.5, which present comparison results for the spectral radii.

LEMMA 2.4. (Theorem 2.1.11, [6]) *Let $A \geq 0$. Then $\alpha x \leq Ax$, $x \geq 0 \Rightarrow \alpha \leq \rho(A)$, and $Ax \leq \beta x$, $x > 0 \Rightarrow \rho(A) \leq \beta$.*

Let us recall the salient aspects of the Perron-Frobenius theory. Let $B$ be a matrix with all entries positive. Perron showed that $\rho(B)$ is an eigenvalue of $B$ and that it is simple, viz., the eigenspace is one dimensional. He also proved that there exists a *unique positive* vector associated with this eigenvalue which is referred to as the *Perron vector*. Now, let $B$ be a nonnegative matrix with at least one zero entry. Then it is known that $\rho(B)$ is again, an eigenvalue (but could be zero) and that there is an associated eigenvector whose entries are all *nonnegative*. Furthermore, if $B$ is nonnegative and irreducible, then $\rho(B) > 0$, is a simple eigenvalue of $B$ and there exists a positive eigenvector corresponding to $\rho(B)$. For proofs of these statements and other relevant details, we refer to the excellent books [11] and [16].

**3. Characterizations of nonnegativity of $A^{\#}$.** In this section, we record the main results of the article. The descriptions of these are as follows: We start with an interesting little result of Fan [9] who showed that if $A - I$ is an invertible $M$-matrix, then ($A$ is invertible and) the matrix $I - A^{-1}$ is also an invertible $M$-matrix. In Theorem 3.1, we obtain an extension of this result for the group inverse. We obtain two consequences of this result. The first is still an extension of Fan's result for inverse positive matrices. The second is the very result of Fan that motivated the generalization we are proving. We then turn our attention to certain interesting results of Barker [1]. He studied generalizations of $M$-matrix properties to matrices which allow splittings of certain types. We prove extensions of his results in Theorem 3.5, Theorem 3.7 and Theorem 3.9. Once again, these results involve the group inverse. The last set of results concern extensions of the corresponding results of [8]. To derive these theorems, we use the notion of a $B_{\#}$-splitting. Theorem 3.14 presents an analogue of Theorem 1.2 for matrices that possess a $B_{\#}$-splitting. We then prove a group inverse analogue of an important result of [8], in Theorem 4.1. As an application of this, certain comparison theorems are proved in Theorem 4.3 and

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Theorem 4.5, extending the corresponding results of [8].

We begin with an extension of the result of Fan, mentioned above.

THEOREM 3.1.  *Let $A \in \mathbb{R}^{n \times n}$ with index 1. Let $F = A - AA^{\#}$ and $G = AA^{\#} - A^{\#}$ be proper splittings of $F$ and $G$, respectively. Then $F^{\#}$ exists. Furthermore, if $AA^{\#} \geq 0$ and $F^{\#} \geq 0$, then $G^{\#}$ exists and $G^{\#} \geq 0$.*

*Proof.* Since $F = A - AA^{\#}$ and $G = AA^{\#} - A^{\#}$ are proper splittings, it follows that

$$R(F) = R(A) = R(AA^{\#}) = R(G)$$

and

$$N(F) = N(A) = N(AA^{\#}) = N(G).$$

Since $A^{\#}$ exists, the subspaces $R(A)$ and $N(A)$ are complementary; so are $R(F)$ and $N(F)$ so that $F^{\#}$ exists. Since we also have the complementarity of the subspaces $R(G)$ and $N(G)$, it follows that $G^{\#}$ exists. Note that

$$GG^{\#} = P_{R(G),N(G)} = P_{R(A),N(A)} = AA^{\#},$$

and so $GG^{\#} \geq 0$.

Let $u \geq 0$ and $v = G^{\#}u \in R(G) = R(A)$ so that $AA^{\#}v = v$. Then $Gv = GG^{\#}u \geq 0$ as $GG^{\#} \geq 0$. We show that $v \geq Gv$ and so we would have $v \geq 0$, proving that $G^{\#} \geq 0$. By Lemma 2.1, we have $u = Gv + w$ for some $w \in N(G)$. Thus, we have

$$Gv \in \mathbb{R}^{n}_{+} + N(G) = \mathbb{R}^{n}_{+} + N(F).$$

Let $z = A^{\#}v$. Then $AA^{\#}z = z$ and $Az = AA^{\#}v$ so that

$$Gv = AA^{\#}v - A^{\#}v = Az - z = Az - AA^{\#}z = Fz.$$

So,

$$Fz \in \mathbb{R}^{n}_{+} + N(F) \text{ and } z \in R(A) = R(F).$$

Since $F^{\#} \geq 0$, by Theorem 1.3, we then have $z \geq 0$. So,

$$0 \leq z = A^{\#}v = AA^{\#}v - Gv = v - Gv.$$

We have shown that $v \geq Gv$, as required. □

We have the following consequence of Theorem 3.1 for matrices with positive inverse.

COROLLARY 3.2.  *Let $A \in \mathbb{R}^{n \times n}$ be invertible, $F = A - I$ and $G = I - A^{-1}$. Suppose that $F^{-1}$ exists and $F^{-1} \geq 0$. Then $G^{-1}$ exists and $G^{-1} \geq 0$.*

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*Proof.* Let $Gx = 0$ so that $x = A^{-1}x$. Then $Ax = x$ and so $Fx = 0$, so that $x = 0$. Thus, $G$ is invertible. It now follows that $F = A - I$ and $G = I - A^{-1}$ are (trivial) proper splittings. Theorem 3.1 can be applied now to conclude that $G^{-1} \geq 0$. ∎

In particular, we have the result of Fan [9].

COROLLARY 3.3. (Lemma 2, [9]) *Let $A - I$ be an invertible $M$-matrix. Then $A$ is invertible and the matrix $I - A^{-1}$ is also an invertible $M$-matrix.*

*Proof.* Let us denote $F = A - I$ and $G = I - A^{-1}$. Since $F$ is an invertible $M$-matrix, $F = A - I = sI - B$ where $s > \rho(B)$ and $B \geq 0$. So, $F^{-1} \geq 0$ by Theorem 1.1. Also, $A = (s+1)I - B$ and $s + 1 > s > \rho(B)$. This implies that $A$ is invertible and $A^{-1} \geq 0$. Let $Gx = 0$ so that $x = A^{-1}x$. Then $Ax = x$ and so $Fx = 0$, so that $x = 0$. Thus, $G$ is invertible. It now follows that $F$ and $G$ satisfies all the conditions of Theorem 3.1. So $G^{-1} \geq 0$. Then $\rho(A^{-1}) < 1$ by Theorem 1.1. Hence, $G = I - A^{-1}$ is an invertible $M$-matrix. ∎

Next, we derive certain generalizations of the results of [1]. As mentioned in the beginning of this section, these are group inverse extensions of results on $M$-matrices. These are presented in Theorem 3.5, Theorem 3.7, Theorem 3.9 and Theorem 3.14. We begin with the following result.

THEOREM 3.4. (Proposition 9, [1]) *Let $A = U - V$ be a regular splitting. Then the following statements are equivalent:*

(i) $A^{-1} \geq 0$.

(ii) *The real parts of the eigenvalues of $U^{-1}A$ are positive.*

(iii) *The real eigenvalues of $U^{-1}A$ are positive.*

Next, we prove an extension of this result to group inverses.

THEOREM 3.5. *Let $A \in \mathbb{R}^{n \times n}$ such that $A^{\#}$ exists. Let $A = U - V$ be a pseudo regular splitting. Then the following statements are equivalent:*

(i) $A^{\#} \geq 0$.

(ii) *The real part of any nonzero eigenvalue of $U^{\#}A$ is positive.*

(iii) *Any nonzero real eigenvalue of $U^{\#}A$ is positive.*

*Proof.* $(i) \Rightarrow (ii)$: Suppose that $A^{\#} \geq 0$ and $A = U - V$ is a pseudo regular splitting. Then, by Theorem 1.4, $\rho(U^{\#}V) < 1$. Let $\mu$ be a nonzero eigenvalue of $U^{\#}V$. There exists $0 \neq x$ such that $U^{\#}Vx = \mu x$. Let $x = x^1 + x^2$, where $x^1 \in R(U)$ and $x^2 \in N(U)$. Since $A = U - V$ is a proper splitting, it follows that $N(U) \subseteq N(V)$. So, $U^{\#}Vx^2 = 0$. Consider $U^{\#}Vx = \mu(x^1 + x^2)$. The left hand side vector belongs

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to $R(U^\#) = R(U)$ and so is $x^1$. Hence, $\mu x^2 = 0$. Since $\mu \neq 0$, we have $x^2 = 0$ and so $x^1 = x \neq 0$. Thus, $U^\# V x^1 = \mu x^1$. Also, $U^\# U x = x^1$ and $Ax = Ax^1$. Hence, we have $U^\# Ax^1 = U^\# Ax = U^\# Ux - U^\# Vx = x^1 - U^\# V x^1 = (1 - \mu)x^1$. So, if $\mu$ is a nonzero eigenvalue of $U^\# V$, then $1 - \mu$ is an eigenvalue of $U^\# A$. An entirely similar argument shows that $1 - \mu$ is an eigenvalue of $U^\# V$, if $\mu$ is a nonzero eigenvalue of $U^\# A$. So, if $\mu$ is a nonzero eigenvalue of $U^\# A$, then $|1 - \mu| < 1$. This means that $Re\,\mu > 0$, showing that $(ii)$ holds.

$(ii) \Rightarrow (iii)$: The proof of this part is obvious.

$(iii) \Rightarrow (i)$: Suppose that the nonzero real eigenvalues of $U^\# A$ are positive. We must show that $A^\# \geq 0$. For this, it is enough to show $\rho(U^\# V) < 1$. If $\rho(U^\# V) = 0$, then there is nothing to prove. If possible, let $\rho(U^\# V) = 1$. Then there exists a nonzero vector $x$ such that $U^\# V x = x$. Then $x \in R(U^\#) = R(U)$ and $UU^\# V x = Ux$. Also, $UU^\# V x = Vx$, since $R(V) \subseteq R(U)$. So, $Vx = Ux$. Therefore, $Ax = Ux - Vx = 0$. Thus, $x \in N(A) = N(U)$, so that $x = 0$, a contradiction. So, $\rho(U^\# V) \neq 1$. Since $U^\# V \geq 0$, $\rho(U^\# V)$ is a non-zero eigenvalue of $U^\# V$, by the Perron-Frobenius theorem. Thus, as before, $1 - \rho(U^\# V)$ is a nonzero eigenvalue of $U^\# A$. So, by hypothesis $1 - \rho(U^\# V) > 0$, proving that $\rho(U^\# V) < 1$. By Theorem 1.4, it now follows that $A^\# \geq 0$. $\square$

The following example illustrates Theorem 3.5.

EXAMPLE 3.6. Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Set $U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $U^\# = \frac{1}{3} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$, $V \geq 0$, $R(A) = R(U)$ and $N(A) = N(U)$. Therefore, $A = U - V$ is a pseudo regular splitting. Also, $A^\# = \frac{1}{2} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$ and $U^\# A = \frac{1}{3} \begin{bmatrix} 3 & -4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that the eigenvalues of $U^\# A$ are $0$, $\frac{2}{3}$ and $1$. So, the non-zero real eigenvalues are positive.

As mentioned in the introduction, if $A \in \mathbb{R}^{n \times n}$ is an $M$-matrix, then $A$ is inverse positive if and only if there exists a vector $x \in \mathbb{R}^n$ such that both $x$ and $Ax$ are positive vectors. If we define $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) := e^T x$, $x \in \mathbb{R}^n$, where $e \in \mathbb{R}^n$ has all its coordinates equal to 1, then the latter part of the previous statement could be written as: There exists $x \in \mathbb{R}^n$ such that $h(x)$ and $h(Ax)$ are positive real numbers. In what follows, we generalize this to the nonnegativity of the group inverse, while

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also extending Theorem 10 of [1] and its converse viz., (part of) Proposition 7 of [1].

THEOREM 3.7. *Let $A \in \mathbb{R}^{n \times n}$ be with index 1. Suppose that $R(A) \cap \mathbb{R}^n_+ \neq \{0\}$. Let $A = U - V$ be a proper splitting of $A$ such that $U \geq 0$, $U^{\#} \geq 0$ and $U^{\#}V \geq 0$. Suppose that there exists a linear functional $f$ such that $f(x) \geq 0$ and $f(Ax) > 0$ for every $0 \neq x \in \mathbb{R}^n_+ \cap R(A)$. Then $A^{\#} \geq 0$. Conversely, suppose that $A^{\#} \geq 0$. Then there exists a linear functional $f$ such that $f(x) \geq 0$ for all $0 \neq x \in \mathbb{R}^n_+ \cap R(A)$ and $f(Ax) > 0$.*

*Proof.* Let us observe that the splitting for $A$ given as above is a weak pseudo regular splitting satisfying the additional condition that $U \geq 0$. We have $U^{\#}V \geq 0$. Let $\rho = \rho(U^{\#}V)$ and let $0 \neq y \geq 0$ be an eigenvector corresponding to $\rho$ so that $U^{\#}Vy = \rho y$. Such a vector exists, by the Perron-Frobenius theorem. If $\rho = 0$, then by Theorem 1.4, it follows that $A^{\#} \geq 0$. Suppose that $\rho > 0$. Observe that $y \in R(U^{\#}) = R(A^{\#}) = R(A)$. Thus, $y \in \mathbb{R}^n_+ \cap R(A)$. Let $g$ be a linear functional satisfying the hypothesis. Then $g(y) \geq 0$ and $g(Ay) > 0$. We have $U^{\#}Vy = \rho y$ so that upon premultiplying by $U$ we have $UU^{\#}Vy = \rho Uy$. Since $R(V) \subseteq R(U)$, it then follows that $Vy = \rho Uy$ so that $(\rho U - V)y = 0$. If $\rho \geq 1$ we get $\rho U - V \geq U - V = A$ (it is in step where $U \geq 0$ is used). This implies that $0 = (\rho U - V)y \geq Ay$ and so $g(Ay) \leq 0$, since $g$ is nonnegative on $\mathbb{R}^n_+ \cap R(A)$. This is a contradiction and so $\rho(U^{\#}V) < 1$. Again, it follows from Theorem 1.4 that $A^{\#} \geq 0$.

To prove the converse, let us suppose that $A^{\#} \geq 0$ and $g$ is a strictly positive linear functional on $\mathbb{R}^n_+$. Clearly, $g(x) := e^T x, x \in \mathbb{R}^n$, where $e \in \mathbb{R}^n$ has all its coordinates equal to 1, is one such functional. Then $g(x) > 0$ for all $0 \neq x \in \mathbb{R}^n_+$. This applies in particular, to all the vectors $0 \neq x \in \mathbb{R}^n_+ \cap R(A)$. Let $0 \neq x^* \geq 0$ and $x^* \in R(A)$. Then $A^{\#}x^* \geq 0$ and $x^* = AA^{\#}x^*$. Set $f = g(A^{\#})$. Then

$$f(Ax^*) = g(A^{\#}Ax^*) = g(AA^{\#}x^*) = g(x^*) > 0,$$

showing that $f$ is the required linear functional. $\square$

EXAMPLE 3.8. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Then the index of $A$ is 1 and $e \in R(A) \cap \mathbb{R}^3_+$. Set $U = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then $R(A) = R(U)$ and $N(A) = N(U)$. Further, $U^{\#} = \frac{1}{3} \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \geq 0$ and so $U^{\#}V \geq 0$. Note that

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$A^{\#} = \dfrac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \geq 0$. Define $f(x) = e^T A^{\#} x$, $x \in \mathbb{R}^3$. Let $0 \neq x = (x_1, x_2, x_3) \in$
$R(A) \cap \mathbb{R}^3_+$. Then $f(x) = \frac{1}{2}(4x_1 + x_3) \geq 0$ and $f(Ax) = e^T A^{\#} A x = x_1 + 2x_3 > 0$.

In order to motivate the next result, let us recall the following: Let $A, B \in \mathbb{R}^{m \times n}$ such that $R(A) = R(B)$ and $N(A) = N(B)$. Suppose that $A \leq B$ and $B^{\dagger} \geq 0$. If $int(\mathbb{R}^n_+) \cap \{A\mathbb{R}^n_+ + N(A^T)\} \neq \emptyset$, then $A^{\dagger} \geq B^{\dagger} \geq 0$. The converse also holds. For a proof of this, we refer to [15, Theorem 3.4]. The next result somewhat resembles the situation mentioned above, without the condition involving the interior. Curiously, there is a reversal of the roles of $A$ and $B$ insofar as the nonnegativity of their group inverses are concerned. It is pertinent to point to the fact that this result is a generalization of a corresponding result for invertible matrices proved in [1, Proposition 8]. However, the proof technique is completely different from the proof in [1].

THEOREM 3.9. *Let $A, B \in \mathbb{R}^{n \times n}$ where $A$ has index $1$. Suppose that the following hold:*

(a) *A and B have pseudo regular splittings.*

(b) *$R(A) = R(B)$, $N(A) = N(B)$ and $A \leq B$.*

(c) *$A^{\#} \geq 0$.*

*Then $B^{\#}$ exists and $A^{\#} \geq B^{\#} \geq 0$.*

*Proof.* Since $A$ and $B$ have pseudo regular splittings, there exist matrices $U_A, V_A$, $U_B$ and $V_B$ such that $R(A) = R(U_A), N(A) = N(U_A), R(B) = R(U_B)$ and $N(B) = N(U_B)$. Further,

$$A = U_A - V_A \text{ with } U_A^{\#} \geq 0 \text{ and } V_A \geq 0$$

and

$$B = U_B - V_B \text{ with } U_B^{\#} \geq 0 \text{ and } V_B \geq 0.$$

Also, we have $A \leq B = U_B - V_B \leq U_B$. Thus, $U_B - A \geq U_B - B \geq 0$. Set $Z = U_B$ and $W = U_B - A$. Then $R(Z) = R(U_B) = R(B) = R(A)$ and $N(Z) = N(U_B) = N(B) = N(A)$. Thus, $A = Z - W$ is a proper splitting. Further,

$$Z^{\#} = U_B^{\#} \geq 0 \text{ and } Z^{\#} W = U_B^{\#}(U_B - A) \geq 0.$$

This shows that the above proper splitting is also a weak pseudo regular splitting. Since it is given that $A^{\#} \geq 0$, by Theorem 1.4, we have

$$1 > \rho(Z^{\#}W) = \rho(U_B^{\#}(U_B - A)) \geq \rho(U_B^{\#}(U_B - B)).$$

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Since $U_B - (U_B - B)$ is a pseudo regular splitting of $B$, again by Theorem 1.4, it follows that $B^\# \geq 0$.

We have $A \leq B$. Premultiplying by $A^\# \geq 0$ and post multiplying by $B^\# \geq 0$, we get $A^\# A B^\# \leq A^\# B B^\#$. Since $R(B) = R(A)$ and $N(B) = N(A)$, it follows that

$$A^\# A B^\# = P_{R(A),N(A)} B^\# = P_{R(B),N(B)} B^\# = B^\# B B^\# = B^\#$$

and

$$A^\# B B^\# = A^\# P_{R(B),N(B)} = A^\# P_{R(A),N(A)} = A^\# A A^\# = A^\#.$$

This shows that $A^\# \geq B^\# \geq 0$, completing the proof. $\square$

The following example illustrates Theorem 3.9.

EXAMPLE 3.10.  Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Set $U_A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $V_A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $U_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $V_B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then $U_A^\# = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \geq 0$, $U_B^\# = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \geq 0$, $V_A \geq 0$ and $V_B \geq 0$. It can be verified that $A = U_A - V_A$ and $B = U_B - V_B$ are pseudo regular splittings.

Also, $R(A) = R(B)$, $N(A) = N(B)$, $A \leq B$ and $A^\# = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \geq 0$. Thus, all the conditions of Theorem 3.9 are satisfied. Note that $B^\# = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and that $A^\# \geq B^\# \geq 0$.

In what follows, we consider a more restricted class of matrices that allow what are known as $B_\#$-splittings. This was introduced in [12].

DEFINITION 3.11.  Let $A \in \mathbb{R}^{n \times n}$. A proper splitting $A = U - V$ is called a $B_\#$-splitting if it satisfies the following conditions:

(i) $U \geq 0$.

(ii) $V \geq 0$.

(iii) $U^\#$ exists, $V U^\# \geq 0$.

(iv) $Ax, Ux \in \mathbb{R}_+^n + N(A)$ and $x \in R(A) \Rightarrow x \geq 0$.

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As mentioned in the introduction, the notion of $B_{\#}$-splitting extends the notion of $B$-splitting studied by Peris [14]. The next two results were stated in [12]. Their proofs are similar to the proofs in the case of Moore-Penrose inverses and are skipped.

THEOREM 3.12. *Let* $A \in \mathbb{R}^{n \times n}$. *Consider the following statements:*

(a) $A^{\#}$ *exists and* $A^{\#} \geq 0$.

(b) $Ax \in \mathbb{R}_{+}^{n} + N(A)$ *and* $x \in R(A) \Rightarrow x \geq 0$.

(c) $\mathbb{R}_{+}^{n} \subseteq A\mathbb{R}_{+}^{n} + N(A)$.

(d) *There exists* $x \in \mathbb{R}_{+}^{n}$ *and* $z \in N(A)$ *such that* $Ax + z > 0$.

*Then we have* $(a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

*Suppose that* $A$ *has a* $B_{\#}$-*splitting* $A = U - V$. *Then each of the above statements is equivalent to the following condition:*

(e) $\rho(VU^{\#}) < 1$.

THEOREM 3.13. *Let* $A \in \mathbb{R}^{n \times n}$. *Suppose that* $A^{\#}$ *exists,* $A^{\#} \geq 0$ *and* $R(A) \cap int(\mathbb{R}_{+}^{n}) \neq \emptyset$. *Further, let* $A^{\#}A \geq 0$. *Then* $A$ *possesses a* $B_{\#}$-*splitting* $A = U - V$ *such that* $\rho(VU^{\#}) < 1$.

Next, we derive an extension of Theorem 1.2, mentioned in the introduction.

THEOREM 3.14. *For* $A \in \mathbb{R}^{n \times n}$, *let* $A = U - V$ *be a* $B_{\#}$-*splitting such that no row of* $U$ *is zero. Suppose that there exists* $x > 0$ *such that* $U^{\#}Vx = \lambda x$ *for some* $\lambda < 1$. *Then* $A^{\#} \geq 0$.

*Proof.* We show that there exists $z \in \mathbb{R}_{+}^{n}$ and $w \in N(A)$ such that $Az + w > 0$. It would then follow from $(d)$ of Theorem 3.12, that $A^{\#} \geq 0$.

Let $U^{\#}Vx = \lambda x$ for some $\lambda < 1$. Premultiplying with $U$ and by using the fact that $R(V) \subseteq R(U)$, we have $Vx = \lambda Ux$. If $\lambda = 0$, then $Vx = 0$ so that $Ax = Ux$. Since $U \geq 0$ and has no zero row, and $x > 0$, we have $Ux > 0$. Thus, $Ax + w > 0$ by taking $w = 0 \in N(A)$. So, if $\lambda = 0$, then $(d)$ of Theorem 3.12 holds. Consider the case $0 < \lambda \ (< 1)$. Then $x \in R(U^{\#})$ so that $U^{\#}Ux = x$. We have $Ax = U(I - U^{\#}V)x = (1 - \lambda)Ux$. Thus, $U^{\#}Ax = (1 - \lambda)U^{\#}Ux = (1 - \lambda)x > 0$. Set $y = U^{\#}Ax > 0$. Then by Lemma 2.1, we have $Uy = Ax + z$, for some $z \in N(U^{\#}) = N(A^{\#}) = N(A)$. Again, since $U$ has no zero row, we have $Uy > 0$. Thus, there exists $z \in N(A)$ such that $Ax + z > 0$, as required. $\square$

The following example demonstrates that the converse of the last theorem is not

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true.

EXAMPLE 3.15. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then $A^\# = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \geq 0$. Set

$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then $R(A) = R(U)$ and $N(A) = N(U)$.

Also, $U \geq 0, V \geq 0, U^\# = \dfrac{1}{4} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $VU^\# \geq 0$. Thus, $A = U - V$ is a

$B_\#$-splitting. Further, $U^\# V = \dfrac{1}{4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and the eigenvalues of $U^\# V$ are 0 and

$\frac{1}{2}$. The corresponding eigenvectors are $\begin{bmatrix} k_1 \\ k_2 \\ 0 \end{bmatrix}$ where $k_1, k_2 \in \mathbb{R}$ and $\begin{bmatrix} 0 \\ k_3 \\ k_3 \end{bmatrix}, k_3 \in \mathbb{R}$,

respectively. Thus, there is no vector $x > 0$ such that $U^\# V x = \lambda x$ for any $\lambda$.

However, we show that the converse can be recovered in the presence of an additional condition.

THEOREM 3.16. *For $A \in \mathbb{R}^{n \times n}$, let $A = U - V$ be a $B_\#$-splitting such that no row of $U$ is zero. Suppose that either $U^\# V \geq 0$ and is irreducible or $U^\# V > 0$. If $A^\# \geq 0$, then there exists a vector $x > 0$ such that $U^\# V x = \lambda x$ for some $\lambda < 1$.*

*Proof.* Suppose that $A^\# \geq 0$. Since $A = U - V$ is a $B_\#$-splitting, by Theorem 3.12 $\rho(U^\# V) = \rho(VU^\#) < 1$. Also, we have either $U^\# V \geq 0$ and is irreducible or $U^\# V > 0$. So, by the Perron-Frobenius theory, there exists a unique vector $x > 0$ such that $U^\# V x = \rho(U^\# V)x$, proving the result. $\square$

**4. Comparison results.** In this last part, we will be concerned with comparison results for the two types of splittings discussed here. In the process we obtain generalizations of the results of [8]. The proof of the first result is very similar to the corresponding result there, to fit into the group inverse frame work. However, we prefer to include the proof for a self-contained discussion. This is mainly used in deriving comparison results, viz., Theorem 4.3 and Theorem 4.5. We would like to point out that the results in this section are motivated by purely theoretical considerations. In particular, no claim of superiority is made on any splitting over another. Applications of these comparison results to numerical solutions of linear systems are

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not considered either.

THEOREM 4.1. (Extension of Theorem 3.4, [8]) *Let $A \in \mathbb{R}^{n \times n}$ with index 1. Suppose that no row or column of $A$ is zero. Suppose also that $A^{\#} \geq 0, R(A) \cap int(\mathbb{R}^n_+) \neq \emptyset$ and $A^{\#}A \geq 0$. Then there exists a $B_{\#}$-splitting $A = U - V$, where the matrices $U, V$ and $A$ are further related by the following statements: There exists $x \in \mathbb{R}^n_+ \cap R(U)$ such that $U^{\#}Vx = \rho(U^{\#}V)x$, and $0 \neq Ax \geq 0$. Moreover, if $VU^{\#}$ is not nilpotent then $0 \neq Vw \geq 0$ for some $w \in \mathbb{R}^n_+ \cap R(U)$.*

*Proof.* By Theorem 3.13, there exists a $B_{\#}$-splitting $A = U - V$. Set $\rho^* = \rho(U^{\#}V)$. Let $x$ be a corresponding eigenvector of $U^{\#}V$ so that $U^{\#}Vx = \rho^*x$. We show that the coordinates of $x$ are all nonnegative or all nonpositive. We have $x \in R(U^{\#}) = R(U)$ so that $UU^{\#}x = U^{\#}Ux = x$. Premultiplying the equation $U^{\#}Vx = \rho^*x$ by $U$ and using the fact that $R(V) \subseteq R(U)$, we get $Vx = \rho^*Ux$. Also, $Vx = VU^{\#}Ux$. Set $z = Ux$. Then $VU^{\#}z = \rho^*z$. If $z = 0$, then $x = 0$, a contradiction. So, $z$ is an eigenvector for the matrix $VU^{\#}$. From Theorem 3.13 we have $\rho^* = \rho(U^{\#}V) = \rho(VU^{\#}) < 1$. Note that by the definition of a $B_{\#}$-splitting, we have $VU^{\#} \geq 0$.

First, let us assume that $VU^{\#} > 0$, not just nonnegative. By the result of Perron, there exists a unique positive (Perron) eigenvector $y$ corresponding to the (simple) eigenvalue $\rho^*$ for the positive matrix $VU^{\#}$, i.e., $VU^{\#}y = \rho^*y$. Thus, we have $Ux = \alpha y$ for some $0 \neq \alpha \in \mathbb{R}$. Upon premultiplying by $U^{\#}$, we have $x = \alpha U^{\#}y$. Since $N(A) = N(U) = N(U^{\#})$, it follows that no row or column of $U^{\#}$ is zero. Hence, $U^{\#}y > 0$. So, either the components of $x$ are all negative or all positive. Replacing $x$ by $-x$, if need be, we have $x > 0$ as well as $Ux > 0$. Thus, $x \in int(\mathbb{R}^n_+) \cap R(U)$. Now, we have $Ax = Ux - Vx = (1 - \rho^*)Ux$ and since $\rho^* < 1$ this proves that $Ax > 0$.

To summarize, we have shown that under the hypotheses of the theorem, if the splitting $A = U - V$ satisfies the assumption that $VU^{\#} > 0$ then there exists $x \in \mathbb{R}^n_+ \cap R(U)$ such that $U^{\#}Vx = \rho^*x$, where $\rho^* = \rho(U^{\#}V)$.

To complete the proof, we consider the case $VU^{\#} \geq 0$. Let $E \in \mathbb{R}^{n \times n}$ be the matrix with all entries 1. Observe that, as mentioned earlier, since $U^{\#}$ has no zero row or column, it follows that $EU^{\#} > 0$. Let $\| . \|$ denote any matrix norm. Let $\alpha$ be chosen such that $0 < \alpha < \frac{1}{\|EU^{\#}\|}$. Then the series $\sum_{k=0}^{\infty}(\alpha EU^{\#})^k$ is convergent. We have

$$0 < \sum_{k=0}^{\infty}(\alpha EU^{\#})^k = (I - \alpha EU^{\#})^{-1}.$$

Define $W = (I - \alpha EU^{\#})^{-1}U^{\#}$. Then $W > 0$ and $WUU^{\#} = W$. Note that

$$WUA^{\#} = (I - \alpha EU^{\#})^{-1}U^{\#}UA^{\#} = (I - \alpha EU^{\#})^{-1}A^{\#}AA^{\#} = (I - \alpha EU^{\#})^{-1}A^{\#}.$$

Again, it follows that $WUA^{\#} > 0$. Set $\epsilon_0 = \frac{1}{\|WUA^{\#}\|}$. Let $\epsilon$ be chosen such that

$$0 < \epsilon < \epsilon_0.$$

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Define $A_\epsilon = A - \epsilon WU$. Then

$$A_\epsilon = A - \epsilon WUU^\# U = A - \epsilon WUA^\# A = (I - \epsilon WUA^\#)A.$$

Also, $\| \epsilon WUA^\# \| = \epsilon \| WUA^\# \| < 1$, so that $I - \epsilon WUA^\#$ is invertible. It then follows that $R(A_\epsilon) = R(A) = R(U)$ and $N(A_\epsilon) = N(A) = N(U)$. Thus, $A_\epsilon = U - (V + \epsilon WU)$ is a proper splitting. Observe that since $A^\#$ exists, the subspaces $R(A)$ and $N(A)$ are complementary. So are $R(A_\epsilon)$ and $N(A_\epsilon)$ and so $A_\epsilon^\#$ exists. Next, we show that $A_\epsilon^\# \geq 0$. First, we show that $A_\epsilon^\#$ exists. Let $X = A^\#(I - \epsilon WUA^\#)^{-1}$ and $x \in R(A_\epsilon)$. Then

$$XA_\epsilon x = A^\#(I - \epsilon WUA^\#)^{-1}(I - \epsilon WUA^\#)Ax = A^\# Ax = x.$$

For $y \in N(A_\epsilon) = N(A) = N(A^\#)$, we also have

$$Xy = A^\#(I - \epsilon WUA^\#)^{-1}y = A^\# \sum_{k=0}^\infty (\epsilon WUA^\#)^k y = A^\# y + \sum_{k=1}^\infty (\epsilon WUA^\#)^k y = 0.$$

Hence, $A_\epsilon^\# = A^\#(I - \epsilon WUA^\#)^{-1} = A^\# \sum_{k=0}^\infty (\epsilon WUA^\#)^k \geq 0$.

Also, $R(A_\epsilon) \cap int(\mathbb{R}_+^n) = R(A) \cap int(\mathbb{R}_+^n) \neq \emptyset$ and $A_\epsilon^\# A_\epsilon = A^\# A \geq 0$. Further, $(V + \epsilon WU)U^\# = VU^\# + \epsilon WUU^\# = VU^\# + \epsilon W > 0$.

By what we have already shown, there exists $x_\epsilon \in \mathbb{R}_+^n \cap R(U)$ such that

$$U^\#(V + \epsilon WU)x_\epsilon = \rho(U^\#(V + \epsilon WU))x_\epsilon$$

and $A_\epsilon x_\epsilon > 0$. We may choose $x_\epsilon$ such that its 1-norm satisfy $\| x_\epsilon \|_1 = 1$. Set $\epsilon_k = \frac{1}{k}\epsilon_0$. Then the sequence $x_{\epsilon_k}$, being bounded, has a convergent subsequence with limit $0 \neq x \geq 0$. Observe that since

$$U^\#(V + \epsilon_k WU)x_{\epsilon_k} = \rho(U^\#(V + \epsilon_k WU))x_{\epsilon_k},$$

we have in the limit, the equation $U^\# Vx = \rho(U^\# V)x$. We have $Ax \geq 0$, as well. If $Ax = 0$, then $Vx = 0$ and so $x = 0$, a contradiction. Hence, $Ax \neq 0$.

Let us prove the last part. We have $Ux = \frac{1}{\rho*}Vx$, where we have used $\rho* \neq 0$, since $VU^\#$ is not nilpotent. Thus,

$$0 \leq Ax = (U - V)x = U(I - U^\# V)x = (1 - \rho*)Ux = \frac{1 - \rho*}{\rho*}Vx.$$

Note that since $\rho* < 1$, we have $Vx \geq 0$. If $w = \frac{1 - \rho*}{\rho*}x$ then $Vw \geq 0$. If $Vw = 0$ then $Ax = 0$, a contradiction. Thus, $Vw \neq 0$, completing the proof. $\square$

In the next result, we show that the conclusions of Theorem 4.1 follow easily, if we consider a splitting that is stronger than a $B_\#$-splitting.

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THEOREM 4.2. *Let $A \in \mathbb{R}^{n \times n}$ with index* 1. *Suppose that no row or column of $A$ is zero. Let $A = U - V$ be a proper splitting such that $U \geq 0$, $V \geq 0$ and $U^{\#} \geq 0$. Suppose also that $A^{\#} \geq 0$. Then there exists $x \in \mathbb{R}_+^n \cap R(U)$ such that $U^{\#}Vx = \rho(U^{\#}V)x$ and $0 \neq Ax \geq 0$. Further, if $VU^{\#}$ is not nilpotent then $0 \neq Vw \geq 0$ for some $w \in \mathbb{R}_+^n \cap R(U)$.*

*Proof.* Note that the given splitting is a $B_{\#}$-splitting. Set $\rho^* = \rho(U^{\#}V)$. Since $U^{\#}V \geq 0$, by the Perron-Frobenius theorem there exists a vector $0 \neq x \geq 0$ such that $U^{\#}Vx = \rho^* x$. Premultiplying the equation $U^{\#}Vx = \rho^* x$ by $U$ and using the fact that $R(V) \subseteq R(U)$, we get $Vx = \rho^* Ux$. Now, we have $Ax = Ux - Vx = Ux - \rho^* Ux = (1 - \rho^*)Ux$ and since $\rho^* < 1$ this proves that $Ax \geq 0$. As above, $Ax \neq 0$. The second part may be proved as done earlier. $\square$

Next, we present some applications of Theorem 4.1. These are comparison results for the spectral radii of iteration matrices corresponding to two matrices $A$ and $B$ with $A \leq B$. These also extend Theorem 3.5 and Theorem 4.2 in [8].

THEOREM 4.3. *Let $B \in \mathbb{R}^{n \times n}$ such that $B$ has index 1, $B^{\#} \geq 0$ and no row or column of $B$ is zero. Suppose that $R(B) \cap int(\mathbb{R}_+^n) \neq \emptyset$ and $B^{\#}B \geq 0$. Then there exists a $B_{\#}$-splitting $B = U_B - V$. Let $A \in \mathbb{R}^{n \times n}$ such that $A$ has index 1, $A^{\#} \geq 0$ and let $A = U_A - V$ be a pseudo regular splitting. Further, suppose that $VU_B^{\#}$ is not nilpotent, $U_B^{\#} \geq 0$ and $A \leq B$ with $R(A) = R(B)$ and $N(A) = N(B)$. Then $\rho(U_B^{\#}V) \leq \rho(U_A^{\#}V) < 1$.*

*Proof.* By Theorem 3.13, there exists a $B_{\#}$-splitting $B = U_B - V$. Since $U_B^{\#} \geq 0$, the splitting $B = U_B - V$ is also a pseudo regular splitting. So, $A, B \in \mathbb{R}^{n \times n}$ satisfy all the conditions of Theorem 3.9. Therefore $A^{\#} \geq B^{\#} \geq 0$. Since $A^{\#} \geq 0$ and $A = U_A - V$ is a pseudo regular splitting, by Theorem 1.4 it follows that $\rho(U_A^{\#}V) < 1$. Similarly, $\rho(U_B^{\#}V) < 1$. Next, we show that $\rho(U_B^{\#}V) \leq \rho(U_A^{\#}V)$. Let us denote $G_A = A^{\#}V$ and $G_B = B^{\#}V$. Then, again by Theorem 1.4, $\rho(U_A^{\#}V) = \dfrac{\rho(G_A)}{1 + \rho(G_A)}$ and $\rho(U_B^{\#}V) = \dfrac{\rho(G_B)}{1 + \rho(G_B)}$. Since the function $f(t) = \frac{t}{1+t}$ is strictly increasing for $t \geq 0$, it is enough to show that $\rho(G_B) \leq \rho(G_A)$. For this, we consider $B = U_B - V$ that satisfies all the conditions of Theorem 4.1. So, there exists a vector $0 \neq x \geq 0$ such that $U_B^{\#}Vx = \rho(U_B^{\#}V)x$ and $0 \neq Vx \geq 0$. Then for the same $x$,

$$G_A x = A^{\#}Vx \geq B^{\#}Vx = G_B x = \rho(G_B)x.$$

This implies that $\rho(G_B) \leq \rho(G_A)$, by Lemma 2.4. $\square$

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The following example illustrates Theorem 4.3.

EXAMPLE 4.4. Let $B = \begin{bmatrix} -1 & 3 & -1 \\ 3 & -2 & 3 \\ -1 & 3 & -1 \end{bmatrix}$, then $B^{\#} = \dfrac{1}{14} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \geq 0$.

Also $R(B) \cap int(\mathbb{R}^3_+) \neq \emptyset$ and $B^{\#}B = \dfrac{1}{14} \begin{bmatrix} 7 & 0 & 7 \\ 0 & 14 & 0 \\ 7 & 0 & 7 \end{bmatrix} \geq 0$. Set $U_B = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$

and $V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then $U_B^{\#} = \dfrac{1}{6} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \geq 0$. So, $B = U_B - V$ is a $B_{\#}$-

splitting. Let $A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$ then $A^{\#} = \dfrac{1}{4} \begin{bmatrix} 3 & 4 & 3 \\ 4 & 4 & 4 \\ 3 & 4 & 3 \end{bmatrix} \geq 0$. Set $U_A = $

$\begin{bmatrix} 0 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, then $U_A^{\#} = \dfrac{1}{64} \begin{bmatrix} 4 & 16 & 4 \\ 16 & 0 & 16 \\ 4 & 16 & 4 \end{bmatrix} \geq 0$. So $A = U_A - V$ is a pseudo

regular spliting. $VU_B^{\#} = \dfrac{1}{6} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ is not nilpotent. $U_B^{\#}V = \dfrac{1}{6} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

and $U_A^{\#}V = \dfrac{1}{64} \begin{bmatrix} 8 & 32 & 8 \\ 32 & 0 & 32 \\ 8 & 32 & 8 \end{bmatrix}$. Observe that $0.4714 = \rho(U_B^{\#}V) \leq \rho(U_A^{\#}V) = $

$0.8431 < 1$.

THEOREM 4.5. *Let $A, B \in \mathbb{R}^{n \times n}$ with index 1 such that $A^{\#} - B^{\#} > 0$, $A^{\#} \geq 0$, $B^{\#} \geq 0$ and no row or column of $A$ and $B$ is zero. Suppose that $R(B) \cap int(\mathbb{R}^n_+) \neq \emptyset$ and $B^{\#}B \geq 0$. Then there exists a $B_{\#}$-splitting $B = U_B - V_B$. Let $A = U_A - V_A$ be a pseudo regular splitting. Suppose also that $U_B^{\#} \geq 0$ and $V_A U_A^{\#}$, $V_B U_B^{\#}$ are not nilpotent.*

(*i*) *If $U_B - U_A \leq B - A$, then $\rho(U_B^{\#}V_B) \leq \rho(U_A^{\#}V_A) < 1$.*

(*ii*) *If $U_A^{\#} - U_B^{\#} \geq A^{\#} - B^{\#}$, then $\rho(U_A^{\#}V_A) < \rho(U_B^{\#}V_B) < 1$.*

*Proof.* (*i*) By Theorem 3.13, there exists a $B_{\#}$-splitting $B = U_B - V_B$. Clearly $\rho_1 = \rho(U_A^{\#}V_A) < 1$ and $\rho_2 = \rho(U_B^{\#}V_B) < 1$. It remains to show $\rho(U_B^{\#}V_B) \leq \rho(U_A^{\#}V_A)$. Let us denote $G_A = A^{\#}V_A$ and $G_B = B^{\#}V_B$. Then, as in the proof of Theorem 4.3, it is enough to show $\rho(G_B) \leq \rho(G_A)$. If $U_B - U_A \leq B - A$ then $V_A \geq V_B$. Applying Theorem 4.1 to $B = U_B - V_B$, we get a non-zero vector $x \geq 0$ such that $U_B^{\#}V_B x = \rho_2 x$. For the same $x$, we have

$$ G_A x = A^{\#}V_A x > B^{\#}V_A x \geq B^{\#}V_B x = G_B x = \rho(G_B)x. $$

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This implies that $\rho(G_B) \leq \rho(G_A)$, by Lemma 2.4.

$(ii)$ Consider

$$U_B^\# V_B B^\# = U_B^\#(U_B - B)B^\# = B^\# - U_B^\#,$$

here we have used the fact that $U_B^\# U_B = B^\# B$. Also, since $U_A^\# U_A = A^\# A$, one has

$$A^\# V_A U_A^\# = A^\#(U_A - A)U_A^\# = A^\# - U_A^\#.$$

Therefore,

$$U_B^\# V_B B^\# = B^\# - U_B^\# \geq A^\# - U_A^\# = A^\# V_A U_A^\# \geq 0.$$

Since $V_A U_A^\# \geq 0$ and $U_B^\# V_B \geq 0$, by the Perron-Frobenius theorem there exist nonzero vectors $x \geq 0$ and $y \geq 0$ such that

$$V_A U_A^\# x = \rho_1 x \quad \text{and} \quad y^T U_B^\# V_B = \rho_2 y^T.$$

Thus

$$\rho_2 y^T B^\# x = y^T U_B^\# V_B B^\# x \geq y^T A^\# V_A U_A^\# x = \rho_1 y^T A^\# x.$$

Since $A^\# > B^\#$ and since $x$ and $y$ are both nonzero and $\rho_1 > 0$, we obtain

$$\rho_2 y^T B^\# x > \rho_1 y^T B^\# x.$$

Therefore, $\rho(U_A^\# V_A) < \rho(U_B^\# V_B) < 1$. ☐

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