Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

ZERO-DILATION INDEX OF S_n-MATRIX AND COMPANION MATRIX*

HWA-LONG GAU† AND PEI YUAN WU‡

Abstract. The zero-dilation index $d(A)$ of a square matrix A is the largest k for which A is unitarily similar to a matrix of the form $\begin{bmatrix} 0_k & * \\ * & * \end{bmatrix}$, where 0_k denotes the k-by-k zero matrix. In this paper, it is shown that if A is an S_n-matrix or an n-by-n companion matrix, then $d(A)$ is at most $\lceil n/2 \rceil$, the smallest integer greater than or equal to $n/2$. Those A's for which the upper bound is attained are also characterized. Among other things, it is shown that, for an odd n, the S_n-matrix A is such that $d(A) = (n+1)/2$ if and only if A is unitarily similar to $-A$, and, for an even n, every n-by-n companion matrix A has $d(A)$ equal to $n/2$.

Key words. Zero-dilation index, S_n-Matrix, Companion matrix, Numerical range.

AMS subject classifications. 47A20, 15B99, 15A60, 47A12.

1. Introduction. The *zero-dilation index* $d(A)$ of an n-by-n complex matrix A is defined as the maximum size k of a zero matrix which can be dilated to A or, equivalently, $d(A)$ is the maximum k for which A is unitarily similar to a matrix of the form $\begin{bmatrix} 0_k & * \\ * & * \end{bmatrix}$, where 0_k denotes the k-by-k zero matrix. The study of $d(A)$ was started in [4], based on the previous work [12] of C.-K. Li and N.-S. Sze on higher-rank numerical ranges. In [4], the matrices A with $d(A) = n - 1$ were completely characterized, and the value of the index for a normal matrix or a weighted permutation matrix with zero diagonals was also determined. The same was done for KMS matrices (cf. [8, Theorem 2.1]). The purpose of this paper is to find the upper bound of $d(A)$ and to characterize those A's which attain this bound among two classes of matrices, namely, the S_n-matrices and companion matrices.

Recall that an n-by-n matrix A is said to be of *class* S_n (or simply an S_n-*matrix*) if it is a contraction ($\|A\| \equiv \max_{x\neq 0 \text{ in } \mathbb{C}^n} \|Ax\|/\|x\| \leq 1$) with all eigenvalues in the open unit disc $\mathbb{D}$ of the complex plane and with $\mathrm{rank}(I_n - A^*A) = 1$, where I_n denotes the n-by-n identity matrix. On the other hand, for any monic polynomial

*Received by the editors on December 12, 2015. Accepted for publication on September 18, 2016. Handling Editor: Panayiotis Psarrakos.

†Department of Mathematics, National Central University, Chungli 32001, Taiwan (hlgau@math.ncu.edu.tw). Supported by the Ministry of Science and Technology of the Republic of China under MOST 105-2115-M-008-005.

‡Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan (pywu@math.nctu.edu.tw).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, its associated *companion matrix* A is

$$\begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix}. \tag{1.1}$$

Note that p is the characteristic polynomial of A. Moreover, it is known that both S_n-matrices and companion matrices are nonderogatory and form, under similarity, the building block of the Jordan form of (finite-dimensional) C_0 contractions and the rational form of general matrices, respectively. A special example of both is the n-by-n Jordan block

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

For more of their properties, the reader may consult [1, Section 3.1] and [11, Section 3.3].

In Section 2 below, we prove that if A is an S_n-matrix, then $d(A)$ is at most $\lceil n/2 \rceil$ (cf. Proposition 2.1), and, moreover, if n is odd, then $d(A) = (n+1)/2$ if and only if A and $-A$ are unitarily similar or, equivalently, the eigenvalues of A are of the form $0, \pm b_1, \ldots, \pm b_{(n-1)/2}$ (cf. Theorem 2.2). An analogous result holds for even n (cf. Theorem 2.3). However, a clear-cut condition on the eigenvalues of A in order that $d(A) = n/2$ is lacking. In fact, the known case of $n = 2$ (an S_2-matrix A is such that $d(A) = 1$ if and only if its eigenvalues λ_1 and λ_2 satisfy $|\lambda_1 + \lambda_2| + |\lambda_1 \lambda_2| \leq 1$; cf. Proposition 2.4) seems to indicate that the conditions should involve one or more inequalities of the eigenvalues.

The study of the zero-dilation index for companion matrices is taken up in Section 3. Here the straightforward case is for the even n. We show that if A is an n-by-n companion matrix, then $d(A) \leq \lceil n/2 \rceil$, and if, moreover, n is even, then $d(A) = n/2$ (cf. Theorem 3.2). For an odd n, characterizations of those A's with $d(A) = (n+1)/2$ are similar to the ones for S_n-matrices; this is the case if and only if A and $-A$ are unitarily similar (cf. Theorem 3.3). What is lacking is a condition in terms of the numerical range of A. Recall that the *numerical range* $W(A)$ of an n-by-n matrix A is the subset $\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the plane, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard inner product and norm of vectors in $\mathbb{C}^n$. [10, Chapter 1] is our main

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

reference for properties of the numerical range. An S_n-matrix is determined, up to unitary similarity, by its numerical range (cf. [5, Theorem 3.2]). This is not the case for companion matrices; there are two 3-by-3 (invertible) companion matrices A_1 and A_2 with $W(A_1) = W(A_2)$ which are not unitarily similar (cf. [7, Example 2.1]). Back to our problem, it is unknown whether, for a noninvertible companion matrix A with odd size, the equality $W(A) = -W(A)$ would guarantee the unitary similarity of A and $-A$.

There is another expression for the zero-dilation index, which is in terms of the higher-rank numerical ranges. Recall that the *rank-k numerical range* $\Lambda_k(A)$ ($1 \leq k \leq n$) of an n-by-n matrix A is the subset $\{\lambda \in \mathbb{C} : \lambda I_k \text{ dilates to } A\}$ of the plane. In particular, $\Lambda_1(A)$ is simply the classical numerical range $W(A)$. Obviously, $d(A)$ equals the maximum k for which $\Lambda_k(A)$ contains 0. A more useful description of $\Lambda_k(A)$ was given by Li and Sze in [12, Theorem 2.2], namely,

$$\Lambda_k(A) = \bigcap_{\theta \in \mathbb{R}} \{\lambda \in \mathbb{C} : \operatorname{Re}(e^{i\theta}\lambda) \leq \lambda_k\big(\operatorname{Re}(e^{i\theta}A)\big)\},$$

where $\operatorname{Re} z = (z+\overline{z})/2$ (resp., $\operatorname{Re} B = (B+B^*)/2$) denotes the *real part* of a complex number z (resp., a matrix B), and, for an n-by-n Hermitian matrix C, $\lambda_1(C) \geq \cdots \geq \lambda_n(C)$ denote its ordered eigenvalues. In terms of this description, $d(A)$ can be expressed as

$$\min\{k_\theta : \lambda_{k_\theta}\big(\operatorname{Re}(e^{i\theta}A)\big) \geq 0 > \lambda_{k_\theta+1}\big(\operatorname{Re}(e^{i\theta}A)\big), \theta \in \mathbb{R}\}$$

(cf. [12, Theorem 3.1]). For a Hermitian C, let $i_{\geq 0}(C)$ denote the number of nonnegative eigenvalues of C (counting multiplicity). From above, it follows that

$$d(A) = \min\{i_{\geq 0}\big(\operatorname{Re}(e^{i\theta}A)\big) : \theta \in \mathbb{R}\}. \tag{1.2}$$

This is the expression we use most often in the subsequent discussions. Indeed, the proofs of the upper bounds for $d(A)$ and the attainment of these bounds make use of [4, Corollaries 2.5 and 2.6], which were derived before from (1.2).

For any nonzero complex number z, $\arg z$ is the unique number in $[0, 2\pi)$ satisfying $z = |z|e^{i(\arg z)}$. The diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ is denoted by $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

The study undertaken in this paper reveals more common properties of the S_n-matrices and companion matrices. Hopefully, results of this nature may lead to the further unlocking of the full potential of higher-rank numerical ranges of these two classes of matrices.

2. S_n-matrix.

We start with the following upper bound of $d(A)$ for A an S_n-matrix.

PROPOSITION 2.1. *If A is an S_n-matrix, then $d(A) \leq \lceil n/2 \rceil$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

Proof. Since $e^{i\theta}A$ is also an S_n-matrix for any real θ, its real part $\mathrm{Re}\,(e^{i\theta}A)$ has only simple eigenvalues (cf. [5, Corollary 2.7]). In particular, this implies that $\dim\ker\big(\mathrm{Re}\,(e^{i\theta}A)\big) \le 1$ for all θ. Thus, $d(A) \le \lceil n/2 \rceil$ by [4, Corollary 2.5]. □

For an odd n, the next theorem gives equivalent conditions for the extremum case $d(A) = \lceil n/2 \rceil$.

THEOREM 2.2. *For an S_n-matrix (n odd), the following conditions are equivalent:*

(a) $d(A) = (n+1)/2$,
(b) $\lambda_{(n+1)/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big) = 0$ *for all real* θ,
(c) *A is unitarily similar to a matrix of the form*

$$\left[\begin{array}{ccc|c} & 0_{(n+1)/2} & & A' \\ \hline 1 & & 0 & \\ & \ddots & \vdots & 0_{(n-1)/2} \\ & 1 & 0 & \end{array}\right],$$

where A' is some $(n+1)/2$-by-$(n-1)/2$ matrix,
(d) *A and $-A$ are unitarily similar,*
(e) *the eigenvalues of A are of the form $0, \pm b_1, \ldots, \pm b_{(n-1)/2}$ with $b_1, \ldots, b_{(n-1)/2}$ in $\mathbb{C}$,*
(f) $W(A) = -W(A)$.

Proof. (a) ⇒ (b). This holds for any n-by-n matrix A. If $\lambda_{(n+1)/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big) < 0$ for some real θ, then $i_{\ge 0}\big(\mathrm{Re}\,(e^{i\theta}A)\big) < (n+1)/2$, which implies, by (1.2), that $d(A) < (n+1)/2$, contradicting (a). Similarly, if $\lambda_{(n+1)/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big) > 0$, then $i_{\ge 0}\big(\mathrm{Re}\,(e^{i(\theta+\pi)}A)\big) < (n+1)/2$ and hence $d(A) < (n+1)/2$, again a contradiction. Thus, $\lambda_{(n+1)/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big) = 0$ for all real θ, that is, (b) holds.

(b) ⇒ (a). Under (b), we have $i_{\ge 0}\big(\mathrm{Re}\,(e^{i\theta}A)\big) \ge (n+1)/2$ for all real θ, and thus, $d(A) \ge (n+1)/2$. Then (a) follows from Proposition 2.1.

(a) ⇒ (c). We may assume that $A = \begin{bmatrix} 0_{(n+1)/2} & B \\ C & D \end{bmatrix}$, where B, C and D are $(n+1)/2$-by-$(n-1)/2$, $(n-1)/2$-by-$(n+1)/2$ and $(n-1)/2$-by-$(n-1)/2$ matrices, respectively. Since $I_n - A^*A = \begin{bmatrix} I_{(n+1)/2} - C^*C & * \\ * & * \end{bmatrix}$ has rank one, we have $\mathrm{rank}(I_{(n+1)/2} - C^*C) \le 1$. Note that $\mathrm{rank}\, C^*C = \mathrm{rank}\, C \le (n-1)/2$. Thus, C^*C is unitarily similar to $\mathrm{diag}(c_1, \ldots, c_{(n-1)/2}, 0)$ for some c_j's satisfying $0 \le c_j \le 1$ for all j. Hence, $I_{(n+1)/2} - C^*C$ is unitarily similar to $\mathrm{diag}(1-c_1, \ldots, 1-c_{(n-1)/2}, 1)$. From $\mathrm{rank}(I_{(n+1)/2} - C^*C) \le 1$, we derive that $c_j = 1$ for all j, $1 \le j \le (n-1)/2$. It follows that C^*C is unitarily similar to $\mathrm{diag}(1, \ldots, 1, 0)$. Note that the singular value decomposition of C yields the existence of unitary matrices U and V of sizes $(n-1)/2$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

and $(n+1)/2$, respectively, such that

$$C = U \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} V$$

(cf. [11, Theorem 2.6.3]). If W denotes the n-by-n unitary matrix $V^* \oplus U$, then

$$W^*AW = \begin{bmatrix} 0_{(n+1)/2} & * \\ U^*CV^* & * \end{bmatrix} = \left[\begin{array}{cccc|c} & 0_{(n+1)/2} & & & * \\ \hline 1 & & & 0 & \\ & \ddots & & \vdots & * \\ & & 1 & 0 & \end{array}\right].$$

Since $\|A\| = 1$, the matrix on the right-hand side of the above expression is of the asserted form in (c).

(c) $\Rightarrow$ (d). Let A'' denote the matrix in (c) and let $U = I_{(n+1)/2} \oplus (-I_{(n-1)/2})$. Then $U^*A''U = -A''$. It follows that A is unitarily similar to $-A$.

(d) $\Rightarrow$ (a). The unitary similarity of A and $-A$ implies, by [4, Corollary 2.6], that $d(A) \geq (n+1)/2$, which together with $\dim\ker\big(\operatorname{Re}(e^{i\theta}A)\big) \leq 1$ for all real θ [5, Corollary 2.7] yields $d(A) = (n+1)/2$.

(d) $\Rightarrow$ (e). Let the eigenvalues of A be $\lambda_1, \ldots, \lambda_n$. Then (d) implies the coincidence of $\lambda_1, \ldots, \lambda_n$ and $-\lambda_1, \ldots, -\lambda_n$. In particular, we have

$$\det A = \prod_j \lambda_j = \prod_j (-\lambda_j) = -\prod_j \lambda_j = -\det A.$$

Hence, $\det A = 0$ and, therefore, $\lambda_j = 0$ for some j. We may assume that $\lambda_1 = 0$. The coincidence of $\lambda_2, \ldots, \lambda_n$ and $-\lambda_2, \ldots, -\lambda_n$ implies that either $-\lambda_2 = \lambda_2$ or $-\lambda_2 = \lambda_j$ for some j, $3 \leq j \leq n$. The former yields $\lambda_2 = 0$ and the latter $\{\lambda_2, \lambda_j\} = \{\pm\lambda_2\}$. Hence, either $\lambda_3, \ldots, \lambda_n$ coincide with $-\lambda_3, \ldots, -\lambda_n$ or $\lambda_3, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n$ coincide with $-\lambda_3, \ldots, -\lambda_{j-1}, -\lambda_{j+1}, \ldots, -\lambda_n$. Continuing in this fashion, we obtain the assertion in (e).

(e) $\Rightarrow$ (d). Obviously, (e) implies that the eigenvalues of A and $-A$ coincide. Thus, A and $-A$ are unitarily similar by, say, [6, Corollary 1.3].

(d) $\Leftrightarrow$ (f). This follows by [5, Theorem 3.2]. $\square$

We now turn to the case of even n.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

THEOREM 2.3. *For an S_n-matrix A with n even, the following conditions are equivalent:*

(a) $d(A) = n/2$,
(b) $\lambda_{(n/2)+1}\big(\mathrm{Re}\,(e^{i\theta}A)\big) \le 0 \le \lambda_{n/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big)$ *for all real* θ,
(c) *A is unitarily similar to a matrix of the form*

$$\left[\begin{array}{cccc|c|c} & 0_{n/2} & & & * & * \\ \hline 1 & & & 0 & & 0 \\ & \ddots & & \vdots & 0_{(n/2)-1} & \vdots \\ & & 1 & 0 & & 0 \\ \hline 0 & \cdots & 0 & * & * & * \end{array}\right],$$

(d) *there is an $(n-1)$-by-$(n-1)$ compression B of A, that is, A is unitarily similar to a matrix of the form $\begin{bmatrix} B & * \\ * & * \end{bmatrix}$ such that B and $-B$ are unitarily similar,*
(e) *for any $(n+1)$-by-$(n+1)$ unitary dilation U of A with eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$ arranged so that $\arg\lambda_1 < \cdots < \arg\lambda_{n+1}$, $-\lambda_j$ lies in the circular arc of $\partial\mathbb{D}$ between $\lambda_{(n+2)+j}$ and $\lambda_{(n+2)+j+1}$ for all j (here λ_k is interpreted as $\lambda_{k-(n+1)}$ if $k > n+1$).*

Proof. The proof of (a) $\Leftrightarrow$ (b) is analogous to the one for Theorem 2.2 (a) $\Leftrightarrow$ (b), which we omit.

(a) $\Rightarrow$ (c). As in the proof of the corresponding implication in Theorem 2.2, we assume that $A = \begin{bmatrix} 0_{n/2} & B \\ C & D \end{bmatrix}$, where B, C and D are all of size $n/2$. As before, we have $\mathrm{rank}(I_{n/2} - C^*C) \le 1$. Let C^*C be unitarily similar to $\mathrm{diag}(\lambda_1, \ldots, \lambda_{n/2})$, where the λ_j's satisfy $0 \le \lambda_{n/2} \le \cdots \le \lambda_1 \le 1$. Thus, $I_{n/2} - C^*C$ is unitarily similar to $\mathrm{diag}(1-\lambda_1, \ldots, 1-\lambda_{n/2})$. The rank condition of $I_{n/2} - C^*C$ yields that $\lambda_j = 1$ for all j, $1 \le j \le (n/2) - 1$. Thus, $C = U\,\mathrm{diag}(1, \ldots, 1, \sqrt{\lambda_{n/2}})V$ for some $(n/2)$-by-$(n/2)$ unitary matrices U and V. It follows that A is unitarily similar to a matrix of the form in (c).

(c) $\Rightarrow$ (d). If B is the $(n-1)$-by-$(n-1)$ leading principal submatrix of the matrix in (c), then B is unitarily similar to $-B$ as in the proof of Theorem 2.2 (c) $\Rightarrow$ (d). This proves (d).

(d) $\Rightarrow$ (a). The unitary similarity of B and $-B$ implies that $d(B) \ge n/2$ by [4, Corollary 2.6]. Thus, $d(A) \ge n/2$. But $d(A) \le n/2$ also holds by [4, Corollary 2.5] since $\dim\ker\big(\mathrm{Re}\,(e^{i\theta}A)\big) \le 1$ for all real θ. Therefore, $d(A) = n/2$.

(a) $\Leftrightarrow$ (e). This is a consequence of [3, Theorem 1.2] and [4, Theorem 4.1 (b)]. Indeed, the condition in (e) and [4, Theorem 4.1 (b)] imply that every $(n+1)$-by-$(n+1)$ unitary dilation U of A is such that $d(U) = n/2$. Hence, 0 is in $\Lambda_{n/2}(U)$ for

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

every such U. [3, Theorem 1.2] then yields that 0 is in $\Lambda_{n/2}(A)$. Hence, $d(A) \geq n/2$. We deduce from Proposition 2.1 that $d(A) = n/2$. This proves (a). The converse (a) $\Rightarrow$ (e) is proven by reversing the above arguments. □

Since an S_n-matrix A is uniquely determined by its eigenvalues up to unitary similarity, it is desirable to have an equivalent eigenvalue condition for $d(A) = n/2$ (n even) in the preceding theorem. As the next proposition shows, such a condition may involve one or more inequalities of the eigenvalues.

PROPOSITION 2.4. *Let A be an S_2-matrix with eigenvalues λ_1 and λ_2. Then $d(A) = 1$ if and only if $|\lambda_1 + \lambda_2| + |\lambda_1\lambda_2| \leq 1$.*

Proof. We need to show that 0 is in $W(A)$ if and only if the above inequality holds. Indeed, since A is unitarily similar to the matrix

$$\begin{bmatrix} \lambda_1 & (1-|\lambda_1|^2)^{1/2}(1-|\lambda_2|^2)^{1/2} \\ 0 & \lambda_2 \end{bmatrix}$$

(cf. [6, Corollary 1.3]), its numerical range equals the elliptic disc with foci λ_1 and λ_2 and the lengths of the minor and major axes equal to $(1-|\lambda_1|^2)^{1/2}(1-|\lambda_2|^2)^{1/2}$ and $|1-\lambda_1\overline{\lambda_2}|$, respectively. Thus, 0 is in $W(A)$ if and only if $|\lambda_1| + |\lambda_2| \leq |1-\lambda_1\overline{\lambda_2}|$, the latter being equivalent to $|\lambda_1 + \lambda_2| + |\lambda_1\lambda_2| \leq 1$. □

3. Companion matrix.

We start with the following result on the nullity of the real part of a companion matrix.

THEOREM 3.1. *Let A be an n-by-n companion matrix.*

(a) *If n is odd, then* $\dim\ker\left(\mathrm{Re}\,(e^{i\theta}A)\right) \leq 1$ *for all real θ.*
(b) *If n is even, then* $\dim\ker\left(\mathrm{Re}\,(e^{i\theta}A)\right) \leq 2$ *for all real θ and, moreover,* $\dim\ker\left(\mathrm{Re}\,(e^{i\theta}A)\right) \leq 1$ *for all but at most n many values of θ in $[0, 2\pi)$.*

Note that the assertion in (a) above does not hold for even n. For example, if $A = \begin{bmatrix} 0 & 1 \\ -1 & i \end{bmatrix}$, then

$$\dim\ker\left(\mathrm{Re}\,(e^{i\theta}A)\right) = \begin{cases} 2 & \text{if } e^{i\theta} = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 3.1. Since $e^{i\theta}A$ is unitarily similar to a companion matrix for any real θ (cf. [7, Lemma 2.8]), we need only prove the assertion in (a) and the first

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

assertion in (b) for $\operatorname{Re} A$ (instead of $\operatorname{Re}(e^{i\theta}A)$). If A is of the form (1.1), then

$$\operatorname{Re} A = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & & & -\overline{a}_n \\ 1 & 0 & 1 & & & & \cdot \\ & 1 & \cdot & \cdot & & & \cdot \\ & & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & 1 & -\overline{a}_3 \\ & & & & 1 & 0 & -\overline{a}_2+1 \\ -a_n & \cdot & \cdot & \cdot & -a_3 & -a_2+1 & -a_1-\overline{a}_1 \end{bmatrix}.$$

Since $\operatorname{Re} J_{n-1}$ is the $(n-1)$-by-$(n-1)$ leading principal submatrix of $\operatorname{Re} A$, the eigenvalues of $\operatorname{Re} J_{n-1}$ and $\operatorname{Re} A$ interlace by Cauchy's interlacing theorem (cf. [11, Theorem 4.3.17]). Hence, if $\dim\ker(\operatorname{Re} A) \geq 2$ for odd n (resp., $\dim\ker(\operatorname{Re} A) \geq 3$ for even n), then 0 is an eigenvalue of $\operatorname{Re} J_{n-1}$ with multiplicity at least one (resp., at least two). However, it is known that $\operatorname{Re} J_{n-1}$ has eigenvalues $\cos(j\pi/n), 1 \leq j \leq n-1$ (cf. [9, p. 373]). For an odd (resp., even) n, none of these (resp., exactly one of these) is zero. Thus, the contradiction leads to $\dim\ker(\operatorname{Re} A) \leq 1$ for odd n (resp., $\dim\ker(\operatorname{Re} A) \leq 2$ for even n).

To prove the second assertion in (b), for any real θ, let $x_\theta = [x_1 \ \cdots \ x_n]^T$ in $\mathbb{C}^n$ be such that $\operatorname{Re}(e^{i\theta}A)x_\theta = 0$. Carrying out the matrix multiplication, we obtain a system of $n/2$ equalities:

$$e^{i\theta}x_2 - \overline{a}_n e^{-i\theta}x_n = 0,$$

$$e^{-i\theta}x_j + e^{i\theta}x_{j+2} - \overline{a}_{n-j}e^{-i\theta}x_n = 0, \quad j = 2, 4, \ldots, n-4,$$

and

$$e^{-i\theta}x_{n-2} + (-\overline{a}_2 e^{-i\theta} + e^{i\theta})x_n = 0.$$

It follows that

$$x_2 = \overline{a}_n e^{-2i\theta}x_n,$$

$$x_{j+2} = (\overline{a}_{n-j}x_n - x_j)e^{-2i\theta}, \quad j = 2, 4, \ldots, n-4,$$

and

$$x_{n-2} = (\overline{a}_2 - e^{2i\theta})x_n.$$

Equating the last two expressions of x_{n-2} and then iteratively substituting $x_{n-4}, \ldots$, x_4, x_2 into the resulting equality, we obtain that $e^{i\theta}$ is a root of the equation $x_n p(z) = 0$ for all real θ, where $p(z)$ is the polynomial $\sum_{j=0}^{n/2}(-1)^j a_{n-2j}z^{n-2j}$. If

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

θ is such that $p(e^{i\theta}) \neq 0$, then its corresponding x_n must equal zero. Our assumption $\mathrm{Re}\,(e^{i\theta}A)x_\theta = 0$, where $x_\theta = [x_1 \ \cdots \ x_{n-1} \ 0]^T$, yields that $\mathrm{Re}\,(e^{i\theta}J_{n-1})x'_\theta = 0$ with $x'_\theta \equiv [x_1 \ \cdots \ x_{n-1}]^T$. However, since $\dim\ker\big(\mathrm{Re}\,(e^{i\theta}J_{n-1})\big) = \dim\ker(\mathrm{Re}\,J_{n-1}) = 1$, we conclude that $\dim\ker\big(\mathrm{Re}\,(e^{i\theta}A)\big) \leq 1$. □

Using Theorem 3.1, we can now say something about the zero-dilation index of a companion matrix.

THEOREM 3.2. *If A is an n-by-n companion matrix, then $d(A) \leq \lceil n/2 \rceil$. Moreover, if n is odd (resp., even), then $d(A) = (n+1)/2$ or $(n-1)/2$ (resp., $d(A) = n/2$).*

Proof. That $d(A) \leq \lceil n/2 \rceil$ is a consequence of Theorem 3.1 and [4, Corollary 2.5].

Assume now that n is odd (resp., even) and A is of the form (1.1). Permuting rows and the corresponding columns of A, we can transform A to

$$A' \equiv \left[\begin{array}{cccc|cccc} & & & 0 & & & & \\ & 0_{(n-1)/2} & & \vdots & & I_{(n-1)/2} & & \\ & & & 0 & & & & \\ -a_n & -a_{n-2} & \cdots & -a_1 & -a_{n-1} & -a_{n-3} & \cdots & -a_2 \\ \hline 0 & & & & & & & \\ \vdots & & I_{(n-1)/2} & & & 0_{(n-1)/2} & & \\ 0 & & & & & & & \end{array}\right]$$

$$\left(\text{resp., } A' \equiv \left[\begin{array}{cccc|cccc} & 0_{n/2} & & & & I_{n/2} & & \\ \hline 0 & 1 & & & 0 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots & \vdots & & \vdots \\ & & 0 & 1 & 0 & 0 & \cdots & 0 \\ -a_n & -a_{n-2} & \cdots & -a_2 & -a_{n-1} & -a_{n-3} & \cdots & -a_1 \end{array}\right]\right),$$

where the rows (resp., columns) of A' numbered $1, 2, \ldots, n$ are the rows (resp., columns) of A numbered $1, 3, \ldots, n, 2, 4, \ldots, n-1$ (resp., $1, 3, \ldots, n-1, 2, 4, \ldots, n$), respectively. This shows that $d(A) = d(A') \geq (n+1)/2$ or $(n-1)/2$ depending on whether $a_1 = a_3 = \cdots = a_n = 0$ or otherwise (resp., $d(A) = d(A') \geq n/2$). Together with $d(A) \leq \lceil n/2 \rceil$ for all companion matrices A, we thus obtain $d(A) = (n+1)/2$ or $(n-1)/2$ (resp., $d(A) = n/2$) as asserted. □

The next result gives equivalent conditions for $d(A) = (n+1)/2$ when A is a companion matrix of odd size n.

THEOREM 3.3. *Let A be an n-by-n companion matrix of the form* (1.1). *If n is odd, then the following conditions are equivalent:*

(a) $d(A) = (n+1)/2$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

(b) $\lambda_{(n+1)/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big) = 0$ *for all real* θ,
(c) $\mathrm{Re}\,(e^{i\theta}A)$ *is noninvertible for all real* θ,
(d) $\dim\ker\big(\mathrm{Re}\,(e^{i\theta}A)\big) = 1$ *for all real* θ,
(e) $a_1 = a_3 = \cdots = a_n = 0$,
(f) A *and* $-A$ *are unitarily similar,*
(g) A *is unitarily similar to a matrix of the form*

$$\begin{bmatrix} 0_{(n+1)/2} & * \\ * & 0_{(n-1)/2} \end{bmatrix},$$

(h) *the eigenvalues of* A *are of the form* $0, \pm b_1, \ldots, \pm b_{(n-1)/2}$ *with* $b_1, \ldots, b_{(n-1)/2}$ *in* $\mathbb{C}$.

In this case, A is unitarily irreducible, meaning that it is not unitarily similar to the direct sum of two other matrices.

Proof. (a) $\Leftrightarrow$ (b). The proof is analogous to the one for (a) $\Leftrightarrow$ (b) of Theorem 2.2, except that, in proving (b) $\Rightarrow$ (a), we use Theorem 3.2 instead of Proposition 2.1.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (b). Note that (c) says that 0 is an eigenvalue of $\mathrm{Re}\,(e^{i\theta}A)$ for all real θ. Since $\mathrm{Re}\,(e^{i\theta}J_{n-1})$ is the $(n-1)$-by-$(n-1)$ leading principal submatrix of $\mathrm{Re}\,(e^{i\theta}A)$, their eigenvalues interlace by Cauchy's interlacing theorem (cf. [11, Theorem 4.3.17]). The unitary similarity of $e^{i\theta}J_{n-1}$ and J_{n-1} and [9, p. 373] yield that

$$\lambda_j\big(\mathrm{Re}\,(e^{i\theta}J_{n-1})\big) = \lambda_j\big(\mathrm{Re}\,J_{n-1}\big) = \cos(j\pi/n)$$

for $1 \le j \le n-1$. These together imply that $\lambda_{(n+1)/2}\big(\mathrm{Re}\,(e^{i\theta}A)\big) = 0$ for all θ, that is, (b) holds.

(c) $\Leftrightarrow$ (d) follows by Theorem 3.1 (a).

(a) $\Rightarrow$ (e). Note that, for any n-by-n matrix A (n odd), $d(A) = (n+1)/2$ implies that 0 is an eigenvalue of A (cf. [2, Proposition 2.2]). Hence, if A is of the form (1.1), then $a_n = 0$, and, for any real θ,

$$\mathrm{Re}\,(e^{i\theta}A) = \frac{1}{2}\begin{bmatrix} 0 & e^{i\theta} & 0 & \cdots & \cdots & 0 \\ e^{-i\theta} & 0 & e^{i\theta} & & & -\overline{a}_{n-1}e^{-i\theta} \\ 0 & e^{-i\theta} & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & e^{i\theta} & -\overline{a}_3 e^{-i\theta} \\ \vdots & & & e^{-i\theta} & 0 & -\overline{a}_2 e^{-i\theta} + e^{i\theta} \\ 0 & -a_{n-1}e^{i\theta} & \cdots & -a_3 e^{i\theta} & -a_2 e^{i\theta} + e^{-i\theta} & -2\mathrm{Re}\,(a_1 e^{i\theta}) \end{bmatrix}.$$

Let A_{n-2} denote the $(n-2)$-by-$(n-2)$ submatrix of A obtained by deleting the first two rows and columns of A. Then $\det\big(\mathrm{Re}\,(e^{i\theta}A)\big) = (1/2^n)(-e^{-i\theta})e^{i\theta}\det\big(\mathrm{Re}\,(e^{i\theta}A_{n-2})\big)$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

via expanding by minors along the first column of $\mathrm{Re}\,(e^{i\theta}A)$ and then along the first row of the resulting minor. Since (a) and (c) are proven to be equivalent, from (c) we have $\det\big(\mathrm{Re}\,(e^{i\theta}A)\big)=0$, and hence, $\det\big(\mathrm{Re}\,(e^{i\theta}A_{n-2})\big)=0$ for all real θ, which in turn implies, from the equivalence of (a) and (b), that $d(A_{n-2})=(n-1)/2$. Thus, $a_{n-2}=0$ by [2, Proposition 2.2]. By induction, we obtain $a_j=0$ for $j=n-4,n-6,\ldots,1$, successively.

(e) $\Rightarrow$ (f). For A of the form (1.1) with odd n, $-A$ is unitarily similar to

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & 1 & \\ & & & & & 0 & 1 \\ a_n & -a_{n-1} & a_{n-2} & \cdots & a_3 & -a_2 & a_1 \end{bmatrix}$$

(cf. [7, Lemma 2.8]). Under (e), the latter matrix is exactly A.

(f) $\Leftrightarrow$ (g) follows by [13, Theorem 2.3] and Theorem 3.2.

(f) $\Rightarrow$ (h). The proof is the same as the one for (d) $\Rightarrow$ (e) of Theorem 2.2.

(h) $\Rightarrow$ (e). Under the assumption in (h), the characteristic polynomial of A is $z(z^2-b_1)\cdots(z^2-b_{(n-1)/2}^2)$. Since this is the same as $z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$, we conclude that $a_1,a_3,\ldots,a_n$, the odd-indexed coefficients, are all equal to zero.

(f) $\Rightarrow$ (a). This is seen by [4, Corollary 2.6] since A and $-A$ are unitarily similar and $\dim\ker\big(\mathrm{Re}\,(e^{i\theta}A)\big)\leq 1$ for all real θ by Theorem 3.1 (a).

Finally, the unitary irreducibility of A follows from (h) and [7, Theorem 1.1]. $\square$

As was remarked in Section 1, it is unknown whether, for an n-by-n (n odd) noninvertible companion matrix A, the equality $W(A)=-W(A)$ would imply $d(A)=(n+1)/2$ or, equivalently, that A and $-A$ are unitarily similar. Our final result shows that this is indeed the case for $n=3$. For larger values of (odd) n, we suspect that this may not be true.

PROPOSITION 3.4. *Let A be a 3-by-3 companion matrix. Then $d(A)=2$ if and only if A is noninvertible and $W(A)=-W(A)$.*

For the proof of the sufficiency, we make use of the Kippenhahn polynomial of a matrix. Recall that the *Kippenhahn polynomial* of an n-by-n matrix A is the degree-n real homogeneous polynomial $p_A(x,y,z)=\det(x\mathrm{Re}\,A+y\mathrm{Im}\,A+zI_n)$ in x,y and z, where $\mathrm{Im}\,A=(A-A^*)/(2i)$ is the *imaginary part* of A. It is known that the numerical range of A equals the convex hull of the real points of the dual curve of

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

$p_A(x,y,z) = 0$ in the sense that $W(A) = \{a+ib : a,b$ real and $ax+by+z=0$ is tangent to $p_A(x,y,z)=0\}^\wedge$, where, for any subset $\triangle$ of the complex plane, $\triangle^\wedge$ denotes its convex hull (cf. [14, Theorem 10]).

Proof of Proposition 3.4. If $d(A) = 2$, then A is noninvertible and $W(A)$ is an elliptic disc with foci $\pm b$ ($b \in \mathbb{C}$) by [4, Lemma 3.4] and Theorem 3.3 (h). In particular, we have $W(A) = -W(A)$.

For the converse, assume that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

is noninvertible and $W(A) = -W(A)$. We readily have $a_3 = 0$. It remains to show that $a_1 = 0$. Two cases are considered separately:

(i) Suppose p_A is irreducible. Since $-A$ is unitarily similar to the companion matrix

$$A' \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_2 & a_1 \end{bmatrix},$$

the equality $W(A) = W(-A) = W(A')$ together with the irreducibility of p_A yields that $A = A'$ (cf. [7, Corollary 2.5]). It follows that $a_1 = 0$.

(ii) Suppose p_A is reducible. Then either $p_A = p_1 p_2$, where p_1 (resp., p_2) is a degree-2 irreducible (resp., degree-1) homogeneous polynomial in x, y and z, or $p_A = q_1 q_2 q_3$, where the q_j's are all of degree 1. The latter would imply that A is normal and hence unitary (cf. [7, Corollary 1.2]), which contradicts the noninvertibility of A. Thus, we must have $p_A = p_1 p_2$. The dual curves of $p_1(x,y,z) = 0$ and $p_2(x,y,z) = 0$ are an ellipse and a single point, respectively. That $W(A) = -W(A)$ implies that $W(A)$ can only be an elliptic disc centered at 0. If b_1 and b_2 are the foci of the ellipse $\partial W(A)$, then they are eigenvalues of A satisfying $b_1 + b_2 = 0$ (cf. [14, Theorem 11]). If $b_1 = b_2 = 0$, then $W(A)$ is a circular disc centered at 0, which implies that $A = J_3$ (cf. [7, Theorem 2.9]). Hence, in this case, we have $a_1 = a_2 = a_3 = 0$. On the other hand, if $b_1 = -b_2 \neq 0$, then the eigenvalues of A consist of 0 and $\pm b_1$, in which case, we readily have $a_1 = 0$. Hence, $d(A) = 2$ by Theorem 3.3. $\square$

Acknowledgment. We thank C.-K. Li for an incisive question he asked at 2016 Shanghai International Workshop on Matrix Inequalities and Matrix Equations on June 9, 2016, which leads to an improvement of the results in Theorem 3.2.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 666-678, October 2016

ELA
http:/repository.uwyo.edu/ela

REFERENCES

[1] H. Bercovici. *Operator Theory and Arithmetic in* H^∞. Amer. Math. Soc., Providence, 1988.
[2] M.-D. Choi, D.W. Kribs, and K. Życzkowski. Higher-rank numerical ranges and compression problems. *Linear Algebra Appl.*, 418:828–839, 2006.
[3] H.-L. Gau, C.-K. Li, and P.Y. Wu. Higher-rank numerical ranges and dilations. *J. Operator Theory*, 63:181–189, 2010.
[4] H.-L. Gau, K.-Z. Wang, and P.Y. Wu. Zero-dilation index of a finite matrix. *Linear Algebra Appl.*, 440:111–124, 2014.
[5] H.-L. Gau and P.Y. Wu. Numerical range of $S(\phi)$. *Linear Multilinear Algebra*, 45:49–73, 1998.
[6] H.-L. Gau and P.Y. Wu. Lucas' theorem refined. *Linear Multilinear Algebra*, 45:359–373, 1999.
[7] H.-L. Gau and P.Y. Wu. Companion matrices: reducibility, numerical ranges and similarity to contractions. *Linear Algebra Appl.*, 383:127–142, 2004.
[8] H.-L. Gau and P.Y. Wu. Zero-dilation indices of KMS matrices. *Ann. Funct. Anal.*, 5:30–35, 2014.
[9] U. Haagerup and P. de la Harpe. The numerical radius of a nilpotent operator on a Hilbert space. *Proc. Amer. Math. Soc.*, 115:371–379, 1992.
[10] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, New York, 1991.
[11] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, New York, 2013.
[12] C.-K. Li and N.-S. Sze. Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations. *Proc. Amer. Math. Soc.*, 136:3013–3023, 2008.
[13] C.-K. Li and N.-K. Tsing. Matrices with circular symmetry on their unitary orbits and C-numerical ranges. *Proc. Amer. Math. Soc.*, 111:19–23, 1991.
[14] P.E. Zachlin and M.E. Hochstenbach. On the numerical range of a matrix. *Linear Multilinear Algebra*, 56:185–225, 2008. A translation of: R. Kippenhahn. Über den Wertevorrat einer Matrix. *Math. Nachr.*, 6:193–228, 1951.