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ON 1-SUM FLOWS IN UNDIRECTED GRAPHS*

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Abstract. Let $G = (V, E)$ be a simple undirected graph. For a given set $L \subset \mathbb{R}$, a function $\omega : E \longrightarrow L$ is called an L-flow. Given a vector $\gamma \in \mathbb{R}^V$, ω is a γ-L-flow if for each $v \in V$, the sum of the values on the edges incident to v is $\gamma(v)$. If $\gamma(v) = c$, for all $v \in V$, then the γ-L-flow is called a c-sum L-flow. In this paper, the existence of γ-L-flows for various choices of sets L of real numbers is studied, with an emphasis on 1-sum flows.

Let L be a subset of real numbers containing 0 and denote $L^* := L \setminus \{0\}$. Answering a question from [S. Akbari, M. Kano, and S. Zare. A generalization of 0-sum flows in graphs. *Linear Algebra Appl.*, 438:3629–3634, 2013.], the bipartite graphs which admit a 1-sum $\mathbb{R}^*$-flow or a 1-sum $\mathbb{Z}^*$-flow are characterized. It is also shown that every k-regular graph, with k either odd or congruent to 2 modulo 4, admits a 1-sum $\{-1, 0, 1\}$-flow.

Key words. L-Flow, γ-L-Flow, c-Sum flow, Bipartite graph.

AMS subject classifications. 05C21, 90C05.

1. Introduction. Let $G = (V, E)$ be a simple undirected graph with $n = |V|$ vertices and $m = |E|$ edges. We say that the vertex $v \in V$ and edge $e \in E$ are incident if $e = \{v, u\}$ for some vertex u. In this paper, we consider two functions $\omega : E \to \mathbb{R}$, the *weight*, and $\gamma : V \to \mathbb{R}$. Then γ is called a γ-flow, if $\gamma(v)$ is the sum of the weights of edges adjacent to v: $\gamma(v) = \sum_{e \in E, v \in e} \omega(e)$. (Note that this definition of a flow is different from the classical definition of a flow on graphs stated in [11, 17, 21]. Sometimes in this paper we call ω a flow.) Let $\boldsymbol{\omega} = (\omega(e))_{e \in E}, \boldsymbol{\gamma} = (\gamma(v))_{v \in V}$ be the column vectors in $\mathbb{R}^E$ and $\mathbb{R}^V$, which correspond to the functions ω and γ respectively. Vice versa, the vectors $\boldsymbol{\omega} \in \mathbb{R}^E$ and $\boldsymbol{\gamma} \in \mathbb{R}^V$ induce the functions $\omega : E \to \mathbb{R}$ and $\gamma : V \to \mathbb{R}$. For a given set $L \subseteq \mathbb{R}$, γ is called an *L-flow*, or a γ-L-flow, if $\omega : E \to L$. Thus, an $\mathbb{R}$-flow is just a γ-flow. Let $c \in \mathbb{R}$. Then for a given $\omega : E \to L$, a γ-L-flow is called a *c-sum L-flow* if $\gamma(v) = c$ for all $v \in V$.

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In this paper, we study the existence problem of a γ-L-flow on undirected graphs. The problem of finding c-sum S-flows was studied in the papers [1, 2, 3, 4]. For simplicity of exposition we will assume that G is a connected graph.

2. Existence of γ-interval-flows. Let $G = (V, E)$ be a simple undirected graph. Denote by $A(G) := A = [a_{ve}] \in \mathbb{R}^{V\times E}$ the vertex edge incidence matrix of G. That is $a_{ve} = 1$ if $v \in e$ and $a_{ve} = 0$ otherwise. Observe that the existence of a γ-L-flow is equivalent to the solvability of the system:

$$A(G)\boldsymbol{\omega} = \boldsymbol{\gamma}, \tag{2.1}$$

where $\boldsymbol{\gamma} \in L^E$. Given an interval L the most basic question is whether a graph G has a γ-L-flow or not. If $L = \mathbb{R}$ this is a purely linear algebraic question. (See the proof of Lemma (2.1).) When L is a proper subinterval of $\mathbb{R}$ the solvability of (2.1) is a linear programming problem. In this case, we can apply methods from linear programming to find conditions for its solvability. (See the proof of Lemma (2.2).) In this section, we will first strengthen an existence result from [4] for γ-$\mathbb{R}$-flows and then look at the case when L is a proper subinterval.

2.1. Existence of $\boldsymbol{\gamma}$-$\mathbb{R}$-flows. It is well known that A is unimodular, i.e., all its minors have values 0, 1 or -1, if and only if G is bipartite [10]. (See [8, §6.5] for a textbook reference.) Assume that G is connected. Then rank $A = n$ if G contains an odd cycle and rank $A = n - 1$ if G is bipartite [9, p. 63]. The following result is a more detailed version of the result proved in [4].

LEMMA 2.1. *Let $G = (V, E)$ be a connected graph and $\boldsymbol{\gamma} \in \mathbb{R}^V$ is given. Then*

1. *If G is not bipartite then there exists a γ-$\mathbb{R}$-flow. Furthermore, if $\boldsymbol{\gamma} \in \mathbb{Z}^V$ then there exists a solution $\boldsymbol{\omega}$ such that $2\boldsymbol{\omega} \in \mathbb{Z}^E$.*
2. *Assume that G is bipartite and $V = V_1 \cup V_2$ is the bipartite decomposition of vertices of G. Then there exists a γ-$\mathbb{R}$-flow if and only if*

$$\sum_{v\in V_1} \gamma(v) - \sum_{v\in V_2} \gamma(v) = 0. \tag{2.2}$$

Equivalently, let $\mathbf{y} = (y_v)_{v\in V} \in \mathbb{R}^V$ be a vector such that $y_v = 1$ if $v \in V_1$ and $y_v = -1$ if $v \in V_2$. That is, $\mathbf{y}^\top = (\mathbf{1}_{V_1}^\top, -\mathbf{1}_{V_2}^\top)$. Then $\mathbf{y}$ is a basis of the null space of $A(G)^\top$. Furthermore, if $\boldsymbol{\gamma} \in \mathbb{Z}^V$ and the condition (2.2) holds then there exists a solution $\boldsymbol{\omega} \in \mathbb{Z}^E$.

Proof.

1. Assume that G is a connected nonbipartite graph. Let T be a spanning tree of G. So $T = (V, E(T))$ is bipartite, and assume that $V = V_1 \cup V_2$ is the

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bipartition of V. Since G is nonbipartite there exists $e \in E \setminus E(T)$ such that e connects two vertices in V_i, for some $i \in \{1,2\}$. Let $H = (V, E(T) \cup \{e\})$. So H is a connected nonbipartite graph. Therefore H is unicyclic with an odd cycle. Then rank $A(H) = |V| = n$. Consider the system (2.1), where $\omega(f) = 0$ for each $f \in E \setminus E(H)$. Hence, it is enough to prove the theorem in this case for $G = H$. We claim that $\det A(H) = \pm 2$.

Suppose first that H is a Hamiltonian cycle. We can assume that $E(H) = \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}, \{v_n.v_1\}\}$. Then H supports exactly two cyclic permutations σ and σ^{-1}, where $\sigma(v_i) = v_{i+1}$ for $i = 1, \dots, n$ and $v_{n+1} = v_1$. The signs of these permutations are $+$, hence $\det A(H) = 2$.

Assume now H is not a Hamiltonian cycle. Without loss of generality we may assume that v_n is a vertex of degree 1 connected to the last edge in H. Expanding $\det A(H)$ by the last row we obtain that $\det A(H) = \det A(H_1)$, where H_1 is a connected unicyclic graph with an odd cycle on $n-1$ vertices. Continue in this manner to deduce that $\det A(H) = \det C = 2$ for some odd Hamiltonian cycle C, after renaming the names of the vertices and edges in H. Now use Cramer's rule and the fact that $A(H)$ is unimodular to deduce that $2\boldsymbol{\omega}_H \in \mathbb{Z}^n$.

2. Assume that G is bipartite and $V = V_1 \cup V_2$ is the bipartite decomposition of V. Clearly, $\mathbf{y}^\top A(G) = 0$. Since rank $A(G) = n-1$ then $\mathbf{y}$ spans the null space of $A(G)^\top$. Hence, the system (2.1) is solvable if and only if the condition (2.2) holds.

   Let $\boldsymbol{\gamma} \in \mathbb{Z}^V$ and assume that the condition (2.2) holds. We now construct a solution $\boldsymbol{\omega} \in \mathbb{Z}^E$. Let T' be a spanning tree of G. Let $\boldsymbol{\omega}$ be the unique solution of (2.1) such that $\omega(e) = 0$ if $e \notin E(T')$. Recall that rank $A(T') = n-1$. Since $\mathbf{y}$ spans the null space of $A(T')^\top$ it follows any $n-1$ rows of $A(T')$ are linearly independent. Let B be a square submatrix of A obtained by deleting a row in $A(T')$ corresponding to a vertex $v \in V$. Denote by $\boldsymbol{\gamma}'$ the vector obtained from $\boldsymbol{\gamma}$ be deleting coordinate $\gamma(v)$. As $A(T')$ is unimodular it follows that $\det B = \pm 1$. Hence, the solution of the system $A(T')\boldsymbol{\omega}' = \boldsymbol{\gamma}$ is given by $\boldsymbol{\omega}' = B^{-1}\boldsymbol{\gamma}' \in \mathbb{Z}^{E(T')}$. As $\omega(e) = \omega'(e)$ for each $e \in E(T')$ we deduce that $\omega \in \mathbb{Z}^E$. $\square$

2.2. Linear programming conditions for the existence of γ-interval-flows. In this section, we apply linear programming methods to study the conditions for existence of γ-L-flows, where L is an interval of $\mathbb{R}$. For simplicity of exposition we assume that L is the closed bounded interval $[a, b]$. Our methods and arguments are close to those given in [8].

We denote $[n] = \{1, \dots, n\}$. We will identify

$$V \equiv [n], \quad E \equiv [m], \quad \mathbb{R}^V \equiv \mathbb{R}^n, \quad \mathbb{R}^E \equiv \mathbb{R}^m,$$

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and no ambiguity will arise. Let $\mathbf{1}_m = \mathbf{1}_E$ be a column vector with $m = |E|$ coordinates equal to 1. For two vectors $\mathbf{x} = (x_1, \ldots, x_n)^\top, \mathbf{z} = (z_1, \ldots, z_n)^\top \in \mathbb{R}^m$ we denote $\mathbf{x} \leq \mathbf{y}$ if $x_j \leq y_j$ for $j = 1, \ldots, m$.

We are looking for a solution of (2.1) such that

$$a\mathbf{1}_m \leq \boldsymbol{\omega} \leq b\mathbf{1}_m. \tag{2.3}$$

Denote by I_m the identity matrix of order m and by $\mathbb{R}_+$ the set of nonnegative real numbers. Let $\mathbf{d}(G) = (\deg(v))_{v\in V} \in \mathbb{R}^V$ be the degree sequence of G. Note that $\mathbf{d}(G) = A(G)\mathbf{1}_m$.

Lemma 2.2. *The following conditions are equivalent:*

1. *System* (2.1) *with conditions* (2.3) *is solvable.*
2. *The system*

$$A\boldsymbol{\omega}' = \boldsymbol{\gamma} - a\mathbf{d}(G), \quad \mathbf{0} \leq \boldsymbol{\omega}' \leq (b-a)\mathbf{1}_m \tag{2.4}$$

   is solvable.
3.

$$\max\{(\boldsymbol{\gamma} - a\mathbf{d}(G))^\top \mathbf{z},\ \mathbf{z} \in \mathbb{R}^n, A^\top \mathbf{z} \leq \mathbf{w}\} \leq (b-a)\mathbf{1}_m^\top \mathbf{w}, \tag{2.5}$$

   for each $\mathbf{w} \in \mathbb{R}_+^m$.

In particular, for a simple undirected graph $G = (V, E)$ with no isolated vertices, the following are equivalent:

1. *G has a c-$[a, b]$-flow.*
2. *If G has a nonnegative $\mathbf{c}\mathbf{1}_m - a\mathbf{d}(G)$-flow such that the value of this flow on each edge is at most $b - a$.*
3. *For each $\mathbf{w} \in \mathbb{R}_+^m$ one has the inequality*

$$\max\{(c\mathbf{1}_n - a\mathbf{d}(G))^\top \mathbf{z},\ \mathbf{z} \in \mathbb{R}^n, A^\top \mathbf{z} \leq \mathbf{w}\} \leq (b-a)\mathbf{1}_m^\top \mathbf{w}.$$

Proof. We first prove the first part of the lemma. The equivalence of conditions *1* and *2* follows straightforward by noting that ω is a solution satisfying (2.1) and (2.3) if and only if $\boldsymbol{\omega}' = \boldsymbol{\omega} - a\mathbf{1}_m$ satisfies (2.4).

It is left to show that the conditions *1* and *3* are equivalent. Clearly, the system (2.1) satisfying the conditions (2.3) is equivalent to the following conditions:

$$F\mathbf{x} \leq \mathbf{f},\ \mathbf{x} \in \mathbb{R}^m,\ \text{where } F = \begin{bmatrix} A \\ -A \\ I_m \\ -I_m \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} \gamma \\ -\gamma \\ b\mathbf{1}_m \\ -a\mathbf{1}_m \end{bmatrix}.$$

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Farkas' lemma asserts [8] that the above system is solvable if and only if the following implication holds:

$$\mathbf{y} \in \mathbb{R}_+^{2(n+m)} \text{ and } \mathbf{y}^\top F = \mathbf{0}^\top \Rightarrow \mathbf{y}^\top \mathbf{f} \geq 0,$$

where $\mathbf{y}^\top = (\mathbf{y}_1^\top, \mathbf{y}_2^\top, \mathbf{y}_3^\top, \mathbf{y}_4^\top), \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n, \mathbf{y}_3, \mathbf{y}_4 \in \mathbb{R}^m$. The equation $\mathbf{y}^\top F = \mathbf{0}^\top$ is equivalent to

$$\mathbf{y}_4 = \mathbf{y}_3 - A^\top \mathbf{z}, \quad \mathbf{z} = \mathbf{y}_2 - \mathbf{y}_1.$$

The condition $\mathbf{y} \geq \mathbf{0}$ is equivalent to the inequalities

$$\mathbf{y}_3 \geq \mathbf{0}, \quad \mathbf{y}_3 \geq A^\top \mathbf{z}.$$

(Note that if the above conditions hold, one can always choose $\mathbf{y}_1, \mathbf{y}_2 \geq \mathbf{0}$ such that $\mathbf{z} = \mathbf{y}_2 - \mathbf{y}_1$.) Clearly, these conditions are satisfiable for $\mathbf{y}_3 \geq 0$ and $\mathbf{z} = 0$. Finally, the condition $\mathbf{y}^\top \mathbf{f} \geq 0$ is equivalent to the following inequality

$$\mathbf{z}^\top \boldsymbol{\gamma} - a\mathbf{z}^\top A \mathbf{1}_m \leq (b-a)\mathbf{y}_3^\top \mathbf{1}_m.$$

Set $\mathbf{w} = \mathbf{y}_3$ and recall that $A\mathbf{1}_m = \mathbf{d}(G)$ to deduce the first part of the lemma.

The second part of the lemma follows straightforward from the first part by assuming that $\boldsymbol{\gamma} = c\mathbf{1}_m$. $\square$

Condition (2.5) can be stated as the following nonlinear inequality in $\mathbf{z} \in \mathbb{R}^n$. Let $\mathbf{z} = (z_1, \ldots, z_n)^\top$ be an arbitrary vector in $\mathbb{R}^n$. Define $\mathbf{w}(\mathbf{z}) = (w_1(\mathbf{z}), \ldots, w_m(\mathbf{z}))^\top \in \mathbb{R}_+^m$ as follows:

$$w_j(\mathbf{z}) = \max(0, (A^\top \mathbf{z})_j) \text{ for } j = 1, \ldots, m.$$

Then condition (2.5) is equivalent to

$$(\boldsymbol{\gamma} - a\mathbf{d}(G))^\top \mathbf{z} \leq (b-a)\mathbf{1}_m^\top \mathbf{w}(\mathbf{z}) \text{ for each } \mathbf{z} \in \mathbb{R}^n.$$

We now give the condition for the existence of nonnegative solutions of (2.1).

Lemma 2.3. *System* (2.1) *with* $\boldsymbol{\gamma} \neq \mathbf{0}$ *has a nonnegative solution if and only if*

$$\min\{\boldsymbol{\gamma}^\top \mathbf{z},\ \mathbf{z} \in \mathbb{R}^n, A^\top \mathbf{z} \geq \mathbf{0}\} = 0. \tag{2.6}$$

Proof. As in the proof of Lemma 2.2 the existence of nonnegative solutions of system (2.1) is equivalent to the system

$$F\mathbf{x} \leq \mathbf{f},\ \mathbf{x} \in \mathbb{R}^m,\ \text{where } F = \begin{bmatrix} A \\ -A \\ -I_m \end{bmatrix},\ \mathbf{f} = \begin{bmatrix} \gamma \\ -\gamma \\ \mathbf{0} \end{bmatrix}.$$

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The above system is solvable if and only if each nonnegative solution of $\mathbf{y}^\top F = \mathbf{0}^\top$ satisfies the inequality $\mathbf{y}^\top \mathbf{f} \geq 0$. Let $\mathbf{y}^\top = (\mathbf{y}_1^\top, \mathbf{y}_2^\top, \mathbf{y}_3)$, where $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$ and $\mathbf{y}_3 \in \mathbb{R}^m$. Then the condition $\mathbf{y} \geq \mathbf{0}$ and $F^\top \mathbf{y} = 0$ are equivalent to the condition that $\mathbf{y}_3 = A^\top \mathbf{z} \geq 0$, where $\mathbf{z} = \mathbf{y}_1 - \mathbf{y}_2$. The condition $\mathbf{y}^\top \mathbf{f} \geq 0$ is equivalent to $\boldsymbol{\gamma}^\top \mathbf{z} \geq 0$. Note that if we choose $\mathbf{z} = \mathbf{0}$ then $\mathbf{y}_3 = \mathbf{0}$ and $\boldsymbol{\gamma}\mathbf{z} = 0$. This implies (2.6). $\square$

3. The range of a 1-flow. Let G be a given graph. Assume that a γ-R-flow is feasible, i.e., there exists $\omega : V \to \mathbb{R}$ such that γ is induced by ω. Denote by $L(\omega)$ the minimal closed interval that contains all the values of a feasible ω, and by $|L(\omega)|$ its length. Then $\omega^\star$ and $L(\omega^\star)$ are called *optimal* and *optimal range* respectively, if $|L(\omega^\star)| \leq |L(\omega)|$ for all feasible ω.

In this section, we consider the following problems for 1-flows: What are possible values of a feasible ω; find optimal flows; characterize graphs which have a 1-sum flow with $L(\omega)$ contained in some given interval L.

First we will look at 1-sum flows on trees, which have a unique 1-sum flow or none at all, and find the optimal range for this class of graphs. After that we do the same for graphs with a single cycle, and then give some bounds for the range of 1-sum flows on general graphs. After this we instead look at conditions guaranteeing that a graph has a 1-sum $[-1, 1]$-flow, or a non-negative flow .

3.1. 1-sum flows on trees. For a given graph $G = (V, E)$ and the weight function $\omega : E \to \mathbb{R}$, for each subset Q of E we denote by $\omega(Q) := \sum_{e \in Q} \omega(e)$. We agree that $\omega(\emptyset) = 0$. Recall that a bipartite graph $G = (V_1 \cup V_2, E)$ is called *balanced* if $|V_1| = |V_2|$.

In this section, we analyze the the values of 1-flows on a tree $T = (V, E)$ with n vertices, i.e., $n = |V|$ and we let $V = [n] = \{1, \ldots, n\}$. Recall that $m = |E| = n - 1$ and T is bipartite. Let $A = A(T)$. Then the system (2.1) is solvable if and only if the condition (2.2) holds. Assume that (2.2) holds.

We now estimate the coordinates of the solution of (2.1). We perform the following *pruning* procedure of a tree T. Let $T_1 = T$ and $P_1 \subset V$ be leaves. If $T_1 = K_2 = K_{1,1}$ or the star $K_{1,n-1}$ then we are done. Otherwise, let T_2 be the subtree of T_1 obtained by deleting the leaves P_1 and the corresponding $|P_1|$ edges. Denote by $E(P_1) \subset E(T)$ the subset of edges attached to P_1. We now continue this process on T_2. We obtain a sequence of subtrees $T_1 \supset T_2 \supset \cdots \supset T_k$, where $T_k = K_{1,n_k-1}$. The leaves of $T_i = (V_i, E_i), n_i = |V_i|$ are P_i. Then $E(P_i) := E_i \setminus E_{i+1}$ for $i = 1, \ldots, k$. $(E_{k+1} = \emptyset.)$ Note

$$p_1 = |P_1| \geq p_2 = |P_2| \geq \cdots \geq p_k = |P_k| = \max(2, n_k - 1).$$

Indeed, if we delete all leaves of T_1 which are neighbors of u, then it is possible that

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u is not a leaf of T_2. On the other hand if u is a leaf in T_2 then u is not a leaf in T_1 and u has at least a leaf neighbor in T_1.

We consider the system (2.1). Let $\boldsymbol{\gamma}^{(1)} = (\gamma^{(1)}(v))_{v\in V_1} = \boldsymbol{\gamma}$ and $\omega_1(e) = \gamma^{(1)}(v)$ for $e \in E(P_1)$ and $v \in e$. The values of $\omega_1(e)$ is the value of $\omega(e)$, where e is the unique edge in T_1 that contains the vertex $v \in P_1$.

Let $\boldsymbol{\gamma}^{(i)} = (\gamma^{(i)}(v))_{v\in V_i}$ and $\omega_i(e), e \in E(P_i)$ be defined recursively as follows for $i = 2,\ldots,k$:

$$\gamma^{(i)}(v) = \gamma^{(i-1)}(v) \text{ for } v \in V_i \text{ not connected to } P_{i-1}$$
$$\gamma^{(i)}(v) = \gamma^{(i-1)}(v) - \sum_{e\in E(P_{i-1}),v\in e} \omega_{i-1}(e) \text{ for } v \in V_i \text{ connected to } P_{i-1}$$
$$\omega_i(e) = \gamma^{(i)}(v) \text{ for } e \in E(P_i) \text{ and } v \in P_i. \tag{3.1}$$

It is easy to see that each γ_l can appear at most in one of the coordinates of $\gamma^{(j)}$ with coefficient ± 1. (This is also follows from the condition (2.2).) Now consider T_k. Assume first that $T_k = K_2$. So $P_k = \{u, v\}$. In order to be able to solve the original system one needs that condition $\gamma^{(k)}(u) - \gamma^{(k)}(v) = 0$. Assume $T_k = K_{1,n_k-1}$, where $n_k \geq 3$. Let u be the center of the star. Then the solvability condition is:

$$\gamma^{(k)}(u) = \sum_{v\in P_k} \gamma^{(k)}(v).$$

In both cases, since each $\gamma(w)$ appears exactly once in some degree of T_k with coefficient ± 1, we deduce that this is equivalent to the fact that a basis to the null space of $A(T)^\top$ is $\mathbf{y}^\top = (\mathbf{1}_{V_1}^\top, -\mathbf{1}_{V_2}^\top)$.

Theorem 3.1. *Assume that a tree T has 1-flow, i.e., T is a balanced bipartite graph. Let $T = T_1 \supset \cdots \supset T_k$ be the subtrees defined as above. Assume that $P_i \subset V(T_i)$ be the leaves of T_i, and $\omega_i : E(P_i) \to \mathbb{R}$ are defined as in* (3.1) *for $i = 1,\ldots,k$. Then the following conditions hold:*

1. *The 1-flow is unique and integer valued.*
2. *The value of ω_1 for each $e \in E(P_1)$ is 1. Hence, $\omega(E(P_1)) = p_1$.*
3. *If i is even then $\omega_i(e) \leq 0$ for $e \in E(P_i)$. If i is odd then $\omega_i(e) \geq 1$ for $e \in E(P_i)$.*
4.

$$(-1)^i\omega(E(P_i)) \geq \sum_{j=0}^{i-1} (-1)^j p_{i-j} \text{ for } i = 2,\ldots,k \tag{3.2}$$

5. *Let $V(T) = V_1(T) \cup V_2(T)$ be the bipartite decomposition of the balanced tree T. Then both $V_1(T)$ and $V_2(T)$ contain a leaf.*

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6. *If $p_1 = 2$ then T is a path and ω is a $\{0,1\}$-flow.*
7. *If $p_1 = 3$ then T has the shape "T", i.e., T is obtained by gluing an end point of one path to the inner point of another path. Furthermore, ω is a $\{0,1\}$-flow.*
8. *Assume that $p_1 \geq 4$. Then $n \geq 6$ and the flow is a 1-sum $\{1 - \lfloor \frac{p_1}{2} \rfloor, 2 - \lfloor \frac{p_1}{2} \rfloor, \dots, \lfloor \frac{p_1}{2} \rfloor\}$-flow.*
9. *In particular, the flow is in $[2 - \frac{n}{2}, \frac{n}{2} - 2]$. The lower bound achieved only for the unique tree $T_{\min}$, where $T_{\min}$ is K_2 with appended $\frac{n-2}{2}$ vertices to each vertex of K_2. For $T_{\min}$ we obtain that the flow is a 1-sum $\{\frac{4-n}{2}, 1\}$-flow. The upper bound is obtained for the unique tree $T_{\max}$, $T_{\max}$ the path on 4 vertices, PL_4, with $\frac{n-4}{2}$ vertices appended to each leaf of PL_4. For $T_{\max}$ we obtain the flow is a 1-sum $\{\frac{6-n}{2}, 1, \frac{n-4}{2}\}$-flow.*
10. *The other optimal tree T', different from $T_{\max}$ and $T_{\min}$, on $n \geq 8$ vertices is obtained as follows. Take the path $PL_4 := v_1 - v_2 - v_3 - v_4$, Add $\frac{n-4}{2}$ leaves at v_1, $\frac{n-6}{2}$ leaf at v_4 and one leaf at v_2. Then this flow is a 1-sum $\{\frac{6-n}{2}, \frac{8-n}{2}, 1, \frac{n-8}{2}, \frac{n-6}{2}\}$-flow.*

Proof.

1. This follows from part *2.* of Lemma 2.1.
2. Self evident.
3. Use (3.1), the fact that $\boldsymbol{\gamma} = \mathbf{1}_n$ and each $\omega_1(e) = 1$ for $e \in E(P_1)$ to deduce that that each $\omega_2(e) \leq 0$ for $e \in E(P_2)$. Continuing in this manner, using (3.1) we deduce the claim.
4. Let $E'(P_i)$ be the subset of all edges in $E(P_i)$ which are connected to P_{i+1}. (Note that some leaves in T_i may be connected to nonleaf vertices in T_{i+1}.) Then summing the 1-flow on all vertices in P_i, for $i \geq 2$ we get

$$p_i = \omega_{i-1}(E'(P_{i-1})) + \omega_i(E(P_i)). \tag{3.3}$$

   Let $i = 2$. As $\omega_1(e) = 1$ for each $e \in E(P_1)$ we deduce that $p_2 \leq \omega_1(E(P_1)) + \omega_2(E(P_2)) = p_1 + \omega_2(E(P_2))$. This establishes (3.2) for $i = 2$.
   Assume now that $i = 3$. As $\omega_2(e) \leq 0$ for each $e \in E(P_2)$ the equality (3.3) and the inequality yields the inequality (3.2) for $i = 2$ yields:

$$p_3 \geq \omega_2(E(P_2)) + \omega_3(E(P_3)) \geq p_2 - p_1 + \omega_3(E(P_2)).$$

   This establishes (3.2) for $i = 3$. Continuing in this manner we deduce (3.2) for $i = 4, \dots, k$.
5. Assume to the contrary that $V_1(T)$ does not have a leaf. So $n - 1 = |E(T)| \geq 2|V_1(T)| = n$ as T is a balanced bipartite. This is impossible. Hence, $V_1(T)$ contains a leaf. Similarly, $V_2(T)$ contains a leaf.
6. Straightforward.

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7. Straightforward.
8. $\omega \in \mathbb{Z}^{n-1}$ it is enough to show that ω is a 1-sum $[1-\lfloor \frac{p_1}{2}\rfloor, \lfloor \frac{p_1}{2}\rfloor]$-flow. We will prove the claim on induction on n. In view of *6.-7.* the claim holds for $n = 2, 4$. Assume that the claim holds for all even n, where $n \le 2N$. Assume that $n = 2N+2$. In view of *6.-7.* we assume that $p \ge 4$. Let T be a balanced tree on $2N+2$ vertices with $p \ge 4$ leaves. Let $\omega : E(T) \to \mathbb{Z}$ be the unique 1-flow on T. Let $u \in V_1(T), v \in V_2(T)$ be two leaves of T. Assume that $\{u, u_1\}, \{v, v_1\} \in E(T)$. Take the path Q in T connecting u and v given by $e_1 = \{u, u_1\} - e_2 - \cdots - e_{2l+1} = \{v_1, v\}$. Note that there is the following flow on Q:
$$\theta(e_1) = \theta(e_3) = \cdots = \theta(e_{2l+1}) = 1, \quad \theta(e_2) = \cdots = \theta(e_{2l}) = -1.$$
Let T' be the tree obtained from T by deleting the vertices u, v. Denote $F = \{e_2, \ldots, e_{2l}\} \subset E(T')$ and assume that T' has p' leaves. Let $\omega' : T' \to \mathbb{Z}$ be the unique 1-flow on T'. Then $\omega'(e) = \omega(e)$ if $e \in E(T')\{F\}$ and $\omega'(e_j) = \omega(e_j) - \theta(e_j)$ for $j = 2, \ldots, 2l$. So $\omega'(e) - 1 \le \omega(e) \le \omega'(e) + 1$ for each $e \in E(T')$.
Suppose first that $\deg(u_1), \deg(v_1) \ge 3$. Then $p' = p - 2$. By induction hypothesis
$$2 - \lfloor \frac{p}{2} \rfloor = \mathbf{1} - \lfloor \frac{p-2}{2} \rfloor \le \omega'(e) \le \lfloor \frac{p-2}{2} \rfloor = -1 + \lfloor \frac{p}{2} \rfloor.$$
This proves *8.* in this case.
Suppose now that $\deg(u_1) = 2$. Then $\omega(e_1) = 1$ and $\omega(e_2) = 0$. Delete vertices u, u_1 in T to obtain a balanced tree T'' with $2N$ vertices and p'' pendant vertices. Clearly $p'' \le p$. Also the 1-flow on T'' coincides. Use the induction hypothesis to deduce *8.*

9-10. Clearly the maximal number of leaves in a balanced tree is $p_1 = n - 2$. This equality is achieved only for the tree $T_{\min}$. Apply *8.* to deduce that the value of each 1-flow on a balanced tree on n vertices is not less than $\frac{4-n}{2}$. For $T_{\min}$ the 1-flow is a $\{\frac{4-n}{2}, 1\}$-flow. Other balanced trees on n vertices have at most $n-4$ leaves. Use *8.* to deduce that the value of each 1-flow on a balanced tree on n vertices is not more than $\frac{n-4}{2}$. There are four nonisomorphic balanced trees with $n-4$ leaves: $T_{\max}, T'$ and S_1, S_2. S_1 is obtained from $T_{\min}$ by deleting one leaf in $V_1(T_{\min})$ and adjoining one vertex of a leaf in $V_2(T_{\min})$. S_2 is obtained from $T_{\max}$ by removing one leaf from $V_1(T_{\max})$ and from $V_2(T_{\max})$ and adjoining these two leaves to v_2 and v_3, respectively. For $T_{\max}$ the 1-flow is $\{\frac{6-n}{2}, 1, \frac{n-4}{2}\}$-flow. For T' the 1-flow is a $\{\frac{6-n}{2}, \frac{8-n}{2}, 1, \frac{n-8}{2}, \frac{n-6}{2}\}$-flow. For S_1 the 1-flow is a $\{\frac{6-n}{2}, 0, 1\}$-flow. For S_2 the 1-flow is a $\{\frac{8-n}{2}, 1, \frac{n-8}{2}\}$-flow.
If T has at most $n-5$ leaves then *8* implies that the range of 1-flow is in $[\frac{8-n}{2}, \frac{n-6}{2}]$. $\square$

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3.2. The range of 1-sum flows on unicyclic graphs. We can also find a bound for the range of a 1-sum flow on a connected unicyclic graph, i.e., a graph which is obtained from a tree by adding a single edge. As for trees we call a vertex of degree one a leaf, and just as for trees the number of leaves turns out to control the range of the 1-sum flows. The bound also depends strongly upon whether the graph is bipartite or not, with bipartite graphs giving us a narrower range, and in each case we find graphs for which the stated bound is optimal.

THEOREM 3.2. *Let $G = (V, E)$ be a connected unicyclic graph, with $|V| = n = |E|$, which has a 1-sum flow. Assume that G has $p \geq 0$ leaves.*

Then one of the following conditions holds:

1. *$p = 0$. In this case, G is a cycle and has a 1-sum $\{\frac{1}{2}\}$-flow*
2. *$p = 1$. If G has a 1-sum flow, then it has a 1-$[0, 1]$-flow*
3. *$p \geq 2$ and G is not bipartite. Then G has a 1-sum $[1 - p, p]$-flow.*
   This bound is optimal for the graph obtained by taking the disjoint union of a triangle and $K_{1,p+1}$ and joining one vertex on the triangle to one of the leaves of the $K_{1,p+1}$.
4. *$p \geq 2$ and G is a balanced bipartite graph. Then G admits a 1-sum $[1 - \lfloor \frac{p}{2} \rfloor, \lfloor \frac{p}{2} \rfloor]$-flow. The above range is smallest possible for the graph obtained by taking two copies of $K_{1,p/2}$ and joining the two high degree vertices by a six vertex path, giving a total of $p + 8$ vertices in the graph, and then adding an edge so that the middle 4 vertices of the path form a 4-cycle.*

Proof.

1. Set the value on each edge to $\frac{1}{2}$.
2. If $p = 1$ then G consists of a cycle C joined to a path P by a single edge $e = \{u, v\}$, where $u \in C$. The flow on the path is uniquely determined, and is locally a flow with only values 0 and 1. If the flow on e is 0 we can set the weight on every edge in C to $\frac{1}{2}$ and we are done. If the flow on e is 1 then $C \setminus u$ is a path with an even number of vertices, since a 1-sum flow exists, and we can set the weight on a perfect matching in that path to 1 and 0 on the remaining edges, and so we have a flow on G with only weights 0 and 1.
3. We inductively assume that the theorem is true for smaller n and p. Let u and v be two leaves of G. If u is adjacent to a vertex w of degree 2 then $G' = G \setminus \{u, w\}$ has a 1-sum $[1-p, p]$-flow, by induction on n, and by setting the weight on the edge $\{u, w\}$ to 1 we can extend this to a 1-sum $[1-p, p]$-flow on G, and we can follow the same procedure if v is adjacent to a vertex of degree 2. Hence, we can assume that u and v are not adjacent to vertices of degree 2.
   Since G is not bipartite there exists a walk W of odd length in G from u to v.

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By induction on p the graph $G' = G \setminus \{u, v\}$ has a 1-sum flow of the desired range. We can now build a 1-sum flow on G by setting the flow on the edges incident to u and v to 1, and then alternatingly subtract and add 1 to the weight of the edges along w. In this way, we get a 1-sum flow on G, and since G' had two less leaves than G and our modification changed each weight by at most 2, which happens if the edge was traversed twice by the walk W, we get a flow of the desired range.

4. In this case, G is a balanced bipartite graph containing a single even cycle C. Let u and v be two leaves of G belonging to different parts of the bipartition. If either one of them, say u, is adjacent to a vertex w of degree 2 then $G' = G \setminus \{u, w\}$ is also a balanced bipartite graph and, by induction on n, it has a 1-sum flow with the desired range. By setting the weight on the edge $\{w, u\}$ to 1 and the weight on the other edge incident to w to 0, extend this to 1-sum flow of the desired range on G. Hence, we can assume that u and v are not adjacent to vertices of degree 2.
Since u and v are in different parts there exists a path from u to v in G of odd length. By induction on p the graph $G' = G \setminus \{u, v\}$ has a 1-sum flow of the desired range. We can now build a 1-sum flow on G by setting the flow on the edges incident to u and v to 1, and then alternatingly subtract and add 1 to the weight of the edges along w. In this way, we get a 1-sum flow on G, and since G' had two less leaves than G and our modification changed each weight by at most 1 we get a flow of the desired range. $\square$

3.3. The range of 1-sum flows for general connected graphs. The following lemma, which is straightforward, gives a simple result on the length of the interval a 1-sum L-flow.

LEMMA 3.3. *If G has a k-regular spanning subgraph then G has a 1-sum $[0, \frac{1}{k}]$-flow.*

We know that random graphs with positive density have k-factors [18], but there are of course dense graphs which do not have a 1-factor. As all connected graphs have a spanning tree, and nonbipartite connected graphs have a spanning connected unicyclic subgraph, we obtain the following.

LEMMA 3.4. *Let $G = (V, E), |V| = n, |E| = m$ be a connected graph. Then there exists a 1-flow if and only G is not a bipartite nonbalanced graph. If a 1-flow exists, then there exists a flow of the following type:*

1. *G is a balanced bipartite graph with $n \geq 8$. Then there exists an integer valued flow with values $\{2 - \frac{n}{2}, \ldots, \frac{n}{2} - 2\}$.*
2. *G is nonbipartite graph. Then there exists a flow x such that $2x \in \mathbb{Z}^m$. Moreover for $n \geq 6$ its values are in the interval $[5 - n, n - 5]$.*

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Proof. Lemma 2.1 yields that a connected graph G has a 1-flow if and only if G is not a bipartite nonbalanced graph.

1. Let $T = (V, E(T))$ be a spanning tree of G. Hence, T is balanced. Let $\omega_T : E(T) \to \mathbb{R}$ be a 1-flow on T. Theorem 3.1 yields that ω_T has an integer valued flow with values $\{2 - \frac{n}{2}, \dots, \frac{n}{2} - 2\}$. Extend ω_T to a flow ω on G by letting $\omega(e) = 0$ for $e \in E \setminus E(T)$.
2. G has a spanning connected unicyclic subgraph $H = (V, E(H))$. Let $\omega_H : E(H) \to \mathbb{R}$ be a 1-flow on H. Theorem 3.2 yields that there exists a flow ω_H such that $2\omega_H$ is integer valued. Moreover for $n \geq 6$ the values of $2\omega_H$ are in the interval $[5 - n, n - 5]$. Extend ω_H to a flow ω on G by letting $\omega(e) = 0$ for $e \in E \setminus E(H)$. $\square$

This result can be sharpened a bit by including information about the independence number $\alpha(G)$ of G. In [7] it was proven that unless G is a cycle, a complete graph, or a balanced complete bipartite graph it has a spanning tree the end vertices of which form an independent set in G. Using this we get the following.

COROLLARY 3.5. *Let $G = (V, E)$ be a connected graph with independence number $\alpha(G)$.*

1. *If G is k-regular then G has a 1-sum $\{\frac{1}{k}\}$-flow.*
2. *If G is not regular and not bipartite then G has a 1-sum $[1-\alpha(G), \alpha(G)]$-flow.*
3. *If G is not regular, but is bipartite and balanced, then G has a 1-$[\lfloor \frac{\alpha(G)}{2} \rfloor, -\lfloor \frac{\alpha(G)}{2} \rfloor]$-flow.*

These bounds are quite far from the actual range for most graphs, since we know that for any fixed r a random graph with minimum degree at least r almost surely has an r-factor [18], and hence a 1-sum $[0, \frac{1}{r}]$-flow.

Assume that a connected balanced bipartite graph G has k disjoint spanning trees. As each spanning tree T of G is a balanced bipartite it has a 1-sum flow ω_T. Then the convex combination of all these flows with the coeffcient $\frac{1}{k}$ gives rise to a 1-sum flow ω on G. In this case, we can reduce the bounds in Corollary 3.4 by a factor of $\frac{1}{k}$. Here we recall that Nash-Williams [14] and Tutte [22] have characterized the graphs which have k disjoint spanning trees, and so their characterization together with Corollary 3.4 give us a collection of graph classes with smaller ranges for their 1-sum flows.

3.4. Nonnegative 1-flows. Let $\Omega_n \subset \mathbb{R}_+^{n \times n}$ be the set of doubly stochastic matrices. That is $A = [a_{ij}]_{i,j=1}^n$ is a nonnegative matrix such that each row and column has sum 1. Denote by $\mathcal{P}_n \subset \Omega_n$ the group of $n \times n$ permutation matrices. Recall the classical result of G. Birkhoff [6], which is also called Birkhoff-von Neumann theorem [15]. Namely, the extreme points of doubly stochastic matrices are the permutation

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matrices.

Let $\Omega_{n,s} \subset \Omega_n$ be the subset of symmetric doubly stochastic matrices. The following result is due to M. Katz [13].

THEOREM 3.6. *Let $\Omega_{n,s}$ be the set of symmetric doubly stochastic matrices. Then $A \in \Omega_{n,s}$ is an extreme point of $\Omega_{n,s}$ if and only if $A = \frac{1}{2}(Q + Q^\top)$ for some permutation matrix $Q \in \mathcal{P}_n$. Equivalently, there exists a permutation matrix $P \in \mathcal{P}_n$ such that $A = PBP^\top$, where $B = \operatorname{diag}(B_1, \ldots, B_t)$ and each B_j is a doubly stochastic symmetric matrix of the following form:*

1. *The 1×1 matrix $[1]$.*
2. *$A(K_2)$.*
3. *$\frac{1}{2}A(C)$, where C is a cycle.*

COROLLARY 3.7. *Let $\Omega_{n,s,0}$ be the set of symmetric doubly stochastic matrices with zero diagonal. Then $A \in \Omega_{n,s,0}$ is an extreme point of $\Omega_{n,s,0}$ if and only if $A = \frac{1}{2}(Q + Q^\top)$ for some permutation matrix $Q \in \mathcal{P}_n$ which does not fix any $i \in [n]$. Equivalently, there exists a permutation matrix $P \in \mathcal{P}_n$ such that $A = PBP^\top$, where $B = \operatorname{diag}(B_1, \ldots, B_t)$ and each B_j is a doubly stochastic symmetric matrix of the forms 2 or 3 given in Theorem 3.6.*

Let H be a simple graph. H is called 1-factor, or perfect matching, if each connected component is K_2. H is called a $\{1, 2\}$-factor if each connected component of H is either K_2 or a cycle. G has a 1-factor, or perfect matching, if G has a spanning subgraph which is a 1-factor. G has a $\{1, 2\}$-factor if G has a spanning subgraph which is a $\{1, 2\}$-factor.

THEOREM 3.8. *Let $G = (V, E)$ be a simple graph. Then G has a 1-$[0, 1]$-flow if and only if one of the following conditions hold:*

1. *G is not bipartite, and G has a $\{1, 2\}$-factor.*
2. *G is bipartite, and G has a 1-factor.*

Furthermore, G has a 1-$(0, 1]$-flow if and only if for each $e \in E$ one of the following conditions holds:

1. *Assume that G is not bipartite. Then there exists a $\{1, 2\}$-factor of G that contains e.*
2. *Assume that G is bipartite. Then there exists a 1-factor of G that contains e.*

Proof. Suppose first that G is not bipartite. Assume that $n = |V|$. View $V = [n] = \{1, \ldots, n\}$ and E as a subset of all pairs $\{i, j\}$, where $i \neq j \in [n]$. Clearly, G has 1-$[0, 1]$ if and only if there exists $C = [c_{ij}]_{i,j=1}^n \in \Omega_{n,s,0}$, such that $c_{ij} = 0$ if $(i, j) \notin E$.

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Corollary 3.7 yields that C is a convex combination of $A = [a_{ij}]$ such that $a_{ij} = 0$ if $\{i,j\} \notin E$. Take such an extreme point. Corollary 3.7 implies that A corresponds to a $\{1,2\}$-factor of G.

Conversely, assume that H is a $\{1,2\}$-factor of G. Let $\omega_H : H \to \{\frac{1}{2}, 1\}$ be the following flow on H. On each edge of the connected component K_2 of H the value of ω_H is 1. On each edge of the cycle in H the value of the edge is $\frac{1}{2}$. Extend this flow to $\hat\omega_H : E \to \{0, \frac{1}{2}, 1\}$ by letting $\hat\omega_H(e) = 0$ for $e \notin E \setminus E(H)$. Note that H induces a unique extremal point $A(H) \in \Omega_{n,s,0}$.

Assume that G has a 1-$[0,1]$-flow. Denote by $\Omega_{n,s,0}(G)$ all the symmetric doubly stochastic matrices corresponding to the 1-$[0,1]$-flow on G. Let $A_1, \ldots, A_M$ be all the extremal points of of $\Omega_{n,s,0}(G)$. So $A_i = A(H_i)$ where H_i is a $\{1,2\}$-factor of G. So any 1-$[0,1]$-flow is a convex combination of $A(H_1), \ldots, A(H_M)$. Suppose there exists a 1-$(0,1]$-flow ω on G. Let $e \in E$. Since $\omega(e) > 0$ it follows that e is contained in some H_i. Vice versa, suppose $H_1, \ldots, H_M$ are all M $\{1,2\}$-factors of G. Assume that each $e \in E$ is contained in some H_i. Consider the 1-flow $\omega = \frac{1}{M}\sum_{i=1}^{M} \hat\omega_{H_i}$. Then ω is a 1-$(0,1]$-flow.

Assume now that G is a bipartite graph. So a 1-$\mathbb{R}$-flow exists if and only if G is balanced bipartite graph $G = (V_1 \cup V_2, E)$. Let $V_1 = \{u_1, \ldots, u_n\}, V_2 = \{v_1, \ldots, v_n\}$. So each edge $e \in E$ is of the form $\{u_i, v_j\}$. Then a 1-$[0,1]$-flow ω on G corresponds to $A = [a_{ij}]_{i,j=1}^n \in \Omega_n$ where $a_{ij} = 0$ if $\{u_i, v_j\} \notin E$. Recall Birkhoff's theorem which shows that $\mathcal{P}_n$ is the set of extreme points on Ω_n.

Assume first that G has a 1-$[0,1]$-flow ω. Then $A \in \Omega_n$ represents ω. So $A = \sum_{j=1}^{M} a_j P_j$ where each $P_j \in \mathcal{P}_n$, $a_j > 0$ and $\sum_{j=1}^{M} a_j = 1$. Hence, each P_j represents a 1-factor of G. Vice versa, assume that G has a 1-factor H. The arguments above imply that $\hat\omega_H$ is a 1-$[0,1]$-flow on G. As in the non-bipartite graph we deduce that G has a 1-$(0,1]$-flow if and only if each edge is covered by some 1-factor of G. ∎

The fundamental works of Tutte give necessary and sufficient conditions for the existence of 1 and $\{1,2\}$-factors [19, 20].

Corollary 3.9. *Let G be a graph and $\delta(G) \geq 2$. If G has no even cycle, then G admits a 1-$[0,1]$-flow.*

Proof. We claim that $G = (V, E)$ has a $\{1,2\}$-factor. We prove this claim by induction on $n = |V(G)|$. For $n = 3$ the claim is trivial. Consider the block decomposition of G. It is well-known that every block of G is K_2 or an odd cycle, see [23]. Now, choose a leaf block of G. Obviously, it is an odd cycle C on $2l+1$ vertices. Suppose first that C has a common vertex v, with another odd cycle C'. Remove all vertices of C except v. The remaining graph G' satisfies the assumption of the corollary. By the induction hypothesis G' has a $\{1,2\}$-factor. The subgraph of C on

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$2l$ vertices has a 1-factor. Hence, G has a $\{1,2\}$-factor.

It is left to discuss the case where the leaf cycle has one vertex v of degree 3 which is common with a K_2-block. Consider the the shortest path, P, between v and another vertex of degree at least 3, say $w \notin V(C)$. Remove all the vertices on C and the path P except the vertex w. The remaining graph $G' = (V', E')$ has a $\{1,2\}$-factor by induction. Consider now the subgraph G_1 of G on the vertices $V \setminus V'$. If the length of path is odd then G_1 has a $\{1,2\}$-factor consisting of C and a matching, where the matching may be empty. If P is even then G_1 has a 1-factor. Hence, G has a $\{1,2\}$-factor. □

3.5. Existence of 1-sum $[-1,1]$-flows on graphs with $\delta(G) \geq 2$. In this section, we consider graphs which admit a 1-sum $[-1,1]$-flow. This class clearly extends the class of graphs which have a 1-sum $[0,1]$-flow, but the inclusion of negative edge weights adds more flexibility. We assume that G is connected graph with the minimal degree $\delta(G)$ at least 2. Lemma 2.2 gives a necessary and sufficient conditions on the existence of these flows but at the moment we do not have a good interpretation of this result in terms of structural properties of the graphs.

However, we can show that not all graphs have a 1-sum $[-1,1]$-flow, and in fact that given an integer t there are graphs of arbitrarily high edge-connectivity which does not have a 1-sum $[-t,\infty)$-flow.

We start out with a simple example and then proceed with the generalization to higher connectivity.

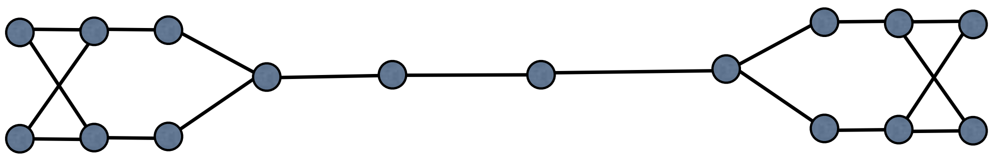

FIG. 3.1. *A bipartite graph with no 1-sum* $[-1,1]$*-flow.*

EXAMPLE 3.10. The graph G on 16 vertices with $\delta(G) = 2$, $\Delta(G) = 3$, which is given in Figure 3.1, does not have 1-sum $[-1,1]$ flow. A direct computation shows that the center edge of G has weight 2 in all 1-sum flows and that the narrowest range is given by a 1-sum $[-1,2]$-flow.

EXAMPLE 3.11. For two positive integers t and s, there is an s-edge connected bipartite graph G which admits a 1-sum $\mathbb{R}$-flow but admits no 1-sum $[-t,\infty)$-flow.

Consider two disjoint copies of $K_{s,s(1+t)+1}$. Call the vertex parts of the first one by (X,Y) and the second one by (X',Y'), where $|X| = |X'| = s$ and $|Y| = |Y'| =$

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$s(1+t)+1$. Choose an arbitrary vertex $v \in X$ and join v to all vertices in X' with the edges $e_1, \ldots, e_s$ and call the resulting graph by G. We claim that G is the desired graph. Clearly, G is an s-edge connected bipartite graph. Note that G is a balanced bipartite graph and so it admits a 1-sum $\mathbb{R}$-flow. By contradiction assume that f is a 1-sum $[-t, \infty)$-flow of G. Then we have

$$s = \sum_{i=1}^{s} s(x_i) = \sum_{1 \le i \le s, 1 \le j \le s(1+t)+1} f(x_i y_j) + \sum_{i=1}^{s} f(e_i) = s(1+t) + 1 + \sum_{i=1}^{s} f(e_i),$$

where $s(x_i)$ is the sum values of all edges incident with x_i. This implies that

$$\sum_{i=1}^{s} f(e_i) = -st - 1.$$

We know that for each i, $f(e_i) \ge -t$, a contradiction.

Problem 3.12. *Characterize the graphs which admit a 1-sum* $[-1,1]$*-flow.*

4. 1-sum L-flows when L is not an interval. As mentioned in the introduction the problem of finding a 1-sum L-flow when L is an interval is a linear programming problem. As soon as L is not an interval we are no longer working with a convex problem and many of the tools we have used so far do not apply. Nonetheless we shall prove some results for two cases of this type. First we will consider the real line with the single point 0 removed, a second we will look at the case when L consist of just a finite list of real numbers.

4.1. 1-$\mathbb{R}^*$-flows. In [4], the following question was proposed.

Question 4.1. *Determine a necessary and sufficient condition under which a bipartite graph admits a* 1*-sum* $\mathbb{R}^*$*-flow or a* 1*-sum* $\mathbb{Z}^*$*-flow.*

In this section, we give an answer to this question.

It is not hard to see that if a graph G admits a 1-sum $\mathbb{Z}$-flow, then the order of G should be even. In [4] it has been proved that a connected bipartite graph admits a 1-sum $\mathbb{R}$-flow if and only if it is balanced.

Theorem 4.2. *Let G be a connected balanced bipartite graph. Then G admits a* 1*-sum* $\mathbb{R}^*$*-flow if and only if there does not exist a cut edge whose removal creates a balanced bipartite connected component.*

Proof. First assume that G admits a 1-sum $\mathbb{R}^*$-flow, say ω, and e is a cut edge its removing makes a balanced bipartite connected component. Call this component by H. Assume that (X, Y) be two vertex parts of H and $|X| = |Y|$. We have

$$|X| = \sum_{v \in X} s(v) = \sum_{v \in Y} s(v) + \omega(e) = |Y| + \omega(e),$$

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where $s(v)$ denotes the sum of the values of all incident edges to v. This implies that $\omega(e) = 0$, a contradiction.

Now, assume that for each cut edge, the removal does not make a balanced bipartite connected component. Let $E(G) = \{e_1, \ldots, e_m\}$ and for $i = 1, \ldots, m$, $W_i \subset \mathbb{R}^m$ is the set of all 0-sum flows of G in which the value of e_i are zero. Let $V \subset \mathbb{R}^m$ be the set of all 0-sum flows of G. Clearly, V and W_i are vector spaces over $\mathbb{R}$. By Theorem 3 of [4], there is a 1-sum $\mathbb{R}$-flow for G.

If $V \not\subset \bigcup_{i=1}^m W_i$, then there exists a 0-sum $\mathbb{R}^*$-flow ω' of G. It is obvious that there exists a suitable real number a such that $\omega + a\omega'$ is a 1-sum $\mathbb{R}^*$-flow and we are done.

Now, assume that there exists $J \subseteq \{1, \ldots, m\}$ such that for every $j \in J$, $V \neq W_j$ and for any $j \in \{1, \ldots, m\} \setminus J$, $V = W_j$. Since $\mathbb{R}$ is infinite, it is well-known that $V \not\subset \bigcup_{j \in J} W_j$. So, there exists a vector $\alpha \in V$, such that the jth component of α is non-zero for every $j \in J$. Now, let $j \in \{1, \ldots, m\} \setminus J$. If e_j is not a cut edge, then it is contained in an even cycle. If we assign 1 and -1 to all edges of this cycle, alternatively and assign 0 to all other edges of G, we obtain a vector in $V \setminus W_j$, a contradiction. Hence, e_j is a cut edge of G. By assumption $G \setminus \{e_j\}$ has a non-balanced bipartite component H. Note that since G is balanced and H is not balanced, the other component, H', is not balanced too. Let $H = (A, B)$ be two vertex parts of H and $|A| < |B|$. Without loss of generality assume that $v \in A$ and e_j is incident with v. Assign 1 to every vertex in $(A \setminus \{v\}) \cup B$ and assign $|B| - |A| + 1$ to v. Then by Theorem 3 of [4], H admits a flow such that $s(v) = |B| - |A| + 1$ and $s(x) = 1$, for each $x \in V(H) \setminus \{v\}$. Similarly, if $H' = (A', B')$ and $|A'| < |B'|$ and e_j is incident with $v' \in A'$, then there exists a flow for H' such that $s(v') = |B'| - |A'| + 1$ and $s(x) = 1$, for each $x \in V(H') \setminus \{v'\}$. Since G is balanced, we have $|A| + |B'| = |A'| + |B|$. This yields that $s(v) = s(v')$. Now, assign $|A| - |B|$ to e_j to obtain a 1-sum flow for e_j. Note that since $V = W_j$, in any 1-sum $\mathbb{R}$-flow of G, the value of e_j should be $|A| - |B|$, which is non-zero. It is not hard to see that there exists a suitable a such that $\omega + a\alpha$ is a 1-sum $\mathbb{R}^*$-flow of G, as desired. □

Let G be a 2-edge connected bipartite graph and $a < 0$ and $b > 0$ be two real numbers and $L = (a, b)$. Then G admits a 1-sum L-flow if and only if G admits a 1-sum L^*-flow. To see this, by Theorem 1 of [2], G has a 0-sum $\mathbb{R}^*$-flow, say ω'. Let ω be a 1-sum L-flow of G. Then if ϵ is small enough, $\omega + \epsilon\omega'$ is a 1-sum $\mathbb{R}^*$-flow of G.

4.2. 1-sum flows with a finite list. We can also let L be a finite list of allowed values, bringing us closer to the situation in the classical study of nowhere-zero flows of graphs [11, 17, 21].

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THEOREM 4.3. *Let k be a positive integer and G be a connected k-regular graph of order n. Then the following hold:*

1. *If k is odd, then G admits a 1-sum $\{-1,0,1\}$-flow.*
2. *If $k \equiv 2\,(mod\,4)$ and n is even, then G admits a 1-sum $\{-1,0,1\}$-flow.*

Proof.

1. First we assign a bipartite graph H to G. Suppose that $V(G) = \{1,\dots,n\}$ and let H be a bipartite graph with two parts $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$. Join x_i and y_j if and only if two vertices i and j are adjacent in G.
   Since H is a k-regular bipartite graph, the edges of H can be decomposed into k, 1-factors, $F_1,\dots,F_k$. Assign $\frac{(-1)^{i-1}}{2}$ to all edges of F_i, $1 \le i \le n$. So, for each vertex $v \in V(H)$, we have $s(v) = \frac{1}{2}$. For two adjacent vertices v_i and v_j in G, assume e_{ij} is the edge between v_i and v_j. Let a_{ij} be the value of the edge x_iy_j, $1 \le i,j \le n$. Assign the value $b_{ij} = a_{ij} + a_{ji}$ to e_{ij}. By our assumption, $b_{ij} \in \{-1,0,1\}$. We have
   $$\sum_{y_j \in N(x_i)} a_{ij} = 0\ , \quad \sum_{x_j \in N(y_i)} a_{ji} = 0.$$
   It is not hard to see that using this assignment we find a 1-sum $\{-1,0,1\}$-flow for G.
2. Since k is even, G is an Eulerian graph and so it is 2-edge connected. Now, by Theorem 3.10, Part (ii) of [5], the edges of G can be decomposed into two spanning $(2k+1)$-regular graphs G_1 and G_2. By Part (i), G_1 has a 1-sum $\{-1,0,1\}$-flow. Now, assign 0 to all edges of G_2. So, G admits a 1-sum $\{-1,0,1\}$-flow, as desired. □

QUESTION 4.4. *Let k be a positive integer divisible by 4. Is it true that every connected k-regular graph of even order admits a 1-sum $\{-1,0,1\}$-flow?*

PROBLEM 4.5. *Characterize the graphs which admit a 1-sum $\{-1,0,1\}$-flow.*

Next we recall the following known results.

THEOREM 4.6.

- *Petersen's Theorem [16]: $G = (V,E)$ is a regular graph of even degree if and only if it is an edge disjoint union of 2-factors.*
- [12] *Let $r \ge 3$ be an odd integer and let k be an integer such that $1 \le k \le \frac{2r}{3}$. Then every r-regular graph has a $[k-1,k]$-factor each component of which is regular.*

THEOREM 4.7. *Let $r \ge 5$ be an odd positive integer. Then every r-regular graph admits a 1-sum $\{-2,-1,1,2\}$-flow. Moreover, every 2-edge connected r-regular graph*

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admits a 1*-sum* $\{-1,1\}$*-flow.*

Proof. First let $r=5$. By Theorem 4.6, G has a $[2,3]$-factor H whose each component is regular. Now, assign -1 to any edge in $E(G)\setminus E(H)$ and 1 to all edges of any 3-regular component and 2 to all edges of any 2-regular component to obtain a 1-sum $\{-2,-1,1,2\}$-flow.

Now, let $r=2t+1\geq 7$. We have $1\leq t+1\leq \frac{2(2t+1)}{3}$. By Theorem 4.6, G has a $[t,t+1]$-factor whose each component is regular. Let H be the union of all t-regular components and K be the union of all $(t+1)$-regular components of G. First assume that t is odd. Assign 1 to all edges in $E(G)\setminus(E(H)\cup E(K))$. Also, Assign -1 to all edges of H. Since $t+1$ is even, by Petersen Theorem, K has a 4-regular factor say L. Since L is a union of two 2-factors, it admits a -2-sum $\{-2,-1,1,2\}$-flow. Now, assign -1 to all edges of $E(K)\setminus E(L)$ to obtain a 1-sum $\{-2,-1,1,2\}$-flow for G. Now, let t be even. Assign -1 to all edges in $E(G)\setminus(E(H)\cup E(K))$. Assign 1 to all edges of K. Since t is even, H has a 2-factor, say L. Assign 1 to all edges in $E(H)\setminus E(L)$ and 2 to all edges in $E(L)$ to obtain a 1-sum $\{-2,-1,1,2\}$-flow for G.

The last part is an immediate consequence of Theorem 8 of [24]. ☐

Theorem 4.8. *Let* $r\geq 3(r\neq 5)$ *be an odd positive integer. Then every* 2*-edge connected* r*-regular graph admits a* 0*-sum* $\{-2,-1,1,2\}$*-flow.*

Proof. Let G be an r-regular graph. We consider three cases:

1. $r=3t+0$. By Theorem 3.10, Part (v) of [12], G has a t-factor. Thus, $E(G)$ can be decomposed into one t-factor and one $2t$-factor. Assign 2 and -1 to each edge of t-factor and $2t$-factor, respectively to obtain a 0-sum $\{-2,-1,1,2\}$-flow for G.
2. $r=3t+1$. Since r is odd, $t+1$ is odd. By Theorem 3.10, Part (v) of [12], G has a $(t+1)$-factor. By Petersen's Theorem $E(G)$ can be decomposed into one $(t+1)$-factor, one $(2t-4)$-factor and two 2-factors F_1 and F_2. Now, assign 2, -1, -2 and -1 to each edge of $(t+1)$-factor, $(2t-4)$-factor, F_1 and F_2, respectively, to obtain a 0-sum $\{-2,-1,1,2\}$-flow for G.
3. $r=3t+2$. Since r is odd, $t+2$ is odd. By Theorem 3.10, Part (v) of [12], G has a $(t+2)$-factor. So by Petersen's Theorem $E(G)$ can be decomposed into one $(t+2)$-factor, one $(2t-4)$-factor and one 4-factor. Now, assign $2,-1$ and -2 to each edge of $(t+2)$-factor, $(2t-4)$-factor and 4-factor, respectively to obtain a 0-sum $\{-2,-1,1,2\}$-flow for G. ☐

Question 4.9. *Does every* 2*-edge connected* 5*-regular graph admit a* 0*-sum* $\{-2,-1,1,2\}$*-flow?*

Given a natural k number, a *c-sum k-flow* is a c-sum flow with values from the

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set $\{\pm 1, \ldots, \pm(k-1)\}$. It is not hard to see if e is a cut edge of a graph G, then in any 0-sum k-flow of G, the value of e should even. Now, let r be an odd positive integer and G be an r-regular graph containing a vertex v such that all edges incident with v is a cut edge. Thus, G does not admit a 0-sum 4-flow. In [4] and [1], it was proved that every r-regular graph ($r \geq 3$) admits a 0-sum 5-flow.

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