

AN ITERATIVE METHOD TO SOLVE A NONLINEAR MATRIX EQUATION*

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Abstract. In this paper, an iterative method to solve one kind of nonlinear matrix equation is discussed. For each initial matrix with some conditions, the matrix sequences generated by the iterative method are shown to lie in a fixed open ball. The matrix sequences generated by the iterative method are shown to converge to the only solution of the nonlinear matrix equation in the fixed closed ball. In addition, the error estimate of the approximate solution in the fixed closed ball, and a numerical example to illustrate the convergence results are given.

Key words. Nonlinear matrix equation, Iterative method, Newton's iterative method, Convergence theorem.

AMS subject classifications. 15A24, 15A39, 65F30.

1. Introduction. We consider the nonlinear matrix equation

$$(1.1) \quad X + A^* X^{-n} A = Q,$$

where Q is an $m \times m$ positive definite matrix, A is an arbitrary real $m \times m$ matrix, and n is a positive integer greater than 1.

The solution of the matrix equation (1.1) is a problem of practical importance. In many physical applications, one must solve the system of the linear equation $Mx = f$, where the positive definite matrix M arises from a finite difference approximation to an elliptic partial differential operator (e.g. see [2]). As an example, let

$$M = \begin{pmatrix} I & A \\ A^* & Q \end{pmatrix}.$$

Then $M = \tilde{M} + \text{diag}(I - X^n, 0)$, where

$$\tilde{M} = \begin{pmatrix} X^n & A \\ A^* & Q \end{pmatrix}.$$

*Received by the editors on March 24, 2015. Accepted for publication on September 13, 2016.
 Handling Editor: Bryan L. Shader. Research was supported by National Natural Science Foundation of China (no. 11261014, no. 11301107).

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If X is a positive solution to (1.1), then the LU decomposition of $\tilde{M}$ is

$$\begin{pmatrix} X^n & A \\ A^* & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ A^*X^{-n} & I \end{pmatrix} \begin{pmatrix} X^n & A \\ 0 & X \end{pmatrix},$$

the system $Mx = f$ is transformed to

$$\left[\begin{pmatrix} I & 0 \\ A^*X^{-n} & I \end{pmatrix} \begin{pmatrix} X^n & A \\ 0 & X \end{pmatrix} + \begin{pmatrix} I - X^n & 0 \\ 0 & 0 \end{pmatrix} \right] x = f,$$

and the Sherman-Morrison-Woodbury formula [25] can be applied to compute the solution of $Mx = f$ directly, or based on the iterative formula

$$\begin{pmatrix} I & 0 \\ A^*X^{-n} & I \end{pmatrix} \begin{pmatrix} X^n & A \\ 0 & X \end{pmatrix} x_{k+1} = f - \begin{pmatrix} I - X^n & 0 \\ 0 & 0 \end{pmatrix} x_k, \quad k = 0, 1, 2, \dots$$

to solve the system $Mx = f$. Another example is in control theory, see the references given in [9].

Several authors have studied the existence, the rate of convergence as well as the necessary and sufficient conditions of the positive definite solutions of (1.1) or similar matrix equations. For example, the fixed point iteration methods to solve (1.1) have been investigated in [6, 7, 14, 26]. The properties of the positive definite solutions of (1.1) and the perturbation analysis of the solutions have been introduced in [23, 27]. Some special cases of (1.1) have been investigated. Fixed point iteration methods to solve (1.1) in cases $n = 2$ and $Q = I$ have been studied in [14, 28, 29, 36]. For the case $n = 1$ or cases $n = 1$ and $Q = I$, fixed point iteration methods are considered in [1, 4, 5, 11, 12, 15, 17, 19, 20, 22, 24, 32]. The inversion free variant of the basic fixed point iteration methods are considered in [10, 16, 38]. Cyclic reduction methods are considered in [18, 19, 32]. Properties of the positive definite solutions of (1.1) and perturbation analysis of the solutions have been discussed in [20, 22, 24]. More general cases have also been analyzed [3, 8, 21, 30, 33, 34, 35, 37].

In this paper, we consider Newton's iterative method to solve the nonlinear matrix equation (1.1). We show, for the given the initial matrix X_0 , that the matrix sequences $\{X_k\}_{k=0}^{\infty}$ generated by the iterative method are included in a fixed open ball $B(X_0, \delta)$. We also show that the $\{X_k\}_{k=0}^{\infty}$ generated by the iterative method converges to the only solution of the nonlinear matrix equation (1.1) in the fixed closed ball $\overline{B}(X_0, \delta)$. In addition, the error estimate of the approximate solution in the fixed closed ball $\overline{B}(X_0, \delta)$ is presented, and a numerical example to illustrate the convergence results is given.

We use $\|A\|_F$ and $\|A\|$ to denote the Frobenius norm and the spectral norm of the matrix A , respectively. The notation $\mathbb{R}^{m \times n}$ and $\mathbb{H}^{n \times n}$ denotes, respectively, the set of all complex $m \times n$ matrices and all $n \times n$ Hermitian matrices. A^* stands for

the conjugate transpose of the matrix A . $A \otimes B$ stands for the Kronecker product of the matrices A and B .

2. Covergence of the iterative methods. In this section, we discuss Newton's iteration method to solve the nonlinear matrix equation (1.1), and show that the iterates converge to the only solution of the nonlinear matrix equation (1.1) in a fixed closed ball.

Let

$$F(X) = X + A^* X^{-n} A - Q,$$

the matrix function $F(X)$ is Frechet-differentiable at the nonsingular matrix X and the Frechet-derivate is given by

$$(2.1) \quad F'_X(E) = E - \sum_{i=1}^n A^* X^{-i} E X^{-n-1+i} A.$$

We have the following lemmas.

LEMMA 2.1. *Let X be a nonsingular with $n\|X^{-1}\|^{n+1}\|A\|^2 < 1$. Then the linear operator F'_X is nonsingular, and*

$$\|(F'_X)^{-1}\| \leq \frac{1}{1 - n\|X^{-1}\|^{n+1}\|A\|^2}.$$

Proof. The inequality

$$\begin{aligned} \|F'_X(E)\| &= \|E - \sum_{i=1}^n A^* X^{-i} E X^{-n-1+i} A\| \\ &\geq \|E\| - \left\| \sum_{i=1}^n A^* X^{-i} E X^{-n-1+i} A \right\| \\ &\geq \|E\| - n\|X^{-1}\|^{n+1}\|A\|^2\|E\| \end{aligned}$$

and the assumption $n\|X^{-1}\|^{n+1}\|A\|^2 < 1$ imply that $F'_X(E) = 0$ if and only if $E = 0$, that is, the operator F'_X is an injection. Since F'_X is a operator on the finite dimension linear space $C^{n \times n}$, F'_X is a surjection. Therefore, F'_X is nonsingular, and

$$\|(F'_X)^{-1}\| = \frac{1}{\min\{\|F'_X(H)\|/\|H\| : H \neq 0\}} \leq \frac{1}{1 - n\|X^{-1}\|^{n+1}\|A\|^2}. \quad \square$$

LEMMA 2.2. *For nonsingular matrices X and Y , we have*

$$(2.2) \quad \|F'_X - F'_Y\| \leq n\|A\|^2 \left(\sum_{i=1}^{n+1} \|X^{-1}\|^{n+2-i} \|Y^{-1}\|^i \right) \|X - Y\|.$$

Proof. For an arbitrary matrix E , we have

$$\begin{aligned}
 \|(F'_X - F'_Y)(E)\| &= \left\| \sum_{i=1}^n A^* X^{-i} E X^{-n-1+i} A - \sum_{i=1}^n A^* Y^{-i} E Y^{-n-1+i} A \right\| \\
 &= \left\| \sum_{i=1}^n A^* X^{-i} E X^{-n-1+i} A - \sum_{i=1}^n A^* X^{-i} E Y^{-n-1+i} A \right. \\
 &\quad \left. + \sum_{i=1}^n A^* X^{-i} E Y^{-n-1+i} A - \sum_{i=1}^n A^* Y^{-i} E Y^{-n-1+i} A \right\| \\
 &= \left\| \sum_{i=1}^n A^* X^{-i} E (X^{-n-1+i} - Y^{-n-1+i}) A + \sum_{i=1}^n A^* (X^{-i} - Y^{-i}) E Y^{-n-1+i} A \right\| \\
 &= \left\| \sum_{i=1}^n A^* X^{-i} E \left[\sum_{k=0}^{n-i} X^{-(n-i-k)} (X^{-1} - Y^{-1}) Y^{-k} \right] A \right. \\
 &\quad \left. + \sum_{i=1}^n A^* \left[\sum_{k=0}^{i-1} X^{-k} (X^{-1} - Y^{-1}) Y^{-(i-1-k)} \right] E Y^{-n-1+i} A \right\| \\
 &= \left\| \sum_{i=1}^n A^* X^{-i} E \left[\sum_{k=0}^{n-i} X^{-(n-i-k+1)} (Y - X) Y^{-k-1} \right] A \right. \\
 &\quad \left. + \sum_{i=1}^n A^* \left[\sum_{k=0}^{i-1} X^{-k-1} (Y - X) Y^{-(i-k)} \right] E Y^{-n-1+i} A \right\| \\
 &\leq \|A\|^2 \sum_{i=1}^n \left(\sum_{k=0}^{n-i} \|X^{-1}\|^{n-k+1} \|Y^{-1}\|^{k+1} \right) \|X - Y\| \|E\| \\
 &\quad + \|A\|^2 \sum_{i=1}^n \left(\sum_{k=0}^{i-1} \|X^{-1}\|^{k+1} \|Y^{-1}\|^{n-k+1} \right) \|X - Y\| \|E\| \\
 &= \|A\|^2 \sum_{i=1}^n \left(\sum_{k=0}^{n-i} \|X^{-1}\|^{n-k+1} \|Y^{-1}\|^{k+1} \right) \|X - Y\| \|E\| \\
 &\quad + \|A\|^2 \sum_{i=1}^n \left(\sum_{k=n-i+1}^n \|X^{-1}\|^{n-k+1} \|Y^{-1}\|^{k+1} \right) \|X - Y\| \|E\| \\
 &= \|A\|^2 \sum_{i=1}^n \left(\sum_{k=0}^n \|X^{-1}\|^{n-k+1} \|Y^{-1}\|^{k+1} \right) \|X - Y\| \|E\| \\
 &= n \|A\|^2 \left(\sum_{i=1}^{n+1} \|X^{-1}\|^{n+2-i} \|Y^{-1}\|^i \right) \|X - Y\| \|E\|.
 \end{aligned}$$

Therefore, (2.2) holds. $\square$

The following lemma from [13, 31] will be useful to prove the next theoretical results.

LEMMA 2.3. *Let $A, B \in \mathbb{R}^{n \times n}$ and assume that A is invertible with $\|A^{-1}\| \leq \alpha$. If $\|A - B\| \leq \beta$ and $\alpha\beta < 1$, then B is also invertible, and $\|B^{-1}\| \leq \alpha/(1 - \alpha\beta)$.*

Applying Newton's method to solve the nonlinear matrix equation (1.1), we have

$$(2.3) \quad X_{k+1} = X_k - (F'_{X_k})^{-1}(F(X_k)), \quad k = 0, 1, 2, \dots$$

or equivalently,

$$(2.4) \quad E_k - \sum_{i=1}^n A^* X_k^{-i} E_k X_k^{-n-1+i} A = -F(X_k),$$

$$(2.5) \quad X_{k+1} = X_k + E_k, \quad k = 0, 1, 2, \dots$$

Letting

$$H(X) = (F'_{X_0})^{-1}(F(X)),$$

we see that the Newton iterates for the matrix function $H(X)$ coincide with those of the matrix function $F(X)$ since

$$(2.6) \quad X_{k+1} - X_k = -(H'_{X_k})^{-1}(H(X_k)) = -(F'_{X_k})^{-1}(F(X_k)).$$

LEMMA 2.4. *Let X_0 be a nonsingular matrix such that*

$$(2.7) \quad 0 < \delta = \frac{(n+1)(\|X_0^{-1}\|^n \|A\|^2 + \|Q - X_0\|)}{1 - n\|X_0^{-1}\|^{n+1} \|A\|^2} < \frac{1 - (n\|X_0^{-1}\|^2 \delta^2)^{1/(n+2)}}{\|X_0^{-1}\|}$$

holds, and let

$$B(X_0, \delta) = \{X \mid \|X - X_0\| < \delta\}.$$

Then

- (i) $\|H(X_0)\| \leq \frac{\delta}{n+1}$;
- (ii) $\|H'_X - H'_Y\| \leq \frac{1}{\delta} \|X - Y\|$ for all $X, Y \in B(X_0, \delta)$;
- (iii) $\|(H'_X)^{-1}\| \leq \frac{1}{1 - \|X - X_0\|/\delta}$ for all $X \in B(X_0, \delta)$;
- (iv) $\|H(X) - H(Y) - H'_Y(X - Y)\| \leq \frac{1}{2\delta} \|X - Y\|^2$ for all $X, Y \in B(X_0, \delta)$.

Proof. (i) By the definition of δ and the estimate obtained by Lemmas 2.1 and 2.2, we have

$$\|H(X_0)\| = \|(F'_{X_0})^{-1}(F(X_0))\|$$

$$\begin{aligned} &\leq \|(F'_{X_0})^{-1}\| \|F(X_0)\| \\ &\leq \frac{1}{1 - n\|X_0^{-1}\|^{n+1}\|A\|^2} \cdot (\|X_0^{-1}\|^n\|A\|^2 + \|Q - X_0\|) \\ &= \frac{\|X_0^{-1}\|^n\|A\|^2 + \|Q - X_0\|}{1 - n\|X_0^{-1}\|^{n+1}\|A\|^2} = \frac{\delta}{n+1}. \end{aligned}$$

(ii) Using the estimates obtained by Lemma 2.1 and 2.2, we have

$$\begin{aligned} \|H'_X - H'_Y\| &= \|(F'_{X_0})^{-1}F'_X - (F'_{X_0})^{-1}F'_Y\| \\ &\leq \|(F'_{X_0})^{-1}\| \|F'_X - F'_Y\| \\ &\leq \frac{1}{1 - n\|X_0^{-1}\|^{n+1}\|A\|^2} \cdot n\|A\|^2 \left(\sum_{i=1}^{n+1} \|X^{-1}\|^{n+2-i} \|Y^{-1}\|^i \right) \|X - Y\|. \end{aligned}$$

Since $\|X - X_0\| < \delta$, $\|Y - X_0\| < \delta$ and $\|X_0^{-1}\|\delta < 1 - (n\|X_0^{-1}\|^2\delta^2)^{1/n+2} < 1$, we have by Lemma 2.3 that X and Y are both nonsingular, and $\|X^{-1}\| \leq \frac{\|X_0^{-1}\|}{1 - \|X_0^{-1}\|\delta}$, $\|Y^{-1}\| \leq \frac{\|X_0^{-1}\|}{1 - \|X_0^{-1}\|\delta}$. Therefore,

$$\begin{aligned} \|H'_X - H'_Y\| &\leq \frac{1}{1 - n\|X_0^{-1}\|^{n+1}\|A\|^2} \cdot \frac{n(n+1)\|X_0^{-1}\|^{n+2}\|A\|^2}{(1 - \|X_0^{-1}\|\delta)^{n+2}} \cdot \|X - Y\| \\ &\leq \frac{1}{1 - n\|X_0^{-1}\|^{n+1}\|A\|^2} \cdot \frac{n(n+1)\|X_0^{-1}\|^2(\|X_0^{-1}\|^n\|A\|^2 + \|Q - X_0\|)}{(1 - \|X_0^{-1}\|\delta)^{n+2}} \cdot \|X - Y\| \\ &= \frac{1}{\delta} \cdot \frac{n\|X_0^{-1}\|^2\delta^2}{(1 - \|X_0^{-1}\|\delta)^{n+2}} \cdot \|X - Y\| \\ &\leq \frac{1}{\delta} \|X - Y\|. \end{aligned}$$

(iii) According to the definition of the matrix function $H(X)$, it is easy to get $\|(H'_{X_0})^{-1}\| = 1$. The estimate (ii) implies that $\|H'_X - H'_{X_0}\| \leq \frac{1}{\delta} \|X - X_0\| < 1$ for all $X \in B(X_0, \delta)$. Thus, we have by Lemma 2.3 that $\|(H'_X)^{-1}\| \leq \frac{1}{1 - \|X - X_0\|/\delta}$ for all $X \in B(X_0, \delta)$.

(iv) Using the Newton-Leibniz formula and (iii), we have

$$\begin{aligned} \|H(X) - H(Y) - H'_Y(X - Y)\| &= \left\| \int_0^1 (H'_{(1-t)Y+tX} - H'_Y)(X - Y) dt \right\| \\ &\leq \|X - Y\| \int_0^1 \|H'_{(1-t)Y+tX} - H'_Y\| dt \\ &\leq \|X - Y\|^2 \int_0^1 t dt = \frac{1}{2\delta} \|X - Y\|^2. \quad \square \end{aligned}$$

LEMMA 2.5. Assume that the initial matrix X_0 satisfies (2.7). Then the Newton's iterates X_k , $k \geq 0$, for the matrix function $H(X)$, and hence for the matrix function

$F(X)$, belong to the open ball $B(X_0, \delta)$, and furthermore,

$$\begin{aligned} \|X_k - X_{k-1}\| &\leq \frac{\delta}{2^{k-1}(n+1)}, \quad \|X_k - X_0\| \leq \delta(1 - \frac{1}{2^{k-1}(n+1)}), \\ \|(H'_{X_k})^{-1}\| &\leq 2^{k-1}(n+1), \quad \|H(X_k)\| \leq \frac{\delta}{2^{2k-1}(n+1)^2}. \end{aligned}$$

hold for all $k \geq 1$.

Proof. First, let us check that the above estimates hold for $k = 1$. Clearly, the point $X_1 = X_0 - (H'_{X_0})^{-1}(H(X_0)) = X_0 - H(X_0)$ is well defined since H'_{X_0} is invertible. Also

$$\|X_1 - X_0\| = \|H(X_0)\| \leq \frac{\delta}{n+1} < \delta(1 - \frac{1}{n+1}),$$

and by (iii) of Lemma 2.4,

$$\|(H'_{X_1})^{-1}\| \leq \frac{1}{1 - \|X_1 - X_0\|/\delta} \leq n+1.$$

By definition of X_1 , and by (iv) of Lemma 2.4 again,

$$\|H(X_1)\| = \|H(X_1) - H(X_0) - H'_{X_0}(X_1 - X_0)\| \leq \frac{1}{2\delta}\|X_1 - X_0\|^2 \leq \frac{\delta}{2(n+1)^2}.$$

Assume that the estimates hold for $k = 1, 2, \dots, m$ for some integer $m \geq 1$. The point $X_{m+1} = X_m - (H'_{X_m})^{-1}(H(X_m))$ is thus well defined since H'_{X_m} is invertible. Moreover, by the induction hypothesis and by estimates of (iii) of Lemma 2.4 (for the third and fourth estimates),

$$\|X_{m+1} - X_m\| \leq \|(H'_{X_m})^{-1}\| \|H(X_m)\| \leq \frac{\delta}{2^m(n+1)},$$

$$\begin{aligned} \|X_{m+1} - X_0\| &\leq \|X_m - X_0\| + \|X_{m+1} - X_m\| \\ &\leq \delta(1 - \frac{1}{2^{m-1}(n+1)}) + \frac{\delta}{2^m(n+1)} \\ &= \delta(1 - \frac{1}{2^m(n+1)}), \end{aligned}$$

$$\|(H'_{X_{m+1}})^{-1}\| \leq \frac{1}{1 - \|X_{m+1} - X_0\|/\delta} \leq 2^m(n+1),$$

$$\begin{aligned} \|H(X_{m+1})\| &= \|H(X_{m+1}) - H(X_m) - H'_{X_m}(X_{m+1} - X_m)\| \\ &\leq \frac{1}{2\delta}\|X_{m+1} - X_m\|^2 \leq \frac{\delta}{2^{2m+1}(n+1)^2}. \end{aligned}$$

Hence, the estimates also hold for $k = m + 1$. $\square$

LEMMA 2.6. Assume that the initial matrix X_0 satisfies (2.7), then the Newton's iterates X_k , $k \geq 0$, converges to a zero $\tilde{X}$ of $H(X)$, and hence of $F(X)$, which belongs to the closed ball $\overline{B(X_0, \delta)}$. Moreover,

$$\|X_k - \tilde{X}\| \leq \frac{\delta}{2^{k-1}(n+1)}$$

for all $k \geq 0$.

Proof. The estimates $\|X_k - X_{k-1}\| \leq \frac{\delta}{2^{k-1}(n+1)}$, $k \geq 1$, established in Lemma 2.5 clearly imply that $\{X_k\}_{k=1}^\infty$ is a Cauchy sequence. Since $X_k \in B(X_0, \delta) \subset \overline{B(X_0, \delta)}$ and $\overline{B(X_0, \delta)}$ is a complete metric space (as a closed subset of the Banach space), there exists $\tilde{X} \in \overline{B(X_0, \delta)}$ such that

$$\tilde{X} = \lim_{k \rightarrow \infty} X_k.$$

Since $\|H(X_k)\| \leq \frac{\delta}{2^{2k-1}(n+1)^2}$, $k \geq 1$, by Lemma 2.5, and $H(X)$ is a continuous function,

$$H(\tilde{X}) = \lim_{k \rightarrow \infty} H(X_k) = 0.$$

Hence, the point $\tilde{X}$ is a zero of $F(X)$.

Given integers $k \geq 1$ and $l \geq 1$, we have, again by Lemma 2.5,

$$\|X_k - X_{k+l}\| \leq \sum_{j=k}^{k+l-1} \|X_{j+1} - X_j\| \leq \sum_{j=k}^{\infty} \frac{\delta}{2^j(n+1)} = \frac{\delta}{2^{k-1}(n+1)},$$

so that, for each $k \geq 1$,

$$\|X_k - \tilde{X}\| = \lim_{k \rightarrow \infty} \|X_k - X_{k+l}\| \leq \frac{\delta}{2^{k-1}(n+1)}. \quad \square$$

LEMMA 2.7. Assume that the initial matrix X_0 satisfies (2.7), then the zero $\tilde{X}$ of $H(X)$, and hence of $F(X)$, in the closed ball $\overline{B(X_0, \delta)}$ unique.

Proof. We first show that, if $\tilde{Z} \in \overline{B(X_0, \delta)}$ and $H(\tilde{Z}) = 0$, then

$$\|X_k - \tilde{Z}\| \leq \frac{\delta}{2^{k-1}(n+1)}$$

for all $k \geq 0$. Clearly, this is true if $k = 0$; so assume that this inequality holds for $k = 1, 2, \dots, m$, for some integer $m \geq 0$. Noting that we can write

$$\begin{aligned} X_{m+1} - \tilde{Z} &= X_m - (H'_{X_m})^{-1}(H(X_m)) - \tilde{Z} \\ &= (H'_{X_m})^{-1}(H(\tilde{Z}) - H(X_m) - H'_{X_m}(\tilde{Z} - X_m)), \end{aligned}$$

we infer from Lemma 2.4 and 2.5, and from induction hypothesis that

$$\|X_{m+1} - \tilde{Z}\| \leq \|(H_{X_m})^{-1}\| \frac{1}{2\delta} \|\tilde{Z} - X_m\|^2 \leq \frac{\delta}{2^m(n+1)}.$$

Hence, the inequality $\|X_k - \tilde{Z}\| \leq \frac{\delta}{2^{k-1}(n+1)}$ holds for all $k \geq 1$. Consequently,

$$\lim_{m \rightarrow \infty} \|X_m - \tilde{Z}\| = \|\tilde{X} - \tilde{Z}\| = 0,$$

which shows that $\tilde{Z} = \tilde{X}$. $\square$

Lemmas 2.5-2.7 now imply the following.

THEOREM 2.8. *Assume that the initial matrix X_0 satisfies (2.7). Then the sequence $\{X_k\}_{k=0}^\infty$ defined by (2.3) is such that $X_k \in B(X_0, \delta)$ for all $k \geq 0$ and converges to a solution $\tilde{X} \in \overline{B(X_0, \delta)}$ of the nonlinear matrix equation (1.1). Moreover, for all $k \geq 0$, $\|X_k - \tilde{X}\| \leq \frac{\delta}{2^{k-1}(n+1)}$, and the point $\tilde{X}$ is the only solution of the nonlinear matrix equation (1.1) in $\overline{B(X_0, \delta)}$.*

3. Numerical examples. In this section, we present a numerical example to illustrate the convergence results of the Newton's method to solve the equation (1.1). Our computational experiments were performed on an IBM ThinkPad of mode T410 with 2.5 GHz and 3.0 RAM. All tests were performed in the MATLAB 7.1 with a 32-bit Windows XP operating system. The following example we consider the matrix equation (1.1) with $n = 2$, and use LSQR_M algorithm from [33] to solve the subproblem (2.4) of the Newton's method. For convenience, we let

$$A_1 = -A^*X_k^{-1}, A_2 = -A^*X_k^{-2}, B_1 = X_k^{-2}A, B_2 = X_k^{-1}A, C = -F(X_k).$$

Then the LSQR_M algorithm to solve the subproblem (2.4), that is,

$$E + A_1EB_1 + A_2EB_2 = C$$

can be described as follows.

LSQR_M:(Algorithm for solving the matrix equation (2.4) with $n = 2$)

- (1) *Initialization.* Set initial matrix $E_0 \in H^{n \times n}$. Compute

$$\begin{aligned} \beta_1 &= \|C - E_0 - A_1E_0B_1 - A_2E_0B_2\|_F, \\ U_1 &= (C - E_0 - A_1E_0B_1 - A_2E_0B_2)/\beta_1, \\ \alpha_1 &= \|U_1 + A_1^T U_1 B_1^T + A_2^T U_1 B_2^T\|_F, \\ V_1 &= (U_1 + A_1^T U_1 B_1^T + A_2^T U_1 B_2^T)/\alpha_1, \\ W_1 &= V_1, \bar{\phi}_1 = \beta_1, \bar{\rho}_1 = \alpha_1. \end{aligned}$$
- (2) *Iteration.* For $i = 1, 2, \dots$, until the stopping criteria have been met

- (a) $\beta_{i+1} = \|V_i + A_1 V_i B_1 + A_2 V_i B_2 - \alpha_i U_i\|_F$, $U_{i+1} = (V_i + A_1 V_i B_1 + A_2 V_i B_2 - \alpha_i U_i) / \beta_{i+1}$,
- (b) $\alpha_{i+1} = \|U_{i+1} + A_1^T U_{i+1} B_1^T + A_2^T U_{i+1} B_2^T - \beta_{i+1} V_i\|_F$, $V_{i+1} = (U_{i+1} + A_1^T U_{i+1} B_1^T + A_2^T U_{i+1} B_2^T - \beta_{i+1} V_i) / \alpha_{i+1}$.
- (c) $\rho_i = (\bar{\rho}_i^2 + \beta_{i+1}^2)^{1/2}$, $c_i = \bar{\rho}_i / \rho_i$, $s_i = \beta_{i+1} / \rho_i$, $\theta_{i+1} = s_i \alpha_{i+1}$, $\bar{\rho}_{i+1} = -c_i \alpha_{i+1}$, $\phi_i = c_i \bar{\phi}_i$, $\bar{\phi}_{i+1} = s_i \bar{\phi}_i$.
- (d) $E_i = E_{i-1} + (\phi_i / \rho_i) W_i$,
- (e) $W_{i+1} = V_{i+1} - (\theta_{i+1} / \rho_i) W_i$.

The stopping criteria is used as $|\phi_{k+1} \alpha_{k+1} c_k| \leq 10^{-10}$ here. Other stopping criteria can also be used, readers can see [33] for details.

EXAMPLE 3.1. Let $n = 2$, and given matrices A and Q be as follows.

$$A = \begin{pmatrix} -1.3963 & 1.9188 & -0.0292 & 0.3194 & 0.1592 & -1.1655 & 2.0658 & -0.1693 \\ 0.7079 & 1.6776 & -0.5023 & 1.6029 & -0.6871 & -0.9641 & 0.7161 & -1.9080 \\ -0.4926 & 1.3365 & -0.3212 & 0.0105 & -0.2489 & 0.6592 & -0.2735 & 1.5914 \\ -0.6207 & 0.3987 & -0.6705 & 1.8185 & -1.7459 & -1.1328 & 0.8301 & -0.8441 \\ -0.7252 & 0.4953 & -0.5459 & 1.4551 & -1.5887 & 0.1873 & -1.1764 & 1.0907 \\ 1.1012 & -1.2551 & 0.6380 & 1.1176 & -0.0156 & 1.7247 & 0.7847 & 0.4714 \\ -2.1087 & -1.4742 & -1.4575 & -0.7771 & 0.4571 & 0.4660 & -0.2668 & 1.1529 \\ 1.8423 & -0.9436 & -0.7286 & -0.9480 & -0.5133 & -0.3008 & 0.8891 & -0.0295 \end{pmatrix},$$

$$Q = \begin{pmatrix} 11.5272 & 3.5007 & 1.8948 & -0.5634 & -0.0616 & -1.8747 & -3.6932 & 0.5252 \\ 3.5007 & 18.5143 & 3.5379 & -1.9406 & -2.7188 & -5.8077 & 2.0738 & -2.2306 \\ 1.8948 & 3.5379 & 10.4091 & -0.8632 & 0.7259 & -1.3282 & -0.1856 & 0.0928 \\ -0.5634 & -1.9406 & -0.8632 & 9.4153 & -0.8946 & 0.4670 & 1.9463 & 1.0822 \\ -0.0616 & -2.7188 & 0.7259 & -0.8946 & 11.5623 & 3.9067 & -2.6642 & 1.8856 \\ -1.8747 & -5.8077 & -1.3282 & 0.4670 & 3.9067 & 24.5212 & -1.6249 & -3.9570 \\ -3.6932 & 2.0738 & -0.1856 & 1.9463 & -2.6642 & -1.6249 & 20.0556 & -2.3762 \\ 0.5252 & -2.2306 & 0.0928 & 1.0822 & 1.8856 & -3.9570 & -2.3762 & 14.8961 \end{pmatrix}.$$

By direct compute, we know that Q is a positive definite matrix, and the following estimates hold:

$$\delta = \frac{(n+1)\|Q^{-1}\|^n \|A\|^2}{1 - n\|Q^{-1}\|^{n+1} \|A\|^2} = 1.7778 < \frac{1 - (n\|Q^{-1}\|^2 \delta^2)^{1/(n+2)}}{\|Q^{-1}\|} = 3.0523.$$

Hence, given matrices A and Q satisfy the condition of Theorem 2.8. Using Newton's method and iterate 4 steps, we have

$$X_4 =$$

$$\begin{pmatrix} 11.3891 & 3.5530 & 1.8764 & -0.5452 & -0.0729 & -1.8990 & -3.6976 & 0.5792 \\ 3.5530 & 18.4520 & 3.5457 & -1.9983 & -2.6767 & -5.7772 & 2.0723 & -2.2291 \\ 1.8764 & 3.5457 & 10.3887 & -0.8389 & 0.6987 & -1.3395 & -0.1831 & 0.0960 \\ -0.5452 & -1.9983 & -0.8389 & 9.2778 & -0.7866 & 0.5225 & 1.9495 & 1.1419 \\ -0.0729 & -2.6767 & 0.6987 & -0.7866 & 11.4663 & 3.8653 & -2.6763 & 1.8484 \\ -1.8990 & -5.7772 & -1.3395 & 0.5225 & 3.8653 & 24.4734 & -1.6015 & -3.9995 \\ -3.6976 & 2.0723 & -0.1831 & 1.9495 & -2.6763 & -1.6015 & 19.9757 & -2.3480 \\ 0.5792 & -2.2291 & 0.0960 & 1.1419 & 1.8484 & -3.9995 & -2.3480 & 14.8018 \end{pmatrix},$$

with $\|X - A^*X^{-2}A - Q\|_F = 3.9450 \times 10^{-12}$, $\|X_4 - Q\| = 0.3142 < \delta = 1.7778$.

4. Conclusions. In this paper, Newton's iterative method to solve the nonlinear matrix equation $X + A^*X^{-n}A = Q$ is discussed. For the given initial matrix X_0 , the results that the matrix sequences $\{X_k\}_{k=0}^\infty$ generated by the iterative method are included in the fixed open ball $B(X_0, \delta)$ (Lemma 2.5) and that the matrix sequences $\{X_k\}_{k=0}^\infty$ generated by the iterative method converges to the only solution of the matrix equation $X + A^*X^{-n}A = Q$ in the fixed closed ball $\overline{B}(X_0, \delta)$ (Lemmas 2.6 and 2.7) are proved. In addition, the error estimate of the approximate solution in the fixed closed ball $\overline{B}(X_0, \delta)$ (Lemma 2.6), and a numerical example to illustrate the convergence results are presented.

The advantage of Newton's iterative method to solve the nonlinear matrix equation $X + A^*X^{-n}A = Q$ are that the fixed closed ball in which the unique solution of the matrix equation included can be determined, the unique solution in the fixed closed ball can be obtained, and the expression of the error estimate of the approximate solution in the fixed closed ball can be given.

Many tests show that, if the initial matrix $X_0 = Q$, the sequence $\{X_k\}_{k=0}^\infty$ defined by (2.3) converges to the maximal positive definite solution X_L of the nonlinear matrix equation (1.1). If the initial matrix $X_0 = (AQ^{-1}A^*)^{1/n}$, the sequence $\{X_k\}_{k=0}^\infty$ defined by (2.3) converges to the minimal positive definite solution X_S of the nonlinear matrix equation (1.1). Here the maximal (minimal) solution X_L (X_S) means that for every positive definite solution Y of the nonlinear matrix equation (1.1) satisfies $X_L \geq Y$ ($Y \geq X_S$). That is, $X_L - Y$ ($Y - X_S$) is a positive definite matrix. Unfortunately these results can not be proved here.

The disadvantage is that the rate of convergence is relatively slower than some existing fixed point iteration methods. This is because the inner iteration, that is, LSQR_M algorithm may need to iterate many times to achieve the required accuracy.

Acknowledgement. The authors thank the anonymous referee for valuable suggestions which helped them to improve this article.

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