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ON THE CHARACTERIZATION OF STRONG LINEARIZATIONS OF REGULAR POLYNOMIAL MATRICES*

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Abstract. In the present note, a new characterization of strong linearizations, corresponding to a given regular polynomial matrix, is presented. A linearization of a regular polynomial matrix is a matrix pencil which captures the finite spectral structure of the original matrix, while a strong linearization is one incorporating its structure at infinity along with the finite one. In this respect, linearizations serve as a tool for the study of spectral problems where polynomial matrices are involved. In view of their applications, many linearization techniques have been developed by several authors in the recent years. In this note, a unifying approach is proposed for the construction of strong linearizations aiming to serve as a bridge between approaches already known in the literature.

Key words. Polynomial matrices, Matrix pencils, Strong linearizations, Parametrization of linearizations.

AMS subject classifications. 15A21, 15A22, 15A54, 93B18.

1. Introduction. Linearizations of polynomial matrices play an important role in the study of problems involving polynomial matrices, which arise naturally in several fields of engineering. For instance, in control theory, analysis and synthesis of a variety of control problems can be addressed and solved using the polynomial matrix framework (see for instance [4, 8, 16, 18]). Mechanical systems are also a good example of a topic where polynomial matrices are involved (see e.g. [12]) and the study of the associated polynomial eigenvalue problem plays a central role. Despite the wide development of polynomial matrix theory, most of the reliable numerical techniques for the solution of such problems are available only for first order polynomial matrices, known in the literature as matrix pencils. A very common workaround to avoid this difficulty, is the reduction of a higher order problem to an equivalent first order one. This essentially involves the reduction of a given polynomial matrix to an "equivalent" matrix pencil, known as a *linearization* of the original one. The key property of a

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linearization is that it preserves certain aspects of the structural invariants of the original matrix, allowing this way its spectral structure to be recovered using well established matrix pencil techniques.

Having as a starting point a generalized, block version of the well known Frobenius companion forms (see [7]), a variety of linearizations of a polynomial matrix can be derived using linearization techniques, a field having received the attention of many authors in the recent years (see [1, 11, 13, 15, 19] and the references therein). In general, a linearization of a polynomial matrix, is a matrix pencil which can be usually constructed by inspection from the coefficients of the polynomial matrix. Particularly, the standalone term "linearization", refers to a matrix pencil preserving only the finite eigenstructure of the the original matrix. However, polynomial matrices possess the unique feature of incorporating eigenstructure at infinity. In many applications, the structure at infinity models important aspects of the systems in which they are involved. It is thus desired this type of structure, to be present in the linearized model. This justifies the requirement for linearizations to preserve both the finite and infinite eigenstructure of the polynomial matrix. Such linearizations are known in the literature as *strong linearizations*. In the present paper, we attempt a unification of existing strong linearization techniques, by generalizing the results presented in [3, 20] for the $2-D$ case.

2. Mathematical background. In what follows $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers respectively, while $\mathbb{F}$ will be used to denote either of them. The ring of polynomials in the indeterminate λ with coefficients from the field $\mathbb{F}$, will be denoted by $\mathbb{F}[\lambda]$ and the corresponding set of polynomial matrices of dimensions $p \times q$, by $\mathbb{F}[\lambda]^{p\times q}$. The following definitions can be found in [18].

DEFINITION 2.1. The *degree* of a polynomial matrix $T(\lambda) \in \mathbb{F}[\lambda]^{p\times q}$, denoted by $\deg T(\lambda)$, is the highest among the degrees of the polynomial entries of $T(\lambda)$.

DEFINITION 2.2. A square polynomial matrix $T(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$, is *regular* if there exists $\lambda_0 \in \mathbb{C}$ such that $\det T(\lambda_0) \neq 0$.

Moreover, if $T(\lambda)$ is regular then $T(\lambda)$ is invertible for almost all $\lambda \in \mathbb{C}$. The *finite eigenv*alues or *zeros* of a regular $T(\lambda)$ are the points $\lambda_i \in \mathbb{C}$, for which $\det T(\lambda_i) = 0$.

DEFINITION 2.3. A square polynomial matrix $T(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$, is *unimodular* if $\det T(\lambda) \neq 0$, for all $\lambda \in \mathbb{C}$.

DEFINITION 2.4. The *reverse* or *dual* of a polynomial matrix $T(\lambda) = \sum_{i=0}^{n} T_i\lambda^i \in \mathbb{F}[\lambda]^{p\times q}$, with $T_n \neq 0$, is given by

$$\mathrm{rev}T(\lambda) = \lambda^n T(\lambda^{-1}) = \sum_{i=0}^{n} T_{n-i}\lambda^i. \tag{2.1}$$

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It can be shown that if $\lambda_0 \neq 0$ is a finite eigenvalue of $\mathrm{rev}T(\lambda)$, then λ_0^{-1} is a finite eigenvalue of $T(\lambda)$. In case $\mathrm{rev}T(\lambda)$ has a zero eigenvalue, the polynomial matrix $T(\lambda)$ is said to have an *infinite eigenvalue*. The algebraic, geometric, and partial multiplicities of the infinite eigenvalue of $T(\lambda)$ are defined to be those of the zero eigenvalue of $\mathrm{rev}T(\lambda)$.

We recall now some facts related to the concept of linearization of a regular polynomial matrix. A linearization is essentially a matrix pencil, that is a first order polynomial matrix, capturing the finite eigenstructure of the polynomial matrix being linearized. Its definition is given below.

Definition 2.5 (Linearization, [7]). Let $T(\lambda) = \sum_{i=0}^{n} T_i\lambda^i \in \mathbb{F}[\lambda]^{p\times p}$, $T_n \neq 0$, be a regular polynomial matrix. A matrix pencil of the form $L(\lambda) = \lambda L_1 + L_0$, where $L_i \in \mathbb{F}^{np\times np}$, $i = 0, 1$, is a *linearization* of $T(\lambda)$, if there exist unimodular matrices $U(\lambda), V(\lambda)$, of appropriate dimensions such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(n-1)p} & 0 \\ 0 & T(\lambda) \end{bmatrix}. \tag{2.2}$$

It is worth noting that under certain assumptions, namely when non-trivial infinite eigenstructure is present in $T(\lambda)$, it is possible to obtain matrix pencils $L(\lambda)$ of dimensions smaller than $np \times np$, satisfying (2.2).

We now focus our attention on the block versions of the well known *first and second Frobenius companion forms*, given by

$$C_1(\lambda) = \lambda \begin{bmatrix} I_p & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_n \end{bmatrix} + \begin{bmatrix} 0 & -I_p & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -I_p \\ T_0 & T_1 & \cdots & T_{n-1} \end{bmatrix}, \tag{2.3}$$

and

$$C_2(\lambda) = \lambda \begin{bmatrix} I_p & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_n \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & T_0 \\ -I_p & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & T_{n-2} \\ 0 & \cdots & -I_p & T_{n-1} \end{bmatrix}, \tag{2.4}$$

respectively. These matrix pencils are known [7] to be linearizations of $T(\lambda)$.

A linearization $L(\lambda)$ and the original polynomial matrix $T(\lambda)$ have identical (up to trivial expansion) finite eigenstructures. In the process of seeking linearizations of

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a polynomial matrix, it is often desired to obtain matrix pencils preserving both the finite and infinite eigenstructures of the original matrix. This is the main feature of strong linearizations introduced below.

DEFINITION 2.6 (Strong linearization, [14]). A linearization $L(\lambda)$ of $T(\lambda)$ is a *strong linearization,* if the matrix pencil $\text{rev}L(\lambda) = L_1 + \lambda L_0$ is a linearization of the polynomial matrix $\text{rev}T(\lambda) = \sum_{i=0}^{n} T_{n-i}\lambda^i$.

Both the first and second Frobenius companion forms are known to be strong linearizations of the polynomial matrix $T(\lambda)$ (see for instance [17]). Furthermore, as a direct consequence of the results in [17], every strong linearization is *strictly equivalent* [6] to the first Frobenius companion form, that is, there exist constant invertible matrices U, V such that

$$UL(\lambda)V = C_1(\lambda). \tag{2.5}$$

Clearly, since a strong linearization $L(\lambda)$ is an ordinary linearization as well, it will preserve the finite eigenstructure of the original polynomial matrix $T(\lambda)$. The preservation of the infinite eigenstructure, is evident from the fact that $\text{rev}L(\lambda)$ is a linearization of $\text{rev}T(\lambda)$ and the zero eigenvalue of $\text{rev}T(\lambda)$, gives rise to the infinite eigenvalue of $T(\lambda)$.

A serious drawback of Definition 2.6 is that in order to check whether a matrix pencil is a strong linearization of a given polynomial matrix, one has to verify that two distinct ordinary linearization definitions are satisfied. A more compact characterization of pairs of polynomial matrices sharing isomorphic finite and infinite elementary divisors structures can be found in [9] and [10], where the notion of divisor equivalence is introduced. However, in the next section we propose a new set of conditions, which turn out to be more convenient for the purpose of characterizing the strong linearizations of a regular polynomial matrix.

We conclude this section by introducing the notation

$$\Lambda_k(\lambda) = \begin{bmatrix} 1 & \lambda & \cdots & \lambda^k \end{bmatrix}^T \otimes I_p, \tag{2.6}$$

for $k = 0, 1, 2, \ldots,$ and

$$E_k = e_k \otimes I_p, \tag{2.7}$$

where e_k, $k = 1, 2, \ldots, n$, is the k-th column vector of the identity matrix I_n and $\otimes$ denotes the Kronecker product.

3. Parametrization of strong linearizations. Our aim is to obtain a characterization of all strong linearizations corresponding to a regular polynomial matrix. In

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order to accomplish this task, we introduce conditions relating a given regular polynomial matrix $T(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ with $\deg T(\lambda) = n$ to a potential strong linearization, that is a matrix pencil $L(\lambda) \in \mathbb{F}[\lambda]^{np\times np}$.

THEOREM 3.1. *The matrix pencil $L(\lambda) = \lambda L_1 + L_0$, where $L_i \in \mathbb{F}^{np\times np}$, is a strong linearization of a regular $T(\lambda) = \sum_{i=0}^{n} T_i\lambda^i \in \mathbb{F}[\lambda]^{p\times p}$, with $T_n \neq 0$, if and only if there exist matrices $B_R(\lambda) = \sum_{i=0}^{n-1} B_{R,i}\lambda^i \in \mathbb{F}[\lambda]^{np\times p}$, with $\deg B_R(\lambda) = n-1$, and $K_R \in \mathbb{F}^{np\times p}$ such that*

1. *$L(\lambda)B_R(\lambda) = K_R T(\lambda)$,*
2. *$V = \begin{bmatrix} B_{R,0} & B_{R,1} & \cdots & B_{R,n-1} \end{bmatrix} \in \mathbb{F}^{np\times np}$ is invertible, and*
3. *$L(\lambda)$ is regular.*

Proof. ($\Rightarrow$) Let $L(\lambda)$ be a strong linearization of $T(\lambda)$. This implies (see [17]) that there exist invertible matrices U, V such that

$$UL(\lambda)V = C_1(\lambda), \tag{3.1}$$

where $C_1(\lambda)$ is the first companion form which is also a strong linearization of $T(\lambda)$. Now, in view of the special form of $C_1(\lambda)$, it is easy to verify that

$$C_1(\lambda)\Lambda_{n-1}(\lambda) = E_n T(\lambda) \tag{3.2}$$

holds. Substituting (3.1) into (3.2), we get

$$UL(\lambda)V\Lambda_{n-1}(\lambda) = E_n T(\lambda). \tag{3.3}$$

Setting $B_R(\lambda) = V\Lambda_{n-1}(\lambda)$ and $K_R = U^{-1}E_n$, it is easy to verify that the matrix V coincides with the one in condition 2 and it is invertible, while condition 1 is a direct consequence of (3.3). Finally, to verify that $L(\lambda)$ is indeed regular, it suffices to note that due to (3.1), $L(\lambda)$ is regular if and only if $C_1(\lambda)$ is regular, which in turn is true if and only if $T(\lambda)$ is regular.

($\Leftarrow$) Assume that there exist matrices $B_R(\lambda), K_R$ described in the statement of the theorem satisfying conditions 1 - 3. Notice that $B_R(\lambda) = V\Lambda_{n-1}(\lambda)$, thus condition 1 may be written in the form

$$L(\lambda)V\Lambda_{n-1}(\lambda) = K_R T(\lambda). \tag{3.4}$$

Define now the matrices

$$N_R(\lambda) = \begin{bmatrix} \lambda I_p & -I_p & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \lambda I_p & -I_p \end{bmatrix} = N_{R,0} + \lambda N_{R,1} \in \mathbb{F}[\lambda]^{(n-1)p\times np}, \tag{3.5}$$

$$\bar{T}_R(\lambda) = \begin{bmatrix} T_0, & \cdots & T_{n-2}, & T_{n-1} + \lambda T_n \end{bmatrix} = \bar{T}_{R,0} + \lambda \bar{T}_{R,1} \in \mathbb{F}[\lambda]^{p\times np}, \tag{3.6}$$

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and note that

$$C_1(\lambda) = \begin{bmatrix} N_R(\lambda) \\ \bar{T}_R(\lambda) \end{bmatrix}, \tag{3.7}$$

$$T(\lambda) = \bar{T}_R(\lambda)\Lambda_{n-1}(\lambda). \tag{3.8}$$

Taking into account (3.8), equation (3.4) becomes

$$L(\lambda)V\Lambda_{n-1}(\lambda) = K_R\bar{T}_R(\lambda)\Lambda_{n-1}(\lambda), \tag{3.9}$$

or

$$\left(L(\lambda)V - K_R\bar{T}_R(\lambda)\right)\Lambda_{n-1}(\lambda) = 0, \tag{3.10}$$

which can be expanded to

$$\begin{bmatrix} L_0V - K_R\bar{T}_{R,0}, & L_1V - K_R\bar{T}_{R,1} \end{bmatrix} \begin{bmatrix} E_1 & E_2 & \cdots & E_n & 0 \\ 0 & E_1 & E_2 & \cdots & E_n \end{bmatrix} = 0. \tag{3.11}$$

It is easy to verify that the rows of the the matrix $\begin{bmatrix} N_{R,0}, & N_{R,1} \end{bmatrix}$ form a basis for the left null space of $\begin{bmatrix} E_1 & E_2 & \cdots & E_n & 0 \\ 0 & E_1 & E_2 & \cdots & E_n \end{bmatrix}$, and thus,

$$\begin{bmatrix} L_0V - K_R\bar{T}_{R,0}, & L_1V - K_R\bar{T}_{R,1} \end{bmatrix} = X_R \begin{bmatrix} N_{R,0} & N_{R,1} \end{bmatrix} \tag{3.12}$$

for some $X_R \in \mathbb{F}^{np\times(n-1)p}$. Rewriting the above equation in pencil form, reads

$$L(\lambda)V = X_RN_R(\lambda) + K_R\bar{T}_R(\lambda), \tag{3.13}$$

or

$$L(\lambda)V = \begin{bmatrix} X_R, & K_R \end{bmatrix} \begin{bmatrix} N_R(\lambda) \\ \bar{T}_R(\lambda) \end{bmatrix}. \tag{3.14}$$

Setting $U = \begin{bmatrix} X_R & K_R \end{bmatrix}$ while taking into account (3.7), we may write (3.14) in the form

$$L(\lambda)V = UC_1(\lambda). \tag{3.15}$$

We show now that U is invertible. Indeed, if $x \in \mathbb{F}^{np}$ such that $x^TU = 0$, then premultiplying both sides of (3.15) would give $x^TL(\lambda)V = 0$, for all $\lambda \in \mathbb{C}$, which in turn due to the invertibilty of V (Condition 2), would give $x^TL(\lambda) = 0$, for all $\lambda \in \mathbb{C}$. This last assertion contradicts the regularity assumption of $L(\lambda)$ (Condition 3). Hence, $x = 0$, or equivalently, U is invertible.

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Thus, in view of (3.15), $L(\lambda)$ is strictly equivalent to $C_1(\lambda)$ and hence a strong linearization of $T(\lambda)$. □

A dual version of Theorem 3.1 also holds. Its statement is as follows.

THEOREM 3.2. *The matrix pencil $L(\lambda) = \lambda L_1 + L_0$, where $L_i \in \mathbb{F}^{np\times np}$, is a strong linearization of a regular $T(\lambda) = \sum_{i=0}^{n} T_i\lambda^i$, with $T_n \neq 0$, if and only if there exist matrices $B_L(\lambda) = \sum_{i=0}^{n-1} B_{L,i}\lambda^i \in \mathbb{F}[\lambda]^{p\times np}$, with $\deg B_L(\lambda) = n-1$, and $K_L \in \mathbb{F}^{p\times np}$ such that*

1. *$B_L(\lambda)L(\lambda) = T(\lambda)K_L$,*
2. *$U = \begin{bmatrix} B_{L,0}^T & B_{L,1}^T & \cdots & B_{L,n-1}^T \end{bmatrix}^T \in \mathbb{F}^{np\times np}$ is invertible, and*
3. *$L(\lambda)$ is regular.*

Proof. The proof is similar to that of Theorem 3.1. □

We explore now the relation between the parametrization of linearizations derived from Theorems 3.1 and 3.2 and the vector spaces of linearizations proposed in [15]. The vector spaces $\mathbb{L}_1(T)$, $\mathbb{L}_2(T)$ associated to the polynomial matrix $T(\lambda)$, defined in [15] (with slightly modified notation in order to conform to the current setup), as follows:

$$\mathbb{L}_1(T) = \{L(\lambda) : L(\lambda)\bar{\Lambda}_{n-1}(\lambda) = v \otimes T(\lambda), v \in \mathbb{R}^n\}, \tag{3.16}$$

$$\mathbb{L}_2(T) = \{L(\lambda) : \bar{\Lambda}_{n-1}^T(\lambda)L(\lambda) = w^T \otimes T(\lambda), w \in \mathbb{R}^n\}, \tag{3.17}$$

where $\bar{\Lambda}_k(\lambda) = \mathrm{rev}\Lambda_k(\lambda)$. The vectors v, w are referred to as “right ansatz” and “left ansatz” vectors. Provided that $T(\lambda)$ is regular, it is shown in [15] that almost all pencils $L(\lambda)$, in $\mathbb{L}_1(T)$ and $\mathbb{L}_2(T)$ are linearizations. Particularly, it has been shown that if $T(\lambda)$ is regular, a pencil $L(\lambda) \in \mathbb{L}_i(T)$, $i = 1, 2$, is a strong linearization if and only if $L(\lambda)$ is regular as well.

The vector spaces of linearizations introduced in [15] can be derived as special cases from Theorems 3.1 and 3.2 presented above. It is easy to see that $L(\lambda) \in \mathbb{L}_1(T)$ if and only if conditions 1 and 2 of Theorem 3.1 are met by setting $B_R(\lambda) = \bar{\Lambda}_{n-1}(\lambda)$ and $K_R = v \otimes I_p$. Moreover, if a particular $L(\lambda) \in \mathbb{L}_1(T)$ is regular, that is, if it satisfies condition 3 of Theorem 3.1 as well, then it is a strong linearization of $T(\lambda)$, a result that is in full accordance with the characterization given in [15]. Similar observations can be made about the derivation of the members of $\mathbb{L}_2(T)$ through the use of Theorem 3.2.

Another interesting family of strong linearizations is the one of generalized Fiedler linearizations introduced in [1] as an extension of the construction of companion

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matrices for scalar polynomials, presented by M. Fiedler in [5], to the polynomial matrix case. Generalized Fiedler linearizations can be derived using the technique proposed above as shown in the following example.

EXAMPLE 3.3. Let $T(\lambda)$ be a regular polynomial matrix for $n = 5$. The strong Fiedler linearization corresponding to the index permutation $\mathcal{I} = (3, 4, 1, 2, 0)$ (see [1])

$$L_{\mathcal{I}}(\lambda) = \begin{bmatrix} \lambda I_p & 0 & -I_p & 0 & 0 \\ T_0 & \lambda I_p & T_1 & 0 & 0 \\ 0 & 0 & \lambda I_p & 0 & -I_p \\ 0 & -I_p & T_2 & \lambda I_p & T_3 \\ 0 & 0 & 0 & -I_p & T_4 + \lambda T_5 \end{bmatrix}, \tag{3.18}$$

which is regular, can be recovered using Theorem 3.1, by setting

$$B_R(\lambda) = \begin{bmatrix} (\mu - \lambda^2) I_p \\ \mu(T_2\lambda + T_3\lambda^2 + T_4\lambda^3 + T_5\lambda^4) + T_0\lambda + T_1\lambda^2 \\ (\lambda\mu - \lambda^3) I_p \\ \mu(T_4\lambda^2 + T_5\lambda^3) + T_0 + T_1\lambda + T_2\lambda^2 + T_3\lambda^3 \\ (\lambda^2\mu - \lambda^4) I_p \end{bmatrix}, \tag{3.19}$$

for some $\mu \in \mathbb{F}$ and $K_R = \begin{bmatrix} 0 & \mu I_p & 0 & 0 & -I_p \end{bmatrix}^T$. The matrix V corresponding to the above choice of $B_R(\lambda)$ in condition 2 of Theorem 3.1, can be shown to be invertible for almost every[1] $\mu \in \mathbb{F}$.

Many other choices of $B_R(\lambda)$, beyond the ones shown above are possible. We shall only illustrate this possibility by providing an indicative example. However, a more systematic approach for the derivation of more families of matrix pencils would be desirable. Such a perspective is currently under investigation by the authors.

EXAMPLE 3.4. Let $T(\lambda)$ be a regular polynomial matrix for $n = 5$, and assume further that T_2 is invertible. Now let

$$B_R(\lambda) = \begin{bmatrix} I_p & \lambda I_p & \lambda^2 T_2^T & \lambda^3 I_p & \lambda^4 I_p \end{bmatrix}^T, \tag{3.20}$$

$$K_R = \begin{bmatrix} 0 & 0 & 0 & 0 & I_p \end{bmatrix}^T. \tag{3.21}$$

[1]It can be shown that $V = -V_1(\sqrt{\mu})V_2(-\sqrt{\mu})$, where

$$V_1(\lambda) = \begin{bmatrix} \lambda I_p & 0 & -I_p & 0 & 0 \\ T_0 & \lambda I_p & T_1 & 0 & 0 \\ 0 & 0 & \lambda I_p & -I_p & 0 \\ 0 & -I_p & T_2 & \lambda T_4 + T_3 & \lambda T_5 \\ 0 & 0 & 0 & \lambda I_p & -I_p \end{bmatrix}, \quad V_2(\lambda) = \begin{bmatrix} \lambda I_p & -I_p & 0 & 0 & 0 \\ T_0 & T_1 + \lambda T_2 & \lambda T_3 & \lambda T_4 & \lambda T_5 \\ 0 & \lambda I_p & -I_p & 0 & 0 \\ 0 & 0 & \lambda I_p & -I_p & 0 \\ 0 & 0 & 0 & \lambda I_p & -I_p \end{bmatrix}$$

are themselves Fiedler linearizations of $T(\lambda)$ (see [1]). As long as μ is chosen such that $\pm\sqrt{\mu}$ avoid the eigenvalues of $T(\lambda)$, V is nonsingular.

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The matrix V corresponding to the above choice of $B_R(\lambda)$, in condition 2 of Theorem 3.1, is clearly invertible. With this setup, Theorem 3.1 implies that the matrix pencil

$$L(\lambda) = \begin{bmatrix} \lambda I_p & -I_p & 0 & 0 & 0 \\ 0 & \lambda T_2 & -I_p & 0 & 0 \\ 0 & 0 & \lambda I_p & -T_2 & 0 \\ 0 & 0 & 0 & \lambda I_p & -I_p \\ T_0 & T_1 & I_p & T_3 & T_4 + \lambda T_5 \end{bmatrix} \tag{3.22}$$

is a strong linearization of $T(\lambda)$.

Applying column or row shifted sums (see [15]) on the coefficient matrices of the pencil in the above example, it is easy to verify the latter is neither a member of $\mathbb{L}_1(T)$ nor $\mathbb{L}_2(T)$. Furthermore, it can be easily checked that the above pencil is not a member of the families introduced in [1] and [19].

4. Conclusions. A unified approach to the problem of characterizing and parameterizing all strong linearizations of a given polynomial matrix has been proposed. Since the question of finding strong linearizations of a polynomial matrix involves the preservation of both finite and infinite eigenstructures, a set of necessary and sufficient conditions has been derived to accomplish this task. This type of conditions, is shown to provide a framework wide enough to characterize and parametrize all strong linearizations of a given regular polynomial matrix.

Existing families of strong linearizations such as those presented in [15] fit naturally in the proposed framework. However, as shown in Example 3.3 the derivation of the matrices involved in the parametrization of generalized Fielder pencils [1] is rather complicated. A systematic method for the derivation of all types of generalized Fiedler pencils (see [1, 2, 19]) is under investigation by the authors.

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