Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

LINEAR PRESERVERS OF HADAMARD MAJORIZATION*

SARA M. MOTLAGHIAN†, ALI ARMANDNEJAD†, AND FRANK J. HALL‡

Abstract. Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $D = [d_{ij}] \in \mathbf{M}_n$ with nonnegative entries is called doubly stochastic if $\sum_{k=1}^{n} d_{ik} = \sum_{k=1}^{n} d_{kj} = 1$ for all $1 \leq i, j \leq n$. For $X, Y \in \mathbf{M}_n$, it is said that X is Hadamard-majorized by Y, denoted by $X \prec_H Y$, if there exists an $n \times n$ doubly stochastic matrix D such that $X = D \circ Y$. In this paper, some properties of $\prec_H$ on $\mathbf{M}_n$ are first obtained, and then, the (strong) linear preservers of $\prec_H$ on $\mathbf{M}_n$ are characterized. For $n \geq 3$, it is shown that the strong linear preservers of Hadamard majorization on $\mathbf{M}_n$ are precisely the invertible linear maps on $\mathbf{M}_n$ which preserve the set of matrices of term rank 1. An interesting graph theoretic connection to the linear preservers of Hadamard majorization is exhibited. A number of examples are also provided in the paper.

Key words. Linear preserver, Hadamard majorization, Doubly stochastic matrix.

AMS subject classifications. 15A04, 15A21.

1. Introduction and preliminaries. Majorization is one of the vital topics in mathematics and statistics. It plays a basic role in matrix theory. For instance, the classical majorization relation involving eigenvalues and singular values of matrices produces many norm inequalities, see for example [1], [6] and [11]. For $X, Y \in \mathbf{M}_{n,m}$ (the set of $n \times m$ real matrices), it is said that X is *multivariate majorized* by Y, if there exists a doubly stochastic matrix $D \in \mathbf{M}_n$ such that $X = DY$. In [9] and [10], the authors obtained the following interesting theorems regarding the multivariate majorization. In these results, J denotes the matrix of order n all of whose entries are 1.

THEOREM 1.1. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear map. Then T preserves multivariate majorization if and only if one of the following statements holds.*

(i) *There exist $A_1, \ldots, A_m \in \mathbf{M}_{n,m}$, such that $T(X) = \sum_{j=1}^{m} \mathrm{tr}(x_j) A_j$, where x_j is j^{th} column of X and $\mathrm{tr}(x_j)$ is the summation of all components of x_j.*

(ii) *There exist $R, S \in \mathbf{M}_m$ and permutation matrix $P \in \mathbf{M}_n$ such that $T(X) = PXR + JXS$.*

*Received by the editors on March 11, 2016. Accepted for publication on August 28, 2016. Handling Editor: Bryan L. Shader.

†Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O. Box 7713936417, Rafsanjan, Iran (motlaghian@vru.ac.ir, armandnejad@vru.ac.ir).

‡Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA (fhall@gsu.edu).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

We remark that an equivalent version of Theorem 1.1 was given in Theorem 2.5 of [2]. We also mention that in [3] the authors characterize (considering only row stochastic matrices) the linear operators that strongly preserve matrix majorization, a generalization of multivariate majorization.

THEOREM 1.2. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear map. Then T strongly preserves multivariate majorization if and only if there exist $R, S \in \mathbf{M}_m$ and permutation matrix $P \in \mathbf{M}_n$ such that $R(R + nS)$ is invertible and $T(X) = PXR + JXS$.*

Due to the important role of the Hadamard product in the space of matrices, in this paper, we define a new kind of majorization which is obtained by the Hadamard product and we find its linear preservers.

The following conventions will be fixed throughout the paper:

$\mathbf{M}_n$ is the set of all $n \times n$ real matrices; E_{ij} is the $n \times n$ matrix whose (i, j) entry is one and all other entries are zero; $\mathbb{N}_k$ is the set $\{1, \ldots, k\}$; For index sets $\alpha, \beta \subseteq \mathbb{N}_k$, $A[\alpha, \beta]$ is the submatrix that lies in the rows of A indexed by α and the columns indexed by β; $|\alpha|$ is the cardinal number of a set α; $X^\top$ is the transpose of a given matrix X.

For $X = [x_{ij}], Y = [y_{ij}] \in \mathbf{M}_n$, *the Hadamard product* (entry-wise product) of X and Y is defined by:

$$X \circ Y = Z = [z_{ij}], \quad \text{where } z_{ij} = x_{ij} y_{ij} \text{ for all } 1 \le i, j \le n.$$

It is an interesting fact that $\mathbf{M}_n$ via the Euclidean norm and the Hadamard product forms a commutative Banach algebra.

DEFINITION 1.3. For $X, Y \in \mathbf{M}_n$, we say that X is *Hadamard-majorized* by Y, denoted by $X \prec_H Y$, if there exists a doubly stochastic matrix $D \in \mathbf{M}_n$ such that $X = D \circ Y$.

DEFINITION 1.4. Let $T : \mathbf{M}_n \to \mathbf{M}_n$ be a linear map. We say that T *preserves* (resp., *strongly preserves*) *Hadamard majorization* if $T(X) \prec_H T(Y)$ whenever $X \prec_H Y$ (resp., $T(X) \prec_H T(Y)$ if and only if $X \prec_H Y$).

Some recent works on linear preservers of majorization can be found in [8] and [12]. Notice that when $n = 1$ the relation $\prec_H$ is simply the equality relation. So we may suppose that $n \ge 2$. Since the case $n = 2$ is different from the other cases where $n \ge 3$, in this paper, first we find the (strong) linear preservers of $\prec_H$ on $\mathbf{M}_2$ and then we find the strong linear preservers of $\prec_H$ on $\mathbf{M}_n$ with $n \ge 3$. Comparing with a result of Beasley and Pullman [4], we also show the surprising result that for $n \ge 3$, the strong linear preservers of Hadamard majorization on $\mathbf{M}_n$ are precisely the invertible linear maps on $\mathbf{M}_n$ which preserve the set of matrices of term rank 1. For $n \ge 3$, we find conditions equivalent to a linear map preserving Hadamard

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

majorization. Utilizing this result, we obtain a simplification for the strong linear preservers. In addition, we exhibit an interesting graph theoretic connection to the linear preservers of Hadamard majorization.

2. General properties and strong linear preservers of $\prec_H$. In this section, we first obtain some properties of Hadamard majorization and we present some linear maps on $\mathbf{M}_n$ which preserve $\prec_H$. For a matrix $A \in \mathbf{M}_n$ and a permutation matrix $P \in \mathbf{M}_n$, it is said that A is *dominated by* P if $P \circ A = A$.

The following proposition gives some properties of Hadamard majorization on $\mathbf{M}_n$.

PROPOSITION 2.1. *If $A, B, C, D \in \boldsymbol{M}_n$, then the following statements hold:*

(i) *$A \prec_H A$ if and only if A is dominated by a permutation matrix.*
(ii) *$A \prec_H B$ and $B \prec_H A$ if and only if $A = B$ and A is dominated by a permutation matrix.*
(iii) *$A \prec_H B$ and $B \prec_H C$ do not always imply that $A \prec_H C$ when $n \geq 2$.*
(iv) *$A(B \circ C)D$ may not be equal to $(ABD) \circ (ACD)$, but for permutation matrices P, Q we have $P(B \circ C)Q = (PBQ) \circ (PCQ)$.*

Proof. (i) If $A \prec_H A$, then there exists a doubly stochastic matrix D such that $A = D \circ A$. This implies that $d_{ij} = 1$ whenever $a_{ij} \neq 0$, and hence, A is dominated by a permutation matrix.

(ii) If $A \prec_H B$ and $B \prec_H A$, then $a_{ij} \neq 0$ if and only if $b_{ij} \neq 0$, and in this case, $d_{ij} = 1$. Therefore, $A = B$ is dominated by a permutation matrix. By using (i), the converse is clear.

(iii) For $n \geq 2$, with $A = \frac{1}{n^2}J$, $B = \frac{1}{n}J$ and $C = J$, we have $A \prec_H B$ and $B \prec_H C$ but $A \not\prec_H C$.

(iv) It is easy to see. □

The following examples give some kinds of linear maps on $\mathbf{M}_n$ that preserve Hadamard majorization.

EXAMPLE 2.2. The linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ defined by $T(X) = X^\top$ for all $X \in \mathbf{M}_n$, strongly preserves Hadamard majorization.

EXAMPLE 2.3. Let $P, Q \in \mathbf{M}_n$ be permutation matrices. The linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ defined by $T(X) = PXQ$ for all $X \in \mathbf{M}_n$, strongly preserves Hadamard majorization.

EXAMPLE 2.4. Let $A \in \mathbf{M}_n$. The linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ defined by $T(X) = X \circ A$ for all $X \in \mathbf{M}_n$, preserves Hadamard majorization. Furthermore, T

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

strongly preserves Hadamard majorization if and only of A has no zero entries (see Theorem 2.10).

Later we show that every $T : \mathbf{M}_n \to \mathbf{M}_n$ which strongly preserves Hadamard majorization is a composition of linear maps appearing in the three previous examples (where A has no zero entries), but the following example shows that not every linear map preserving Hadamard majorization is necessarily such a composition.

EXAMPLE 2.5. Let $n \geq 2$. The linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ defined by

$$T\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & & 0 \\ & \ddots & \\ 0 & & x_{11} \end{pmatrix}, \quad \forall\, X \in \mathbf{M}_n,$$

preserves Hadamard majorization but it is not a composition of linear maps appearing in the three previous examples.

THEOREM 2.6. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then the following conditions are equivalent:*

(i) *$T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \leq p, q, r, s \leq n$ with $(p, q) \neq (r, s)$.*
(ii) *There exist a function $f : \mathbb{N}_n \times \mathbb{N}_n \to \mathbb{N}_n \times \mathbb{N}_n$ and a matrix $A \in \boldsymbol{M}_n$ such that for every $X = [x_{i,j}] \in \boldsymbol{M}_n$,*

$$T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \ddots & \vdots \\ x_{f(n,1)} \cdots & x_{f(n,n)} \end{pmatrix} \circ A,$$

where $x_{f(i,j)}$ means x_{pq} if $f(i, j) = (p, q)$.

Proof. For every $1 \leq p, q \leq n$, let $I_{pq} = \{(i, j) : (T(E_{pq}))_{ij} \neq 0\}$.

$(i) \Rightarrow (ii)$. From the assumption (i), it is clear that $I_{pq} \cap I_{rs} = \emptyset$, for all $1 \leq p, q, r, s \leq n$ with $(p, q) \neq (r, s)$. Let $A = T(J)$ where $J \in \mathbf{M}_n$ is as before the matrix whose all entries are 1. Define the function $f : \mathbb{N}_n \times \mathbb{N}_n \to \mathbb{N}_n \times \mathbb{N}_n$ by

$$f(i, j) = \begin{cases} (p, q), & \text{if} \quad (i, j) \in I_{pq}; \\ (i, j), & \text{otherwise.} \end{cases}$$

For every $1 \leq p, q \leq n$, let $J_{pq} \in \mathbf{M}_n$ be the matrix whose (i, j) entry is 1 if $(i, j) \in I_{pq}$ and 0 otherwise. It is easy to see that $T(E_{ij}) = J_{ij} \circ T(J)$, and hence, for $X = [x_{ij}]$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

we have

$$
\begin{aligned}
T(X) &= \sum_{i,j} x_{ij} T(E_{ij}) \;=\; \sum_{i,j} x_{ij} (J_{ij} \circ T(J)) \\
&= \left(\sum_{i,j} x_{ij} J_{ij} \right) \circ A = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \ddots & \vdots \\ x_{f(m,1)} \cdots & x_{f(m,n)} \end{pmatrix} \circ A,
\end{aligned}
$$

as desired.

$(ii) \Rightarrow (i)$. Assuming (ii), observe that for every $1 \leq i, j, k, l \leq n$, $(T(E_{kl}))_{ij} \neq 0$ if and only if $f(i,j) = (k,l)$ and $a_{ij} \neq 0$. Therefore, $T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \leq p, q, r, s \leq n$ with $(p,q) \neq (r,s)$. ☐

For $X = [x_{ij}], Y = [y_{ij}] \in \mathbf{M}_n$, by $< X, Y >$ we mean the usual inner product on $\mathbf{M}_n$, i.e., $< X, Y >= \sum_{i,j=1}^{n} x_{ij} y_{ij}$.

THEOREM 2.7. *Let $n \geq 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. If T preserves Hadamard majorization then there exist a function $f : \mathbb{N}_n \times \mathbb{N}_n \to \mathbb{N}_n \times \mathbb{N}_n$ and a matrix $A \in \boldsymbol{M}_n$ such that for every $X = [x_{i,j}] \in \boldsymbol{M}_n$,*

$$
T(X) = \begin{pmatrix} x_{f(1,1)} \cdots & x_{f(1,n)} \\ \vdots \quad \ddots & \vdots \\ x_{f(n,1)} \cdots & x_{f(n,n)} \end{pmatrix} \circ A, \tag{2.1}
$$

where $x_{f(i,j)}$ means x_{pq} if $f(i,j) = (p,q)$. Furthermore, T is invertible if and only if f is bijective and A has no zero entry.

Proof. By Theorem 2.6, it is enough to show that $T(E_{pq}) \circ T(E_{rs}) = 0$ for every $1 \leq p, q, r, s \leq n$ with $(p,q) \neq (r,s)$. Assume if possible that there exist some $(p,q) \neq (r,s)$ such that $T(E_{pq}) \circ T(E_{rs}) \neq 0$. By the use of Example 2.3, without loss of generality, we may assume that $< T(E_{pq}), E_{11} >= \lambda$ and $< T(E_{rs}), E_{11} >= \mu$ for some nonzero scalars $\lambda, \mu \in \mathbb{R}$. Since $n \geq 3$, there exists a doubly stochastic matrix $D \in \mathbf{M}_n$ such that the (p,q) and the (r,s) entries of D are $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Let $Y = \frac{1}{\lambda} E_{pq} - \frac{1}{\mu} E_{rs}$ and $X = D \circ Y$. Therefore, $X \prec_H Y$ but $T(X) \not\prec_H T(Y)$ which is a contradiction. Moreover, it is clear that T is invertible if and only if f is bijective and A has no zero entry. ☐

The following example shows that the condition $n \geq 3$ is necessary in the previous theorem.

EXAMPLE 2.8. The linear map $T : \mathbf{M}_2 \to \mathbf{M}_2$ defined by $T \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + x_{22} & x_{12} + x_{21} \\ x_{21} + x_{12} & x_{22} + x_{11} \end{pmatrix}$, for all $X \in \mathbf{M}_2$, preserves Hadamard majorization but is

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

not of the form (2.1).

Lemma 2.9. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map that preserves Hadamard majorization. Then the following statements hold:*

(i) *For every $1 \le p, q \le n$, $T(E_{pq})$ is dominated by a permutation matrix.*
(ii) *If $n \ge 3$, for every $1 \le p, q, r, s \le n$ with $p \ne r$ and $q \ne s$, $T(E_{pq})$ and $T(E_{rs})$ do not simultaneously have a nonzero entry in any row and in any column.*

Proof. (i) For every $1 \le p, q \le n$, $E_{pq} \prec_H E_{pq}$, and hence, $T(E_{pq}) \prec_H T(E_{pq})$. This implies that $T(E_{pq})$ is dominated by a permutation matrix by Proposition 2.1.

(ii) For arbitrary but fixed $1 \le p, q, r, s \le n$ with $p \ne r$ and $q \ne s$, let $A = [a_{ij}] = T(E_{pq})$ and $B = [b_{ij}] = T(E_{rs})$. We show that A and B do not simultaneously have a nonzero entry in any row and in any column. If $A = 0$ or $B = 0$ there is nothing to prove. Let $A \ne 0$. By the use of Example 2.3, without loss of generality, we may assume that $a_{11} \ne 0$. We show that the first row and the first column of B are zero. Assume if possible that $b_{1j} \ne 0$ for some $1 \le j \le n$. So $a_{1j} = 0$ by Theorem 2.6 and Theorem 2.7. Let $E = E_{pq} + E_{rs}$. Since $p \ne r$ and $q \ne s$, E is dominated by a permutation matrix, and hence, $E \prec_H E$. So $T(E) = A + B \prec_H T(E) = A + B$, and hence, $A + B$ is dominated by a permutation matrix which is a contradiction. Consequently the first row of B is a zero row. Similarly, we can show that the first column of B is a zero column. ∎

Theorem 2.10. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. If T strongly preserves Hadamard majorization, then T is invertible*

Proof. Assume that there exists a matrix $X \in \mathbf{M}_n$ such that $T(X) = 0$. So $T(X) = T(0) = 0 = I \circ T(0)$, and hence, $T(X) \prec_H T(0)$. Since T strongly preserves Hadamard majorization, $X \prec_H 0$, and hence, $X = 0$. ∎

The linear maps preserving or strongly preserving Hadamard majorization on $\mathbf{M}_2$ are characterized in the following theorem.

Theorem 2.11. *Let $T : \boldsymbol{M}_2 \to \boldsymbol{M}_2$ be a linear map. Then T preserves Hadamard majorization if and only if there exist $\alpha_1, \ldots, \alpha_8 \in \mathbb{R}$ and permutation matrix $P \in \boldsymbol{M}_2$ such that*

$$T(X) = P \begin{pmatrix} \alpha_1 x_{11} + \alpha_2 x_{22} & \alpha_3 x_{12} + \alpha_4 x_{21} \\ \alpha_5 x_{12} + \alpha_6 x_{21} & \alpha_7 x_{11} + \alpha_8 x_{22} \end{pmatrix}, \quad \forall\, X \in \boldsymbol{M}_2. \tag{2.2}$$

Moreover, T strongly preserves Hadamard majorization if and only if

$$\alpha_1\alpha_8 - \alpha_2\alpha_7 \ne 0 \ne \alpha_3\alpha_6 - \alpha_4\alpha_5. \tag{2.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

Proof. First observe that for any 2×2 doubly stochastic matrix D, $d_{11}=d_{22}$ and $d_{12}=d_{21}$. If T is of the form (2.2), for every doubly stochastic matrix $D\in \mathbf{M}_2$ and for every $X\in \mathbf{M}_2$ it is then easy to see that $T(D\circ X)=(PD)\circ T(X)$, and hence, T preserves Hadamard majorization.

Conversely, assume that T preserves Hadamard majorization. We show that $T(E_{11})\circ T(E_{12})=0$. Assume if possible that $T(E_{11})\circ T(E_{12})\neq 0$. Without loss of generality, we may assume that $< T(E_{11}), E_{11} >= \lambda$ and $< T(E_{12}), E_{11} >= \mu$ for some nonzero scalars $\lambda,\mu\in\mathbb{R}$. Consider the doubly stochastic matrix $D=\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3\end{pmatrix}$. Let $Y=\frac{1}{\lambda}E_{11}-\frac{1}{\mu}E_{12}$ and $X=D\circ Y$. Therefore $X\prec_H Y$ but $T(X)\not\prec_H T(Y)$ which is a contradiction. Similarly, we can show that $T(E_{11})\circ T(E_{21})=T(E_{22})\circ T(E_{12})=T(E_{22})\circ T(E_{21})=0$. By Lemma 2.9, for every $1\leq i,j\leq 2$, $T(E_{ij})$ is dominated by a permutation matrix. Since $E_{11}+E_{22}\prec_H E_{11}+E_{22}$ and $E_{12}+E_{21}\prec_H E_{12}+E_{21}$, $T(E_{11})+T(E_{22})$ and $T(E_{12})+T(E_{21})$ are dominated by some permutation matrices. Also it is easy to see that the linear maps $X\mapsto \begin{pmatrix} x_{11} & 0 \\ 0 & x_{12}\end{pmatrix}$, $X\mapsto \begin{pmatrix} x_{11} & 0 \\ 0 & x_{21}\end{pmatrix}$, $X\mapsto \begin{pmatrix} x_{22} & 0 \\ 0 & x_{12}\end{pmatrix}$ and $X\mapsto \begin{pmatrix} x_{22} & 0 \\ 0 & x_{21}\end{pmatrix}$ do not preserve Hadamard majorization. By the above restrictions on T we reach (2.2). Furthermore, by Theorem 2.10, if T strongly preserves Hadamard majorization, then T is invertible. Also observe that T is invertible if and only if (2.3) holds. In this latter case, we have

$$T^{-1}(X)=P\begin{pmatrix} \beta_1x_{11}+\beta_2x_{22} & \beta_3x_{12}+\beta_4x_{21} \\ \beta_5x_{12}+\beta_6x_{21} & \beta_7x_{11}+\beta_8x_{22}\end{pmatrix}, \quad \forall\, X\in\mathbf{M}_2,$$

where $\beta_1=\frac{\alpha_8}{\gamma}$, $\beta_2=\frac{-\alpha_2}{\gamma}$, $\beta_7=\frac{-\alpha_7}{\gamma}$, $\beta_8=\frac{\alpha_1}{\gamma}$, $\beta_3=\frac{\alpha_6}{\delta}$, $\beta_4=\frac{-\alpha_4}{\delta}$, $\beta_5=\frac{-\alpha_5}{\delta}$, $\beta_6=\frac{\alpha_3}{\delta}$, with $\gamma=\alpha_1\alpha_8-\alpha_2\alpha_7$ and $\delta=\alpha_3\alpha_6-\alpha_4\alpha_5$. Hence, T^{-1} has the form (2.2), so that T^{-1} preserves Hadamard majorization. Therefore, T strongly preserves Hadamard majorization. ∎

The following theorem characterizes the linear maps on $\mathbf{M}_n$ which strongly preserve Hadamard majorization. In fact we show that these maps are compositions of maps appearing in the three examples earlier in this section.

THEOREM 2.12. *Let $n\geq 3$ and let $T:\boldsymbol{M}_n\to\boldsymbol{M}_n$ be a linear map. Then T strongly preserves Hadamard majorization if and only if there exist $A\in\boldsymbol{M}_n$ with no zero entry and permutation matrices $P,Q\in\boldsymbol{M}_n$ such that one of the following holds:*

(i) $T(X)=(PXQ)\circ A$, *for all* $X\in\boldsymbol{M}_n$.
(ii) $T(X)=(PX^\top Q)\circ A$, *for all* $X\in\boldsymbol{M}_n$.

Proof. First assume that T strongly preserves Hadamard majorization. By Theorem 2.10, T is invertible and then by Theorem 2.7 there exist a bijection $f:\mathbb{N}_n\times\mathbb{N}_n\to$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

$\mathbb{N}_n \times \mathbb{N}_n$, and a matrix $A \in \mathbf{M}_n$ with no zero entry such that for every $X = [x_{i,j}] \in \mathbf{M}_n$,

$$T(X) = \begin{pmatrix} x_{f(1,1)} & \cdots & x_{f(1,n)} \\ \vdots & \ddots & \vdots \\ x_{f(n,1)} & \cdots & x_{f(n,n)} \end{pmatrix} \circ A.$$

First we show that for every $1 \leq k \leq n$ there exists $1 \leq i_k \leq n$ such that

$$\{f(k,1), \dots, f(k,n)\} = \{(i_k, 1), \dots, (i_k, n)\}, \quad \text{or} \tag{2.4}$$

$$\{f(k,1), \dots, f(k,n)\} = \{(1, i_k), \dots, (n, i_k)\}, \tag{2.5}$$

and also there exists $1 \leq j_k \leq n$ such that

$$\{f(1,k), \dots, f(n,k)\} = \{(1, j_k), \dots, (n, j_k)\}, \quad \text{or} \tag{2.6}$$

$$\{f(1,k), \dots, f(n,k)\} = \{(j_k, 1), \dots, (j_k, n)\}. \tag{2.7}$$

Let $1 \leq p, q, r, s \leq n$. Since T and T^{-1} preserve Hadamard majorization, Lemma 2.9 (ii) implies that the first or the second components of (p,q) and (r,s) are equal if and only if the first or the second components of $f(p,q)$ and $f(r,s)$ are equal. So the first or the second components of $f(k,1)$ and $f(k,2)$ are equal. Without loss of generality, say that their first components are equal (so that their second components are different since f is a bijection). We also know that for $3 \leq m \leq n$, $f(k,m)$ has a common component with each of $f(k,1)$ and $f(k,2)$, which then clearly must be the first component. So for every $1 \leq m \leq n$, the first components of $f(k,1)$ and $f(k,m)$ are the same, and we obtain (2.4). On the other hand, if the second components of $f(k,1)$ and $f(k,2)$ are equal, we similarly obtain (2.5). In a corresponding way, by considering the pairs $(1,k)$ and $(2,k)$ we reach (2.6) or (2.7).

Now, we consider two cases.

Case 1. Assume that (2.4) holds for $k = 1$. We show that in this case, for all $1 \leq k \leq n$, (2.4) and (2.6) hold. For every $2 \leq k \leq n$, $\{(1,1), \dots, (1,n)\} \cap \{(k,1), \dots, (k,n)\} = \emptyset$ and hence $\{(i_1,1), \dots, (i_1,n)\} \cap \{f(k,1), \dots, f(k,n)\} = \emptyset$. Then the only possibility for $\{f(k,1), \dots, f(k,n)\}$ is (2.4), and hence there exists $1 \leq i_k \leq n$ such that $\{f(k,1), \dots, f(k,n)\} = \{(i_k,1), \dots, (i_k,n)\}$.

Also, $\{(1,1), \dots, (1,n)\} \cap \{(1,k), \dots, (n,k)\} = \{(1,k)\}$ which implies that $\{(i_1,1), \dots, (i_1,n)\} \cap \{f(1,k), \dots, f(n,k)\}$ has one element. Then the only possibility for $\{f(1,k), \dots, f(n,k)\}$ is (2.6), and hence there exists $1 \leq j_k \leq n$ such that $\{f(1,k), \dots,$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

$f(n,k)\} = \{(1,j_k),\ldots,(n,j_k)\}$. Let P and Q be the permutation matrices corresponding to the maps $k \mapsto i_k$ and $k \mapsto j_k$ respectively. It is easy to see that (i) holds.

Case 2. Assume that (2.5) holds for $k = 1$. With a similar argument as for Case 1, we may obtain (ii).

Conversely, if T satisfies (i) or (ii), it is easy to see that T strongly preserves Hadamard majorization. □

The *term rank* of a matrix A is the smallest number of lines (a line is either a row or a column) which contain all the nonzero entries of A. The following result is due to Beasley and Pullman.

PROPOSITION 2.13. [4, Corollary 3.1.2] *Suppose that T is an invertible operator on $\boldsymbol{M}_n$. Then T preserves the set of matrices of term rank 1 if and only if T is one of or a composition of some of the following operators.*

(i) $X \mapsto X^\top$.
(ii) $X \mapsto PXQ$ *for some fixed but arbitrary permutation matrices in* $\boldsymbol{M}_n$.
(iii) $X \mapsto X \circ A$ *for some fixed but arbitrary matrix* $A \in \boldsymbol{M}_n$ *with all nonzero entries.*

In view of Theorem 2.12 and Proposition 2.13 we obtain the following surprising connection.

THEOREM 2.14. *Let $n \geq 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then T strongly preserves Hadamard majorization if and only if T is invertible and T preserves the set of matrices of term rank 1.*

REMARK 2.15. In [5], the authors investigated the so-called **PO**, **PSO**, **PSRO**, and **PSCO** properties of matrices. They proved that if T strongly preserves **PO**, **PSO**, **PSRO** or **PSCO** then T preserves the set of matrices of term rank 1, [5, Lemma 2.4]. Consequently, if T strongly preserves **PO**, **PSO**, **PSRO**, or **PSCO**, then T strongly preserves Hadamard majorization.

3. Linear preservers of Hadamard majorization .

In this section, we find the linear maps on $\mathbf{M}_n$ preserving Hadamard majorization.

LEMMA 3.1. *Let $n \geq 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. If T satisfies the conditions*

1. *$T(E_{kl}) \circ T(E_{pq}) = 0$ for every $1 \leq k,l,p,q \leq n$ with $(k,l) \neq (p,q)$,*
2. *$T(J)$ is a $(0,1)$-matrix,*

then $T(X \circ Y) = T(X) \circ T(Y)$ for all $X, Y \in \boldsymbol{M}_n$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

Proof. Since $T(J)$ is a $(0,1)$-matrix, by using (1), it is clear that $T(E_{ij})$ is a $(0,1)$-matrix, and hence, $T(E_{ij}) \circ T(E_{ij}) = T(E_{ij})$ for all $1 \le i,j \le n$. Then, $T(X \circ Y) = T(\sum_{i,j} x_{ij}E_{ij} \circ \sum_{i,j} y_{ij}E_{ij}) = T(\sum_{i,j} x_{ij}y_{ij}E_{ij}) = \sum_{i,j} x_{ij}y_{ij}T(E_{ij}) = \sum_{i,j} x_{ij}T(E_{ij}) \circ \sum_{i,j} y_{ij}T(E_{ij}) = T(X) \circ T(Y)$. ◻

With the use of Lemma 3.1 we obtain the following result, which is of the independent interest.

Theorem 3.2. *Let $n \ge 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. If T preserves Hadamard majorization, then $T(X \circ Y)oT(J) = T(X) \circ T(Y)$ for all $X, Y \in \boldsymbol{M}_n$.*

Proof. Let $A = [a_{ij}] = T(J)$ and $B = [b_{ij}]$, where $b_{ij} = \frac{1}{a_{ij}}$ if $a_{ij} \ne 0$, and 0 otherwise. Consider the linear map $S : \mathbf{M}_n \to \mathbf{M}_n$ defined by $S(X) = T(X) \circ B$. Note that by the use of (2.1), $T(X) = S(X) \circ A$ for all $X \in \mathbf{M}_n$. Also it is clear that $S(J)$ is a $(0,1)$-matrix. Since T preserves Hadamard majorization, S preserves Hadamard majorization. By Theorem 2.6 and Theorem 2.7, S satisfies condition (1) of Lemma 3.1, and hence, $S(X \circ Y) = S(X) \circ S(Y)$ for all $X, Y \in \mathbf{M}_n$. Therefore $S(X \circ Y) \circ A \circ A = S(X) \circ A \circ S(Y) \circ A$, which implies that $T(X \circ Y) \circ A = T(X) \circ T(Y)$. ◻

Lemma 3.3. *For every $r, s \in \mathbb{N}_n$, let m_{rs} be the smallest integer such that every $r \times s$ submatrix of every doubly stochastic matrix $D \in \mathbf{M}_n$ is an $r \times s$ submatrix of a doubly stochastic matrix $D' \in \mathbf{M}_{m_{rs}}$. Then*

$$m_{rs} = \begin{cases} r+s, & \text{if } r+s \le n; \\ n, & \text{if } r+s \ge n. \end{cases}$$

Proof. From the definition of m_{rs}, it is clear that $m_{rs} \le n$. Without loss of generality, we may consider the upper left $r \times s$ submatrices of $n \times n$ doubly stochastic matrices. Put $m = m_{rs}$ and let the doubly stochastic matrices D and D' be partitioned as

$$D = \begin{pmatrix} A & B_1 \\ B_2 & B_3 \end{pmatrix} \in \mathbf{M}_n, \quad D' = \begin{pmatrix} A & C_1 \\ C_2 & C_3 \end{pmatrix} \in \mathbf{M}_m,$$

where A is $r \times s$. Let λ be the sum of the entries of A. So, the sum of the entries of C_1 and the sum of the entries of C_2 are $r - \lambda$ and $s - \lambda$, respectively. Hence, the sum of entries of C_3 is $m - r - s + \lambda$. Since C_3 is a block of a doubly stochastic matrix, $m + \lambda \ge r + s$. Now, we consider two cases:

Case 1. Let $r \le n - s$. Here we consider the doubly stochastic matrix D where $A = 0$ and $B_1 = [I_r \ 0]$, i.e., $D = \begin{pmatrix} 0 & (I_r \ 0) \\ B_2 & B_3 \end{pmatrix}$. So, in this case, λ can be taken to be 0, and hence, $m \ge r + s$. But for every $r \times s$ submatrix A of an $n \times n$ doubly stochastic matrix D, we can form the doubly stochastic matrix

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

$\begin{pmatrix} A & \mathrm{diag}(\alpha_1,\dots,\alpha_r) \\ \mathrm{diag}(\beta_1,\dots,\beta_s) & A^\top \end{pmatrix} \in \mathbf{M}_{r+s}$, where each $\alpha_i = 1 - \sum_{j=1}^{s} a_{ij}$ and each $\beta_j = 1 - \sum_{i=1}^{r} a_{ij}$. So, $m \le r+s$, and thus, $m = r+s$.

Case 2. Let $r \ge n-s$. Here we consider the doubly stochastic matrix D where $B_1 = [I_{n-s}\ 0]^\top$, i.e., $D = \begin{pmatrix} \begin{pmatrix} 0 \\ * \end{pmatrix} & \begin{pmatrix} I_{n-s} \\ 0 \end{pmatrix} \\ B_2 & B_3 \end{pmatrix}$. So, in this case, λ can be taken to be $r-(n-s)$, and hence, $m+r-n+s \ge r+s$ or $m \ge n$. Thus, $m = n$. ☐

The following examples give some kinds of linear maps $T : \mathbf{M}_n \to \mathbf{M}_m$ that preserve Hadamard majorization where $m \le n$. In each of these examples, it is easy to see that the conditions (1) and (2) of Lemma 3.1 hold, so that we have $T(X \circ Y) = T(X) \circ T(Y)$ for all $X, Y \in \mathbf{M}_n$.

Example 3.4. Let $P_1, \dots, P_n$ be permutation matrices such that $P_1 + \cdots + P_n = J$. For every $1 \le i \le n$ the linear map $S : \mathbf{M}_n \to \mathbf{M}_n$ defined by

$$S(X) = x_{i1}P_1 + \cdots + x_{in}P_n, \ \ \forall\, X \in \mathbf{M}_n, \ \text{ or}$$
$$S(X) = x_{1i}P_1 + \cdots + x_{ni}P_n, \ \ \forall\, X \in \mathbf{M}_n,$$

preserves Hadamard majorization. Here, if D is doubly stochastic, then $S(D)$ is doubly stochastic and we have $S(D \circ X) = S(D) \circ S(X)$.

Example 3.5. Let $\alpha, \beta \subseteq \mathbb{N}_n$ and $|\alpha| = r, |\beta| = s$. The linear map $T : \mathbf{M}_n \to \mathbf{M}_n$ defined by $T(X) = \begin{pmatrix} X[\alpha,\beta] & 0 \\ 0 & 0 \end{pmatrix}$ for all $X \in \mathbf{M}_n$, preserves Hadamard majorization. Here, if D is doubly stochastic, then $D[\alpha,\beta]$ can be extended to a doubly stochastic $D' \in \mathbf{M}_n$ and hence $T(D) \circ T(X) = D' \circ T(X)$.

Example 3.6. Let $\alpha, \beta \subseteq \mathbb{N}_n$, $|\alpha| = r, |\beta| = s$ and $m < n$. Then the linear map $T : \mathbf{M}_n \to \mathbf{M}_m$ defined by $T(X) = \begin{pmatrix} X[\alpha,\beta] & 0 \\ 0 & 0 \end{pmatrix}$ for all $X \in \mathbf{M}_n$, preserves Hadamard majorization if and only if $r+s \le m$. If $r+s \le m$, by Case 1 in the proof of Lemma 3.3, if D is doubly stochastic, then $D[\alpha,\beta]$ can be extended to a doubly stochastic $D' \in \mathbf{M}_m$ and hence $T(D) \circ T(X) = D' \circ T(X)$. Then T preserves Hadamard majorization. Lemma 3.3 also can be used to show that if T preserves Hadamard majorization, then $r+s \le m$.

Example 3.7. Let $\alpha, \beta \subseteq \mathbb{N}_n$ and $|\alpha| = r, |\beta| = s$. If $(r+s) \le m \le n$, the linear map $T : \mathbf{M}_n \to \mathbf{M}_m$ defined by $T(X) = \begin{pmatrix} S(X)[\alpha,\beta] & 0 \\ 0 & 0 \end{pmatrix}$ for all $X \in \mathbf{M}_n$, preserves Hadamard majorization where S is the linear map introducing in Example

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

3.4. Note that T is the composition of the linear maps in Examples 3.4 and 3.6.

Example 3.8. Let m, n and k be positive integers such that $mk \leq n$. Let $P_1, \ldots, P_k \in \mathbf{M}_m$ be permutation matrices such that $P_i \circ P_j = 0$, for every $1 \leq i < j \leq n$. The linear map $T : \mathbf{M}_n \to \mathbf{M}_{mk}$ defined by

$$T(X) = \begin{pmatrix} x_{ij_1}P_1 & x_{ij_2}P_2 & \ldots & x_{ij_k}P_k \\ x_{ij_k}P_k & x_{ij_1}P_1 & \ldots & x_{ij_{k-1}}P_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{ij_2}P_2 & x_{ij_3}P_3 & \ldots & x_{ij_1}P_1 \end{pmatrix}, \quad \forall\, X \in \mathbf{M}_n,$$

where $1 \leq i \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$, preserves Hadamard majorization. Here, if D is doubly stochastic, then $T(D)$ can be extended to a doubly stochastic $D' \in \mathbf{M}_{mk}$ and hence $T(D) \circ T(X) = D' \circ T(X)$. In fact for every permutation matrix $P \in \mathbf{M}_n$, there exists permutation matrix $Q \in \mathbf{M}_{mk}$ such that $T(P \circ X) = Q \circ T(X)$.

In the following theorem we exhibit a large class of linear maps which preserve Hadamard majorization.

Theorem 3.9. *Let $m_1, \ldots, m_k$ be some positive integers. Put $n = m_1 + \cdots + m_k$ and for every $1 \leq j \leq k$, let $T_j : \boldsymbol{M}_n \to \boldsymbol{M}_{m_j}$ be a linear map appearing in Examples 3.6, 3.7 or 3.8. Let $P, Q \in \boldsymbol{M}_n$ be permutation matrices and $A \in \boldsymbol{M}_n$. Then the linear map $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ defined by*

$$T(X) = (P \begin{pmatrix} T_1(X) & & & 0 \\ & T_2(X) & & \\ & & \ddots & \\ 0 & & & T_k(X) \end{pmatrix} Q) \circ A, \qquad \forall\, X \in \boldsymbol{M}_n,$$

preserves Hadamard majorization.

Proof. Let $X \prec_H Y$. Then there exist doubly stochastic matrices $D_j \in \mathbf{M}_{m_j}$ such that $T_j(X) = D_j \circ T_j(Y)$, for every $1 \leq j \leq k$. With $D' = D_1 \oplus \cdots \oplus D_k$, we have

$$\begin{aligned} T(X) &= [P(D_1 \circ T_1(Y) \oplus \cdots \oplus D_k \circ T_k(Y))Q] \circ A \\ &= [P(D' \circ [T_1(Y) \oplus \cdots \oplus T_k(Y)])Q] \circ A \\ &= (PD'Q) \circ T(Y). \end{aligned}$$

Therefore, $T(X) \prec_H T(Y)$, and hence, T preserves Hadamard majorization. ☐

Theorem 3.10. (Birkhoff's theorem) *The set of the $n \times n$ doubly stochastic matrices is a convex set with its extreme points the set of $n \times n$ permutation matrices.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

By a *generalized permutation matrix* we mean a square matrix with exactly one nonzero entry in every row and in every column.

THEOREM 3.11. *Let $n \geq 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then T preserves Hadamard majorization if and only if T satisfies the following conditions:*

1. *$T(E_{kl}) \circ T(E_{pq}) = 0$ for every $1 \leq k, l, p, q \leq n$ with $(k, l) \neq (p, q)$,*
2. *For every permutation matrix $P \in \boldsymbol{M}_n$ there exists a $(0, 1)$-matrix $Z \in \boldsymbol{M}_n$ such that $Z \circ T(J) = 0$ and $T(P) + Z$ is a generalized permutation matrix.*

Proof. Let $A = [a_{ij}] = T(J)$ and $B = [b_{ij}]$, where $b_{ij} = \frac{1}{a_{ij}}$, if $a_{ij} \neq 0$ and 0 otherwise. As in the proof of Theorem 3.2, we consider the linear map $S : \mathbf{M}_n \to \mathbf{M}_n$ defined by $S(X) = T(X) \circ B$. As before we have that $S(J)$ is a $(0, 1)$-matrix.

First assume that T preserves Hadamard majorization, and hence, S preserves Hadamard majorization. Then by Theorem 2.6 and Theorem 2.7, (1) holds for both T and S. To prove (2), let $P \in \mathbf{M}_n$ be a permutation matrix. Since $P \circ J \prec_H J$, $S(P \circ J) \prec_H S(J)$ and then $S(P \circ J) = D \circ S(J)$ for some doubly stochastic matrix D. By Lemma 3.1, $S(P) \circ S(J) = D \circ S(J)$. Now, $S(P)$ is a $(0, 1)$-matrix, which by Proposition 2.1 is dominated by a permutation matrix, and hence, up to permutation equivalence, $S(P) = I_k \oplus 0$ for some $0 \leq k \leq n$. Therefore, $(I_k \oplus 0) \circ S(J) = D \circ S(J)$, and this implies that $D = I_k \oplus D'$ and $(0 \oplus D') \circ S(J) = 0$, for some doubly stochastic matrix $D' \in \mathbf{M}_{n-k}$. So there exists a permutation matrix $Q \in \mathbf{M}_{n-k}$ such that $(0 \oplus Q) \circ S(J) = 0$. Now, we have $[S(P) - (I_k \oplus Q)] \circ S(J) = 0$. Put $Z = (I_k \oplus Q) - S(P) = (0 \oplus Q)$, and then $Z + S(P)$ is a permutation matrix and $Z \circ S(J) = 0$. We have $Z \circ T(J) = Z \circ S(J) \circ A = 0$ and $Z + T(P) = (0 \oplus Q) + S(P) \circ A = \operatorname{diag}(a_{11}, \ldots, a_{kk}) \oplus Q$.

Conversely, assume that T satisfies (1) and (2). It is clear that S satisfies (1), and hence, by Theorem 2.6, $T(X) = S(X) \circ A$ for all $X \in \mathbf{M}_n$ and then it is enough to show that S preserves Hadamard majorization. By (2), for every permutation matrix $P \in \mathbf{M}_n$ there exists a $(0, 1)$-matrix $Z \in \mathbf{M}_n$ such that $Z \circ T(J) = 0$ and $T(P) + Z$ is a generalized permutation matrix. By the use of (1) we have $Z \circ T(P) = 0$, and hence, $S(P) + Z$ is a permutation matrix. By Lemma 3.1, we have $S(X \circ Y) = S(X) \circ S(Y)$ for all $X, Y \in \mathbf{M}_n$. Let $X \prec_H Y$. Then there exists a doubly stochastic matrix D such that $X = D \circ Y$, and hence, $S(X) = S(D) \circ S(Y)$. By Birkhoff's theorem, $D = \sum_{i=1}^k \lambda_i P_i$ for some permutation matrices $P_1, \ldots, P_k \in \mathbf{M}_n$ and some positive numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^k \lambda_i = 1$. For every $1 \leq i \leq k$, there exists $Z_i \in \mathbf{M}_n$ such that $Z_i \circ A = 0$ and $S(P_i) + Z_i$ is a permutation matrix. So $D' = \sum_{i=1}^k \lambda_i (S(P_i) + Z_i)$ is a doubly stochastic matrix. Therefore, $S(X) = S(D) \circ S(Y) = S(\sum_{i=1}^k \lambda_i P_i) \circ S(Y) = (\sum_{i=1}^k \lambda_i (S(P_i) + Z_i)) \circ S(Y) = D' \circ S(Y)$, and hence, S preserves Hadamard majorization. $\square$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

In the following result, $\mathbf{P}(n)$ is the set of all $n \times n$ permutation matrices.

Corollary 3.12. *Let $n \geq 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then T strongly preserves Hadamard majorization if and only if T is invertible and T satisfies the following conditions:*

1. *$T(E_{kl}) \circ T(E_{pq}) = 0$ for every $1 \leq k, l, p, q \leq n$ with $(k, l) \neq (p, q)$.*
2. *For every permutation matrix $P \in \boldsymbol{M}_n$, $T(P)$ is a generalized permutation matrix.*

Proof. If T strongly preserves Hadamard majorization, then by Theorem 2.10, T is invertible and by Theorem 3.11, (1) holds, and for every permutation matrix $P \in \mathbf{M}_n$ there exists a $(0, 1)$-matrix $Y \in \mathbf{M}_n$ such that $Y \circ T(J) = 0$ and $T(P) + Y$ is a generalized permutation matrix. But by Theorem 2.7, $T(J)$ has no zero entry, and hence, $Y = 0$ as desired.

Conversely, assume that T is invertible and (1) and (2) hold. Then by Theorem 3.11, T preserves Hadamard majorization. As in the proof of Theorem 3.2, we consider the linear map $S : \mathbf{M}_n \to \mathbf{M}_n$ defined by $S(X) = T(X) \circ B$. Then S satisfies (1) and for every permutation matrix $P \in \mathbf{M}_n$, $S(P)$ is a permutation matrix. Since S is invertible, $S(\mathbf{P}(n)) = \mathbf{P}(n)$, and hence, S^{-1} satisfies (2). For every $1 \leq k, l, p, q \leq n$ with $(k, l) \neq (p, q)$, let $A = S^{-1}(E_{kl})$ and $B = S^{-1}(E_{pq})$. So, by Lemma 3.1, $S(A \circ B) = S(A) \circ S(B) = E_{kl} \circ E_{pq} = 0$. This implies that $A \circ B = 0$, and hence, S^{-1} satisfies (1). Thus, by Theorem 3.11, S^{-1} preserves Hadamard majorization, and hence, S strongly preserves Hadamard majorization. Therefore, T strongly preserves Hadamard majorization. ∎

Note that a (directed) graph D is a pair $(V, \mathcal{E})$, consisting of the set V of nodes and the set $\mathcal{E}$ of edges, which are ordered pairs of elements of V. If $|V| = n$, the *adjacency matrix* of D is the square $n \times n$ matrix A such that a_{ij} is one when $(i, j) \in \mathcal{E}$, and zero otherwise. The reader can see [7] for these notions.

For our purpose we make the following definition. We say that a directed graph D is a *permutation graph* if its adjacency matrix is a permutation matrix. This means that D is the union of distinct simple circles including all the nodes of D.

In the remaining part of the paper, when we say "graph" we mean directed graph.

Remark 3.13. Let $n \geq 3$ and let $T : \mathbf{M}_n \to \mathbf{M}_n$ be a linear map preserving Hadamard majorization. By the use of Theorem 2.7, it can be seen that there exist a subset $\mathcal{E}$ of $\mathbb{N}_n \times \mathbb{N}_n$, a function $\varphi : \mathcal{E} \to \mathbb{N}_n \times \mathbb{N}_n$, and nonzero scalars $\lambda_{ij} \in \mathbb{R}$ such that for every $1 \leq i, j \leq n$, and for all $X \in \mathbf{M}_n$,

$$(T(X))_{ij} = \begin{cases} \lambda_{ij} x_{\varphi(i,j)}, & \text{if } (i, j) \in \mathcal{E}; \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

where $x_{\varphi(i,j)}$ means x_{pq} if $\varphi(i,j) = (p,q)$. If $\lambda_{ij} = 1$ for all $1 \le i,j \le n$, then $T(E_{rs})$ is the adjacency matrix of the graph $(\mathbb{N}_n, \varphi^{-1}\{(r,s)\})$ for every $1 \le r,s \le n$. By the use of previous remark we may state and prove the following theorem that gives a necessary and sufficient condition for linear maps to preserve Hadamard majorization.

THEOREM 3.14. *Let $n \ge 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then T preserves Hadamard majorization if and only if there exist a directed graph $(\mathbb{N}_n, \mathcal{E})$, nonzero scalars $\lambda_{ij} \in \mathbb{R}$, and a function $\varphi : \mathcal{E} \to \mathbb{N}_n \times \mathbb{N}_n$ such that (3.1) holds, and for every permutation graph $(\mathbb{N}_n, \mathcal{P})$ there exists a subset $\mathcal{F}$ of $\mathcal{E}^c$ such that $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}) \cup \mathcal{F})$ is a permutation graph.*

Proof. First assume that T preserves Hadamard majorization. Then by Remark 3.13, there exist a subset $\mathcal{E}$ of $\mathbb{N}_n \times \mathbb{N}_n$, a function $\varphi : \mathcal{E} \to \mathbb{N}_n \times \mathbb{N}_n$, and nonzero scalars $\lambda_{ij} \in \mathbb{R}$ such that (3.1) holds. Without loss of generality (as in the proof of Theorem 3.2, we can consider the linear map $S(X) = T(X) \circ B$) we may assume that $\lambda_{ij} = 1$ for all $1 \le i,j \le n$. Hence, for every permutation graph $(\mathbb{N}_n, \mathcal{P})$ with adjacency matrix P, it is easy to see that $T(P)$ is the adjacency matrix of the graph $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}))$. By the condition (ii) of Theorem 3.11, there exists a $(0,1)$-matrix $Y \in \mathbf{M}_n$ such that $Y \circ T(J) = 0$ and $T(P) + Y$ is a permutation matrix. Now, let $(\mathbb{N}_n, \mathcal{F})$ be the graph with adjacency matrix Y. Since $Y \circ T(J) = 0$, $\mathcal{F} \subset \mathcal{E}^c$ and $T(P) + Y$ is the adjacency matrix of the graph $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}) \cup \mathcal{F})$, as desired.

Conversely, to prove the sufficiency, it is enough to show that T satisfies the conditions (1) and (2) of Theorem 3.11. Without loss of generality, we may assume that $\lambda_{ij} = 1$ for all $1 \le i,j \le n$. Since $T(E_{rs})$ is the adjacency matrix of the graph $(\mathbb{N}_n, \varphi^{-1}\{(r,s)\})$, if $(k,l) \ne (p,q)$ then $\varphi^{-1}\{(k,l)\} \cap \varphi^{-1}\{(p,q)\}) = \emptyset$ and hence $T(E_{kl}) \circ T(E_{pq}) = 0$. Now, let $P \in \mathbf{M}_n$ be a permutation matrix. Consider the permutation graph $(\mathbb{N}_n, \mathcal{P})$ with matrix adjacency P. By the assumption, there exists a subset $\mathcal{F}$ of $\mathcal{E}^c$ such that $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}) \cup \mathcal{F})$ is a permutation graph. Let Y be the adjacency matrix of the graph $(\mathbb{N}_n, \mathcal{F})$. It is easy to see that Y is a $(0,1)$-matrix such that $Y \circ T(J) = 0$ and $T(P) + Y$ is a permutation matrix, in fact it is the adjacency matrix of $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}) \cup \mathcal{F})$. $\square$

COROLLARY 3.15. *Let $n \ge 3$ and let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear map. Then T strongly preserves Hadamard majorization if and only if there exist nonzero scalars $\lambda_{ij} \in \mathbb{R}$, and a bijection $\varphi : \mathbb{N}_n \times \mathbb{N}_n \to \mathbb{N}_n \times \mathbb{N}_n$ such that (3.1) holds, and for every permutation graph $(\mathbb{N}_n, \mathcal{P})$, $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}))$ is a permutation graph.*

Proof. If T strongly preserves Hadamard majorization then by Theorem 2.10, T is invertible, and hence, the function φ in Theorem 3.14 is bijective and $\mathcal{E}^c = \emptyset$. Therefore, for every permutation graph $(\mathbb{N}_n, \mathcal{P})$, $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}))$ is a permutation graph.

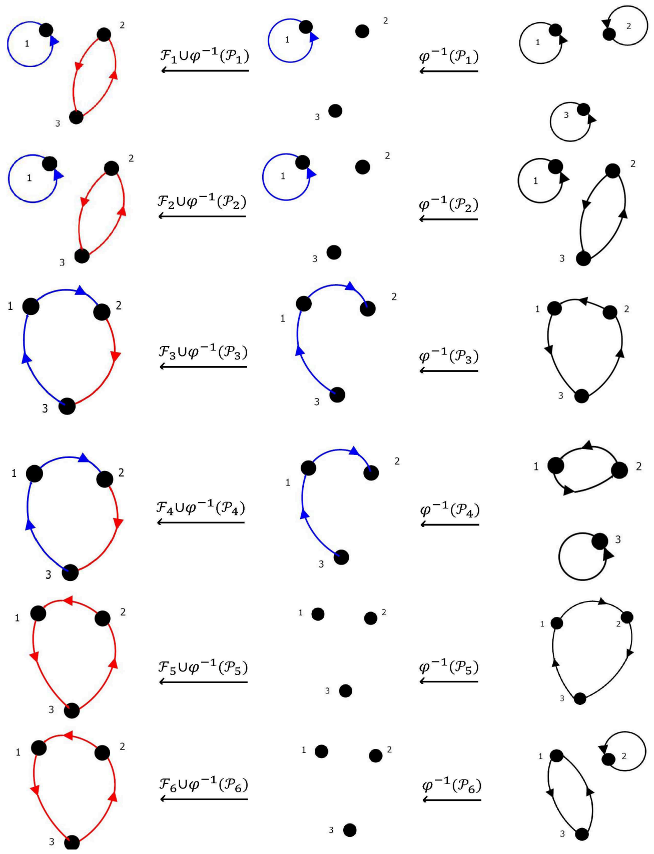
Conversely, assume that there exist nonzero scalars $\lambda_{ij} \in \mathbb{R}$, and a bijection

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

$\varphi : \mathbb{N}_n \times \mathbb{N}_n \to \mathbb{N}_n \times \mathbb{N}_n$ such that (3.1) holds, and for every permutation graph $(\mathbb{N}_n, \mathcal{P})$, $(\mathbb{N}_n, \varphi^{-1}(\mathcal{P}))$ is a permutation graph. So by Theorem 3.14, T preserves Hadamard majorization. Since φ is bijective, for every permutation graph $(\mathbb{N}_n, \mathcal{P})$, $(\mathbb{N}_n, \psi^{-1}(\mathcal{P}))$ is a permutation graph where $\psi = \varphi^{-1}$ is the inverse of φ. Therefore, by Theorem 3.14, T^{-1} preserves Hadamard majorization, and hence, T strongly preserves Hadamard majorization. ☐

Example 3.16. Consider the linear map $T : \mathbf{M}_3 \to \mathbf{M}_3$ defined by $T(X) = \begin{pmatrix} x_{11} & x_{21} & 0 \\ 0 & x_{11} & 0 \\ x_{21} & 0 & 0 \end{pmatrix}$, for all $X \in \mathbf{M}_3$. Then $\mathcal{E} = \{(1,1),(1,2),(2,2),(3,1)\}$ and $\varphi(1,1) = (1,1), \varphi(1,2) = (2,1), \varphi(2,2) = (1,1), \varphi(3,1) = (2,1)$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_6$ be all permutation graphes of order 3. We have



where the blue edges and the red edges belong to the graphs $(\mathbb{N}_3, \mathcal{E})$ and $(\mathbb{N}_3, \mathcal{E}^c)$, respectively. Hence, by Theorem 3.14, T preserves Hadamard majorization.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 593-609, September 2016

ELA
http:/repository.uwyo.edu/ela

Acknowledgment. The authors appreciate the kind remark of the referee.

REFERENCES

[1] T. Ando. Majorization, doubly stochastic matrices, and comparision of eigenvalues. *Linear Algebra Appl.*, 118:163–248, 1989.

[2] L.B. Beasley and S.G. Lee. Linear operators preserving multivariate majorization. *Linear Algebra Appl.*, 304:141–159, 2000.

[3] L.B. Beasley, S.G. Lee, and Y.H. Lee. A characterization of strong preservers of matrix majorization. *Linear Algebra Appl.*, 367:341–346, 2003.

[4] L.B. Beasley and N.J. Pullman. Linear operators that preserve term rank 1. *Math. Proc. R. Ir. Acad., Section A: Mathematical and Physical Sciences*, 91A(1):71–78, 1991.

[5] L.B. Beasley and D.J. Scully. Linear operators which preserve combinatorial orthogonality. *Linear Algebra Appl.*, 201:171–180, 1994.

[6] R. Bhatia. *Matrix Analysis.* Springer-Verlag, New York, 1997.

[7] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, New York, 1991.

[8] M.A. Hadian Nadoshan and A. Armandnejad. B-majorization and its linear preservers. *Linear Algebra Appl.*, 478:218–227, 2015.

[9] A.M. Hasani and M. Radjabalipour. The structure of linear operators strongly preserving majorizations of matrices. *Electron. J. Linear Algebra*, 15:260–268, 2006.

[10] C.K. Li and E. Poon. Linear operators preserving directional majorization. *Linear Algebra Appl.*, 325:15–21, 2001.

[11] A.W. Marshall, I. Olkin, and B.C. Arnold. *Inequalities: Theory of Majorization and Its Applications*, second edition. Springer, New York, 2011.

[12] M. Soleymani and A. Armandnejad. Linear Preservers of even majorization on $\mathbf{M}n, m$. *Linear Multilinear Algebra*, 62(11):1437–1449, 2014.