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THE INVERSE ALONG AN ELEMENT IN RINGS*

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Abstract. Several properties of the inverse along an element are studied in the context of unitary rings. New characterizations of the existence of this inverse are proved. Moreover, the set of all invertible elements along a fixed element is fully described. Furthermore, commuting inverses along an element are characterized. The special cases of the group inverse, the (generalized) Drazin inverse and the Moore-Penrose inverse (in rings with involutions) are also considered.

Key words. Inverse along an element, Drazin inverse, Generalized Drazin inverse, Group inverse, Moore-Penrose inverse, Unitary ring.

AMS subject classifications. 15A09, 16U99, 16W10.

1. Introduction. In [5], the notion of an inverse along an element was introduced. This inverse has the advantage that it encompasses several well known generalized inverses such as the group inverse, the Drazin inverse and the Moore-Penrose inverse. The aforementioned inverse was studied by several authors (see for example [1, 3, 5, 6, 7, 9]).

The main objective of this article is to study several properties of the inverse along an element in unitary rings. In Section 3, after having recalled some preliminary definitions and facts in Section 2, more equivalent conditions that assure the existence of the inverse under consideration will be given. In addition, in Section 4, more characterizations of this inverse will be proved. In Section 5, the set of all invertible elements along a fixed element will be fully described. Furthermore, the special cases of the group inverse, the (generalized) Drazin inverse and the Moore-Penrose inverse (in the presence of an involution) will be considered. In Section 6, the reverse order law will be studied. In Section 7, the commuting inverse along an element will be characterized. In particular, a characterization of group invertible elements through the inverse along an element will be presented. Finally, in Section 8, inverses along elements that are also inner inverses will be considered.

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2. Preliminary definitions and facts. The symbol $\mathbb{N}$ will denote the natural numbers, in particular, if $n \in \mathbb{N}$, then $n \geq 1$. From now on $\mathcal{R}$ will denote a unitary ring with unit 1. Let $\mathcal{R}^{-1}$ be the set of invertible elements of $\mathcal{R}$ and denote by $\mathcal{R}^{\bullet}$ the set of idempotents of $\mathcal{R}$, i.e., $\mathcal{R}^{\bullet} = \{p \in \mathcal{R} \colon p^2 = p\}$. Given $a \in \mathcal{R}$, the following notation will be used: $a\mathcal{R} = \{ax \colon x \in \mathcal{R}\}$, $\mathcal{R}a = \{xa \colon x \in \mathcal{R}\}$, $a^{-1}(0) = \{x \in \mathcal{R} \colon ax = 0\}$, $a_{-1}(0) = \{x \in \mathcal{R} \colon xa = 0\}$.

Recall that $a \in \mathcal{R}$ is *regular*, if there is $z \in \mathcal{R}$ such that $a = aza$. Such an element z is called an *inner* or a *generalized inverse* of a. The set of regular elements of $\mathcal{R}$ will be denoted by $\hat{\mathcal{R}}$. Next follows one of the main definitions of this article.

Definition 2.1. Given $a \in \mathcal{R}$, an element $b \in \mathcal{R}$ is an *outer inverse* of a if $b = bab$.

Remark 2.2. Consider a, $b \in \mathcal{R}$ such that b is an outer inverse of a. Then, the following statements can be easily proved:

(i) ab, $ba \in \mathcal{R}^{\bullet}$.

(ii) $bR = ba\mathcal{R}$, $\mathcal{R}ab = \mathcal{R}b$.

(iii) $b^{-1}(0) = (ab)^{-1}(0)$, $b_{-1}(0) = (ba)_{-1}(0)$.

(iv) When b is also an inner inverse of a, i.e., $a = aba$, the following statements are equivalent:
 (1) $a = aba$,
 (2) $\mathcal{R} = b\mathcal{R} \oplus a^{-1}(0)$,
 (3) $\mathcal{R} = \mathcal{R}b \oplus a_{-1}(0)$.

Recall that when $b \in \mathcal{R}$ is an outer and an inner inverse of $a \in \mathcal{R}$, b is said to be a *normalized generalized inverse* of a. It is well known that if z is a generalized inverse of a, then zaz is a normalized generalized inverse of a.

Next the definition of the key notion of this article will be recalled (see [5, Definition 4]).

Definition 2.3. Consider a, $d \in \mathcal{R}$. The element a is *invertible along* d, if there exists $b \in \mathcal{R}$ such that b is an outer inverse of a, $b\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}b = \mathcal{R}d$.

Recall that, in the conditions of Definition 2.3, according to [5, Theorem 6], if such $b \in \mathcal{R}$ exists, then it is unique. Therefore, the element b satisfying Definition 2.3 will be said to be the *inverse of* a *along* d. In this case, the inverse under consideration will be denoted by $a^{\parallel d}$. According to [5, Theorem 7] (see also [7, Theorem 2.1]), a necessary and sufficient condition for $a \in \mathcal{R}$ to be invertible along $d \in \mathcal{R}$ is that ad is group invertible (see the definition below) and $\mathcal{R}d \subset \mathcal{R}ad$, or equivalently, da is group invertible and $d\mathcal{R} \subset da\mathcal{R}$. Moreover, $a^{\parallel d} = d(ad)^{\sharp} = (da)^{\sharp}d$, where $(ad)^{\sharp}$ (respectively, $(da)^{\sharp}$) is the group inverse of ad (respectively, of da) (see the

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notation below). In addition, according to [7, Theorem 2.2], $a^{\|d}$ exists if and only if $dad\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}dad = \mathcal{R}d$. These existence criteria will be used to prove more equivalent conditions to the existence of the inverse along an element (see Theorem 4.1).

Note that if $\tilde{d} \in \mathcal{R}$ is such that $d\mathcal{R} = \tilde{d}\mathcal{R}$ and $\mathcal{R}d = \mathcal{R}\tilde{d}$, then a is invertible along d if and only if a is invertible along $\tilde{d}$, in addition, in this case $a^{\|d} = a^{\|\tilde{d}}$. In particular, given $d \in \mathcal{R}^{-1}$, a necessary and sufficient condition for $a \in \mathcal{R}$ to be invertible along d is that $a \in \mathcal{R}^{-1}$; moreover, in this case $a^{\|d} = a^{-1}$. In fact, since $1\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}1 = \mathcal{R}d$, a is invertible along d if and only if a is invertible along 1, which is equivalent to $a \in \mathcal{R}^{-1}$, and in this case $a^{\|1} = a^{-1}$.

Moreover, according to [7, p. 3], if $a^{\|d}$ exists, then d is regular. In particular, if d is not regular, no $a \in \mathcal{R}$ has an inverse along d. Thus, without loss of generality it will be assumed that $d \in \hat{\mathcal{R}}$. However, the next remark shows that more conditions for d can be assumed without loss of generality.

Remark 2.4. Consider $d \in \hat{\mathcal{R}}$. Note that if $d = 0$, then any $a \in \mathcal{R}$ is invertible along 0; in fact, in this case $a^{\|d} = 0$. Then suppose that $d \neq 0$ and that there exists $\overline{d} \in \mathcal{R}$ such that $d\overline{d}d = d$. Thus, $d(1 - \overline{d}d) = 0 = (1 - d\overline{d})d$. Therefore, if $d \in \mathcal{R}$ is not a zero divisor (there is no $z \in \mathcal{R}$, $z \neq 0$, such that $zd = 0$ or $dz = 0$), then $d \in \mathcal{R}^{-1}$. Consequently, in the general case, it is possible to assume that $d \in \hat{\mathcal{R}} \setminus (\mathcal{R}^{-1} \cup \{0\})$, with d a zero divisor. However, if the ring $\mathcal{R}$ has no zero divisors, for example if $\mathcal{R}$ is a field, an integral domain or a polynomial ring over an integral domain, an inverse along an element $d \in \mathcal{R}$ exists if and only if $d \in \mathcal{R}^{-1} \cup \{0\}$, in which case this situation has been characterized.

Next follow the definitions of several generalized inverses such as the group inverse, the (generalized) Drazin inverse and the Moore-Penrose inverse. These classes of invertible elements are particular cases of the inverse studied in this article.

Consider $a \in \mathcal{R}$. The element $a \in \mathcal{R}$ will be said to be *group invertible,* if there exists a (necessarily unique) $b \in \mathcal{R}$ such that

$$a = aba, \quad b = bab, \quad ab = ba$$

(see for example [8]). When $a \in \mathcal{R}$ is group invertible, its group inverse will be denoted by $a^\sharp$. Clearly, $a^\sharp$ is group invertible and $(a^\sharp)^\sharp = a$. According to [5, Theorem 11], a necessary and sufficient condition for $a \in \mathcal{R}$ to be group invertible is that a is invertible along a, moreover, in this case $a^\sharp = a^{\|a}$. Next some of the main properties of group invertible elements will be recalled. To this end, let $\mathcal{R}^\sharp$ stand for the set of all group invertible elements of the ring $\mathcal{R}$. In addition, recall that if $p \in \mathcal{R}^\bullet$, then $p\mathcal{R}p$ is a subring of $\mathcal{R}$ with unit p.

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REMARK 2.5. Consider $a \in \mathcal{R}^\sharp$.

(i) Note that

$$a\mathcal{R} = aa^\sharp\mathcal{R} = a^\sharp a\mathcal{R} = a^\sharp\mathcal{R},$$
$$\mathcal{R}a = \mathcal{R}aa^\sharp = \mathcal{R}a^\sharp a = \mathcal{R}a^\sharp,$$
$$aa^\sharp\mathcal{R}aa^\sharp = a^\sharp a\mathcal{R}a^\sharp a = a\mathcal{R}a = a^\sharp\mathcal{R}a^\sharp.$$

(ii) Recall that according to [8, Lemma 3], $a \in \mathcal{R}^\sharp$ if and only if there exists $p \in \mathcal{R}^\bullet$ such that $a + p \in \mathcal{R}^{-1}$, $ap = pa = 0$. Now, using this result, it is not difficult to prove that a necessary and sufficient condition for a to be group invertible is that there exists $p \in \mathcal{R}^\bullet$ such that $a = (1-p)a(1-p)$ and $a \in ((1-p)\mathcal{R}(1-p))^{-1}$. In this case, letting b be the inverse of a in $(1-p)\mathcal{R}(1-p)$, then $a^\sharp = b$. Moreover, according to [8, Corollary 2], the idempotent, p_a, involved in the definition is unique and $p_a = 1 - a^\sharp a$. In particular, $p_a = p_{a^\sharp}$.

(iii) Let $a \in \mathcal{R}$ and suppose that a has a commuting generalized inverse b, i.e., $aba = a$ and $ab = ba$. Then, it is not difficult to prove that $a \in \mathcal{R}^\sharp$ and $a^\sharp = bab$.

(iv) Let $a \in \mathcal{R}^\sharp$ and consider $n \in \mathbb{N}$. Then, since $n \geq 1$, an easy calculation proves that $a^n \in \mathcal{R}^\sharp$, $(a^n)^\sharp = (a^\sharp)^n$ and $p_{a^n} = p_a$. Note that if $a \in \mathcal{R}^{-1}$, then $p_{a^n} = p_a = 0$.

Consider $a \in \mathcal{R}$. The element a is said to be *Drazin invertible*, if there exists a (necessarily unique) $x \in \mathcal{R}$ such that

$$a^m xa = a^m, \quad xax = x, \quad ax = xa,$$

for some $m \in \mathbb{N}$ (see for example [2, 8]). In this case, the solution of these equations will be denoted by a^d and $\mathcal{R}^D$ will stand for the set of all Drazin invertible elements of $\mathcal{R}$. In addition, the smallest m for which the above equations hold is called the *Drazin index* of a and it will be denoted by $\mathrm{ind}(a)$. Note that $\mathrm{ind}(a) = 1$ if and only if a is group invertible. On the other hand, it is not difficult to prove that if $a \in \mathcal{R}^D$ and $\mathrm{ind}(a) = k$, then $a^k \in \mathcal{R}^\sharp$, $(a^k)^\sharp = (a^d)^k$ and $p_{a^k} = 1 - aa^d$. Furthermore, according to [5, Theorem 11], $a \in \mathcal{R}$ is Drazin invertible if and only if a is invertible along a^m, for some $m \in \mathbb{N}$. Moreover, in this case, $a^d = a^{\| a^m}$.

Next the definition of generalized Drazin invertible elements will be recalled. However, to this end some preparation is needed.

An element $a \in \mathcal{R}$ is said to be *quasinilpotent*, if for every $x \in \mathrm{comm}(a)$, $1 + xa \in \mathcal{R}^{-1}$, where $\mathrm{comm}(a) = \{x \in \mathcal{R} \colon ax = xa\}$ (see [4, Definition 2.1]). The set of all quasinilpotent elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{qnil}$. Note that if $\mathcal{R}^{nil}$ denotes the set of nilpotent elements of $\mathcal{R}$, then $\mathcal{R}^{nil} \subset \mathcal{R}^{qnil}$ (see [4]).

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Recall that $a \in \mathcal{R}$ is said to be *generalized Drazin invertible,* if there exists $b \in \mathcal{R}$ such that

$$b \in \mathrm{comm}^2(a), \quad ab^2 = b, \quad a^2b - b \in \mathcal{R}^{qnil},$$

where $\mathrm{comm}^2(a) = \{x \in \mathcal{R}\colon xy = yx \text{ for all } y \in \mathrm{comm}(a)\}$ (see [4, Definition 4.1]). The set of all generalized Drazin invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{gD}$. Note that a necessary and sufficient condition for $a \in \mathcal{R}^{gD}$ is that there exists $p \in \mathrm{comm}^2(a) \cap \mathcal{R}^{\bullet}$ such that $ap \in \mathcal{R}^{qnil}$ and $a + p \in \mathcal{R}^{-1}$ ([4, Theorem 4.2]). Moreover, this idempotent, which is unique ([4, Proposition 2.3]), is called the *spectral idempotent* of a and is denoted by a^{π}. Furthermore, $a \in \mathcal{R}^{gD}$ has at most one generalized Drazin inverse ([4, Theorem 4.2]), which will be denoted by a^D. In addition, in this case, $a^{\pi} = 1 - aa^D = 1 - a^D a$ ([4, p. 142]).

On the other hand, note that $\mathcal{R}^{\sharp} \subset \mathcal{R}^D \subset \mathcal{R}^{gD}$. Moreover, if $a \in \mathcal{R}^D$, then $a^D = a^d$ and $a^{\pi} = 1 - aa^d$ ([4, Proposition 4.9 and Remark 4.10]), In particular, if $a \in \mathcal{R}^{\sharp}$, then $p_a = a^{\pi}$.

Recall that according to [6, Theorem 8], if $a \in \mathcal{R}^{gD}$, then a is invertible along $d = 1 - a^{\pi}$ and $a^{\| 1 - a^{\pi}} = a^D$.

The last generalized inverse that will be recalled in this section is the Moore-Penrose inverse.

An involution ${}^*\colon \mathcal{R} \to \mathcal{R}$ is an anti-isomorphism:

$$(a + b)^* = a^* + b^*, \quad (ab)^* = b^* a^*, \quad (a^*)^* = a,$$

where a, $b \in \mathcal{R}$.

An element $a \in \mathcal{R}$ is said to be *Moore-Penrose invertible,* if there exists a (necessarily unique) $b \in \mathcal{R}$ such that

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

The Moore-Penrose inverse of a is denoted by $a^{\dagger}$ and the set of all Moore-Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{\dagger}$ (see for example [4]). Recall that according to [5, Theorem 11], a necessary and sufficient condition for $a \in \mathcal{R}^{\dagger}$ is that a is invertible along a^*. Moreover, in this case $a^{\| a^*} = a^{\dagger}$.

In addition, recall that $a \in \mathcal{R}^{\dagger}$ *is said to be EP*, if $aa^{\dagger} = a^{\dagger} a$ (see [4]). Let $\mathcal{R}^{EP}$ be the set of all EP elements in $\mathcal{R}$.

3. Equivalent conditions for the inverse along an element. In this section, new conditions equivalent to the ones in Definition 2.3 will be given.

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First of all note that if $\mathcal{R}$ is a ring and $b,\ d \in \mathcal{R}$ are such that $b\mathcal{R} = d\mathcal{R}$, then $b_{-1}(0) = d_{-1}(0)$; similarly from $\mathcal{R}b = \mathcal{R}d$ it can be derived that $b^{-1}(0) = d^{-1}(0)$. These conditions will be used to prove the invertibility along an element. Next follows a preliminary result.

Proposition 3.1. *Consider $b,\ d \in \mathcal{R}$.*

(a) *Let $a \in \mathcal{R}$ be such that b is an outer inverse of a. Then*
   (i) *if $b^{-1}(0) \subseteq d^{-1}(0)$, then $d = dab$; in particular $\mathcal{R}d \subseteq \mathcal{R}b$;*
   (ii) *if $b_{-1}(0) \subseteq d_{-1}(0)$, then $d = bad$; in particular $d\mathcal{R} \subseteq b\mathcal{R}$.*

(b) *Suppose that $d \in \hat{\mathcal{R}}$. Then*
   (iii) *if $d^{-1}(0) \subseteq b^{-1}(0)$, then $\mathcal{R}b \subseteq \mathcal{R}d$;*
   (iv) *if $d_{-1}(0) \subseteq b_{-1}(0)$, then $b\mathcal{R} \subseteq d\mathcal{R}$.*

Proof. (i) Since $b(1 - ab) = 0$, $1 - ab \in b^{-1}(0) \subseteq d^{-1}(0)$. Thus, $d(1 - ab) = 0$.

(ii) Apply a similar argument to the one used in the proof of statement (i).

(iii) Let $\overline{d} \in \mathcal{R}$ be such that $d\overline{d}d = d$. Then d is an outer inverse of $\overline{d}$. Therefore, according to statement (i), $\mathcal{R}b \subseteq \mathcal{R}d$.

(iv) Apply a similar argument to the one used in the proof of statement (iii), using statement (ii) instead of statement (i). □

In the following theorems, equivalent conditions to the ones in Definition 2.3 will be proved. These conditions will be presented in two different theorems to show when it is necessary to assume the regularity of the element $d \in \mathcal{R}$.

Theorem 3.2. *Consider $a,\ b,\ d \in \mathcal{R}$ be such that b is an outer inverse of a. Then, the following statements are equivalent.*

(i) *b is the inverse of a along d.*

(ii) *$\mathcal{R}d = \mathcal{R}b$, $b\mathcal{R} \subseteq d\mathcal{R}$ and $b_{-1}(0) \subseteq d_{-1}(0)$.*

(iii) *$b\mathcal{R} = dR$, $\mathcal{R}b \subseteq \mathcal{R}d$ and $b^{-1}(0) \subseteq d^{-1}(0)$.*

(iv) *$\mathcal{R}b \subseteq \mathcal{R}d$, $b\mathcal{R} \subseteq d\mathcal{R}$, $b_{-1}(0) \subseteq d_{-1}(0)$ and $b^{-1}(0) \subseteq d^{-1}(0)$.*

Proof. It is clear that statement (i) implies that each of the other statements holds.

On the other hand, to prove that each of statements (ii)–(iv) implies that b is the inverse of a along d, note that according to Proposition 3.1 (i) (respectively, Proposition 3.1 (ii)), if $b^{-1}(0) \subseteq d^{-1}(0)$ (respectively, if $b_{-1}(0) \subseteq d_{-1}(0)$), then $\mathcal{R}d \subseteq \mathcal{R}b$ (respectively, $d\mathcal{R} \subseteq b\mathcal{R}$). □

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THEOREM 3.3. *Consider a, b, $d \in \mathcal{R}$ be such that b is an outer inverse of a and $d \in \hat{\mathcal{R}}$. Then, the following statements are equivalent.*

(i) *b is the inverse of a along d.*
(ii) $\mathcal{R}d = \mathcal{R}b$, $d\mathcal{R} \subseteq b\mathcal{R}$ *and* $d_{-1}(0) \subseteq b_{-1}(0)$.
(iii) $b\mathcal{R} = dR$, $\mathcal{R}d \subseteq \mathcal{R}b$ *and* $d^{-1}(0) \subseteq b^{-1}(0)$.
(iv) $\mathcal{R}b = \mathcal{R}d$, $b_{-1}(0) = d_{-1}(0)$.
(v) $\mathcal{R}b \subseteq \mathcal{R}d$, $d\mathcal{R} \subseteq b\mathcal{R}$, $b^{-1}(0) \subseteq d^{-1}(0)$ *and* $d_{-1}(0) \subseteq b_{-1}(0)$.
(vi) $\mathcal{R}d \subseteq \mathcal{R}b$, $b\mathcal{R} \subseteq d\mathcal{R}$, $b_{-1}(0) \subseteq d_{-1}(0)$ *and* $d^{-1}(0) \subseteq b^{-1}(0)$.
(vii) $\mathcal{R}d \subseteq \mathcal{R}b$, $d\mathcal{R} \subseteq b\mathcal{R}$, $d^{-1}(0) \subseteq b^{-1}(0)$ *and* $d_{-1}(0) \subseteq b_{-1}(0)$.
(viii) $b\mathcal{R} = dR$, $b^{-1}(0) = d^{-1}(0)$.
(ix) $\mathcal{R}b \subseteq \mathcal{R}d$, $b^{-1}(0) \subseteq d^{-1}(0)$ *and* $b_{-1}(0) = d_{-1}(0)$.
(x) $\mathcal{R}d \subseteq \mathcal{R}b$, $d^{-1}(0) \subseteq b^{-1}(0)$ *and* $b_{-1}(0) = d_{-1}(0)$.
(xi) $d\mathcal{R} \subseteq b\mathcal{R}$, $d_{-1}(0) \subseteq b_{-1}(0)$ *and* $b^{-1}(0) = d^{-1}(0)$.
(xii) $b\mathcal{R} \subseteq d\mathcal{R}$, $b_{-1}(0) \subseteq d_{-1}(0)$ *and* $b^{-1}(0) = d^{-1}(0)$.
(xiii) $b^{-1}(0) = d^{-1}(0)$ *and* $b_{-1}(0) = d_{-1}(0)$.

Proof. As in the proof of Theorem 3.2, statement (i) implies all the other statements.

To prove that statements (ii)–(xiii) imply that b is the inverse of a along d, note the following facts. According to Proposition 3.1 (i) (respectively, Proposition 3.1 (ii)), if $b^{-1}(0) \subseteq d^{-1}(0)$ (respectively, if $b_{-1}(0) \subseteq d_{-1}(0)$), then $\mathcal{R}d \subseteq \mathcal{R}b$ (respectively, $d\mathcal{R} \subseteq b\mathcal{R}$). On the other hand, according to Proposition 3.1 (iii) (respectively, Proposition 3.1 (iv)), if $d^{-1}(0) \subseteq b^{-1}(0)$ (respectively, if $d_{-1}(0) \subseteq b_{-1}(0)$), then $\mathcal{R}b \subseteq \mathcal{R}d$ (respectively, $b\mathcal{R} \subseteq d\mathcal{R}$). □

4. Further invertible elements. In this section, using ideas similar to the ones in [5, Theorem 7] (see also [7, Theorem 2.1]), more equivalent conditions to the existence of an inverse along an element will be given. Moreover, thanks to these characterizations, given $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ such that a is invertible along d, more elements invertible along d will be constructed.

First of all, note that if $d \in \hat{\mathcal{R}}$ and $\overline{d}$ is a generalized inverse of d, then $d\overline{d}\mathcal{R}d\overline{d}$ (respectively, $\overline{d}d\mathcal{R}\overline{d}d$) is a subring of $\mathcal{R}$ with identity $d\overline{d}$ (respectively, $\overline{d}d$).

THEOREM 4.1. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ with generalized inverse $\overline{d}$. Then, the following conditions are equivalent.*

(i) *The element a is invertible along d.*
(ii) $dad\overline{d} \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$.

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(iii) $\overline{d}dad \in (\overline{d}d\mathcal{R}\overline{d}d)^{-1}$.

(iv) $dad\overline{d}$ *is group invertible and* $p_{dad\overline{d}} = 1 - d\overline{d}$.

(v) $\overline{d}dad$ *is group invertible and* $p_{\overline{d}dad} = 1 - \overline{d}d$.

Furthermore, if any of these statements holds, then given $x \in d\overline{d}\mathcal{R}d\overline{d}$ *(respectively,* $y \in \overline{d}d\mathcal{R}\overline{d}d$*) the inverse of* $dad\overline{d}$ *(respectively, of* $\overline{d}dad$*) in the subring* $d\overline{d}\mathcal{R}d\overline{d}$ *(respectively, in the subring* $\overline{d}d\mathcal{R}\overline{d}d$*),* $xd = a^{\|d}$ *(respectively,* $dy = a^{\|d}$*).*

Proof. Suppose that $a^{\|d}$ exists. To prove statement (ii), first note that $dad\overline{d} = d\overline{d}dad\overline{d} \in d\overline{d}\mathcal{R}d\overline{d}$. Moreover, according to [5, Theorem 7], $d\overline{d}\mathcal{R} = d\mathcal{R} = da\mathcal{R}$. In particular, there is $u \in \mathcal{R}$ such that $d\overline{d} = dau$. Furthermore, da is group invertible ([5, Theorem 7]). As a result,

$$d\overline{d} = dau = da(da(da)^\sharp ud\overline{d}).$$

Thus, $z = da(da)^\sharp ud\overline{d} = d\overline{d}da(da)^\sharp ud\overline{d} \in d\overline{d}\mathcal{R}d\overline{d}$ and $d\overline{d} = daz = dad\overline{d}z$.

On the other hand, according again to [5, Theorem 7], $\mathcal{R}d = \mathcal{R}ad$ and ad is group invertible. In particular, there is $v \in \mathcal{R}$ such that $d = vad$. Thus,

$$d\overline{d} = vad\overline{d} = (d\overline{d}v(ad)^\sharp ada)dad\overline{d} = (d\overline{d}v(ad)^\sharp adad\overline{d})dad\overline{d}.$$

Then, $w = d\overline{d}v(ad)^\sharp adad\overline{d} \in d\overline{d}\mathcal{R}d\overline{d}$ and $d\overline{d} = wdad\overline{d}$. Consequently, $dad\overline{d} \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$.

Suppose that statement (ii) holds. Let $x \in d\overline{d}\mathcal{R}d\overline{d}$ such that

$$dad\overline{d}x = d\overline{d} = xdad\overline{d}.$$

Then, it will be proved that $xd = a^{\|d}$. First of all, note that

$$xdaxd = xdad\overline{d}xd = d\overline{d}xd = xd.$$

In addition, clearly $\mathcal{R}xd \subseteq \mathcal{R}d$ and since $x = d\overline{d}x$, $xd\mathcal{R} \subseteq d\mathcal{R}$. Moreover, since $xdad\overline{d} = d\overline{d}$, $xdad = d$, which implies that $d\mathcal{R} \subseteq xd\mathcal{R}$. Similarly, since $dad\overline{d}x = d\overline{d}$, $dad\overline{d}xd = d$, which implies that $\mathcal{R}d \subseteq \mathcal{R}xd$.

To prove the equivalence between statements (ii) and (iv), apply Remark 2.5 (ii).

The equivalence among the statements (i), (iii) and (v) can be proved using similar arguments. ∎

Remark 4.2. Under the same hypotheses in Theorem 4.1 and using the same notation in this theorem, note that since $xd = a^{\|d}$, $(dad\overline{d})^{-1}_{d\overline{d}\mathcal{R}d\overline{d}} = (dad\overline{d})^\sharp = x = a^{\|d}\overline{d}$. Similarly, since $dy = a^{\|d}$, $(\overline{d}dad)^{-1}_{\overline{d}d\mathcal{R}\overline{d}d} = (\overline{d}dad)^\sharp = y = \overline{d}a^{\|d}$.

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Given $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ such that a is invertible along d, applying Theorem 4.1 new elements invertible along d can be created.

COROLLARY 4.3. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ with generalized inverse $\overline{d}$. Then, the following statements are equivalent.*

(i) *a is invertible along d.*
(ii) *$ad\overline{d}$ is invertible along d.*
(iii) *$\overline{d}da$ is invertible along d.*

Furthermore, if any of these statements hold, then

$$a^{\| d} = (ad\overline{d})^{\| d} = (\overline{d}da)^{\| d}.$$

Proof. Note that $dad\overline{d}d\overline{d} = dad\overline{d}$. Therefore, according to Theorem 4.1, statements (i) and (ii) are equivalent.

A similar argument proves the equivalence between statements (i) and (iii).

According to Theorem 4.1, if $x \in d\overline{d}\mathcal{R}d\overline{d}$ is an inverse of $dad\overline{d}$, then $a^{\| d} = xd$. In particular, since $dad\overline{d}d\overline{d} = dad\overline{d}$, $a^{\| d} = (ad\overline{d})^{\| d}$. A similar argument, using the inverse of $\overline{d}dad \in \overline{d}d\mathcal{R}\overline{d}d$, proves that $a^{\| d} = (\overline{d}da)^{\| d}$. ∎

COROLLARY 4.4. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ with generalized inverse $\overline{d}$. Then, if $a^{\| d}$ exists and x, $y \in \mathcal{R}$, the following statements hold.*

(i) *$a + x(1 - d\overline{d})$ is invertible along d.*
(ii) *$a + (1 - \overline{d}d)y$ is invertible along d.*
(iii) *$a + x(1 - d\overline{d}) + (1 - \overline{d}d)y$ is invertible along d.*

Moreover, $a^{\| d} = (a + x(1 - d\overline{d}))^{\| d} = (a + (1 - \overline{d}d)y)^{\| d} = (a + x(1 - d\overline{d}) + (1 - \overline{d}d)y)^{\| d}$.

Proof. Since $ad\overline{d} = (a + x(1 - d\overline{d}))d\overline{d}$, according to Corollary 4.3, $(a + x(1 - d\overline{d}))^{\| d}$ exists and $a^{\| d} = (a + x(1 - d\overline{d}))^{\| d}$. A similar argument proves that $a + (1 - \overline{d}d)y$ is invertible along d and $a^{\| d} = (a + (1 - \overline{d}d)y)^{\| d}$. Statement (iii) and the remaining identity can be derived applying statements (i) and (ii). ∎

Given $a \in \mathcal{R}^{-1}$, $(a^{-1})^{-1} = a$. The following proposition will show how this property can be reformulated for the inverse along an element.

PROPOSITION 4.5. *Consider $a \in \mathcal{R}$ and d, $\overline{d} \in \hat{\mathcal{R}}$ such that a is invertible along d and $\overline{d}$ is a normalized generalized inverse of d. Let x, $y \in \mathcal{R}$. Then, the following statements hold.*

(i) *$a^{\| d} + x(1 - \overline{d}d) + (1 - d\overline{d})y$ is invertible along $\overline{d}$.*

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(ii) $(a^{\|d} + x(1 - \overline{d}d) + (1 - d\overline{d})y)^{\|\overline{d}} = (a^{\|d})^{\|\overline{d}} = \overline{d}dad\overline{d}$.

Proof. Since a is invertible along d, according to Theorem 4.1, $v = dad\overline{d} \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$, and if w is the inverse of v in $(d\overline{d}\mathcal{R}d\overline{d})^{-1}$, then $a^{\|d} = wd$.

Note that since $\overline{d}$ is a normalized generalized inverse of d, both $\overline{d}a^{\|d}\overline{d}d = \overline{d}wd$ and $\overline{d}vd$ belong to $\overline{d}d\mathcal{R}\overline{d}d$. Furthermore, two direct calculations prove that $\overline{d}a^{\|d}\overline{d}d$ is invertible in $\overline{d}d\mathcal{R}\overline{d}d$ with inverse $\overline{d}vd$. Therefore, according to Theorem 4.1, $a^{\|d}$ is invertible along $\overline{d}$ and $(a^{\|d})^{\|\overline{d}} = \overline{d}vd\overline{d} = \overline{d}dad\overline{d}$. To conclude the proof, apply statement (iii) of Corollary 4.4. ☐

To end this section, the case $d \in \mathcal{R}^\bullet$ is considered.

Corollary 4.6. *Consider $a \in \mathcal{R}$, $p \in \mathcal{R}^\bullet$, x, $y \in \mathcal{R}$ and $m = x(1-p)+(1-p)y$. Then, the following statements are equivalent.*

(i) *a is invertible along p.*
(ii) *ap is invertible along p.*
(iii) *pa is invertible along p.*
(iv) *$pap \in (p\mathcal{R}p)^{-1}$.*
(v) *pap is group invertible and $p_{pap} = 1 - p$.*

Moreover, in this case, $a + m$, $ap + m$, and $pa + m$ are invertible along p and

$$(a + m)^{\|p} = (ap + m)^{\|p} = (pa + m)^{\|p} = (pap)^\sharp.$$

Furthermore, if a is invertible along p, then $(pap)^\sharp + m$ is invertible along p and $((pap)^\sharp + m)^{\|p} = ((pap)^\sharp)^{\|p} = pap$.

Proof. Apply Theorem 4.1, Corollary 4.3, Corollary 4.4 and Proposition 4.5. ☐

5. The set of invertible elements along a fixed $d \in \hat{\mathcal{R}}$.

In this section, given a regular element $d \in \mathcal{R}$, the set of all invertible elements along d will be fully characterized. Moreover, some special cases will be also considered. To this end, the set under consideration will be denoted by $\mathcal{R}^{\|d}$, i.e., $\mathcal{R}^{\|d} = \{a \in \mathcal{R} \colon a^{\|d} \text{ exists}\}$. Note that if $\tilde{d} \in \hat{\mathcal{R}}$ is such that $d\mathcal{R} = \tilde{d}\mathcal{R}$ and $\mathcal{R}d = \mathcal{R}\tilde{d}$, then $\mathcal{R}^{\|d} = \mathcal{R}^{\|\tilde{d}}$. In addition, recall that any $p \in \mathcal{R}^\bullet \setminus \{0,\ 1\}$ leads to the Pierce decomposition $\mathcal{R} = p\mathcal{R}p \oplus p\mathcal{R}(1-p) \oplus (1-p)\mathcal{R}p \oplus (1-p)\mathcal{R}(1-p)$. Next follows the main theorem of this section.

Theorem 5.1. *Consider d and $\overline{d} \in \mathcal{R}$ such that $d \in \hat{\mathcal{R}}$ and $\overline{d}$ is a generalized inverse of d.*

(i) *Then, the following identity holds:*

$$\mathcal{R}^{\|d} = \overline{d}(d\overline{d}\mathcal{R}d\overline{d})^{-1} + (1 - \overline{d}d)\mathcal{R}d\overline{d} \oplus \mathcal{R}(1 - d\overline{d}).$$

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Moreover, given $x,\, y \in \mathcal{R}$ *and* $v \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$ *with inverse* w*, then*

$$(\overline{d}v)^{\|d} = (\overline{d}v + (1-\overline{d}d)xd\overline{d} + y(1-d\overline{d}))^{\|d} = wd.$$

(ii) *In addition,*

$$\mathcal{R}^{\|d} = (\overline{d}d\mathcal{R}\overline{d}d)^{-1}\overline{d} + \overline{d}d\mathcal{R}(1-d\overline{d}) \oplus (1-\overline{d}d)\mathcal{R}.$$

Furthermore, given $s,\, t \in \mathcal{R}$ *and* $z \in (\overline{d}d\mathcal{R}\overline{d}d)^{-1}$ *with inverse* u*, then*

$$(z\overline{d})^{\|d} = (z\overline{d} + \overline{d}ds(1-d\overline{d}) + (1-d\overline{d})t)^{\|d} = du.$$

(iii) *In particular, if* $p \in \mathcal{R}^{\bullet} \setminus \{0,1\}$*, then*

$$\mathcal{R}^{\|p} = (p\mathcal{R}p)^{-1} + p\mathcal{R}(1-p) \oplus (1-p)\mathcal{R}p \oplus (1-p)\mathcal{R}(1-p),$$

and if $r \in (p\mathcal{R}p)^{-1}$ *with inverse* l *and* $m \in p\mathcal{R}(1-p) \oplus (1-p)\mathcal{R}p \oplus (1-p)\mathcal{R}(1-p)$*, then*

$$r^{\|p} = (r+m)^{\|p} = l.$$

Proof. (i) Let $v \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$ and consider $a = \overline{d}v$. Thus, $dad\overline{d} = d\overline{d}vd\overline{d} = v$. Consequently, according to Theorem 4.1, $a^{\|d}$ exists and $a^{\|d} = wd$. Moreover, according to Corollary 4.4, $(a + (1-\overline{d}d)xd\overline{d} + y(1-d\overline{d}))^{\|d}$ exists and $a^{\|d} = (a + (1-\overline{d}d)xd\overline{d} + y(1-d\overline{d}))^{\|d}$.

To prove the converse, let $a \in \mathcal{R}$ such that $a^{\|d}$ exists. Then

$$a = \overline{d}dad\overline{d} + (1-\overline{d}d)ad\overline{d} + a(1-d\overline{d}).$$

However, $\overline{d}dad\overline{d} = \overline{d}(dad\overline{d})$ and, according to Theorem 4.1, $dad\overline{d} \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$.

(ii) Apply a similar argument to the one used in the proof of statement (a).

(iii) Given $d = p \in \mathcal{R}^{\bullet} \setminus \{0,1\}$, consider $\overline{d} = p$. Then apply what has been proven. ∎

Remark 5.2. Under the same hypotheses in Theorem 5.1 and using the same notation in this Theorem, note the following facts.

(i)

$$\begin{aligned}(1-\overline{d}d)\mathcal{R}d\overline{d} \oplus \mathcal{R}(1-d\overline{d}) &= \overline{d}d\mathcal{R}(1-d\overline{d}) \oplus (1-\overline{d}d)\mathcal{R} \\ &= (1-\overline{d}d)\mathcal{R}d\overline{d} \oplus \overline{d}d\mathcal{R}(1-d\overline{d}) \oplus (1-\overline{d}d)\mathcal{R}(1-d\overline{d}).\end{aligned}$$

(ii) $\overline{d}(d\overline{d}\mathcal{R}d\overline{d})^{-1} \subseteq \overline{d}d\mathcal{R}d\overline{d}$. In addition,

$$\overline{d}(d\overline{d}\mathcal{R}d\overline{d})^{-1} \cap (1-\overline{d}d)\mathcal{R}d\overline{d} \oplus \mathcal{R}(1-d\overline{d}) = \emptyset.$$

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(iii) In particular, given $p \in \mathcal{R}^\bullet \setminus \{0, 1\}$,

$$(p\mathcal{R}p)^{-1} \cap (p\mathcal{R}(1-p) \oplus (1-p)\mathcal{R}p \oplus (1-p)\mathcal{R}(1-p)) = \emptyset.$$

(iv) The elements w, u and l in Theorem 5.1 can be presented as in Remark 4.2 (w and u) and Corollary 4.6 (l). For example, $w = (\overline{d}v)^{\|d}\overline{d}$, $u = \overline{d}(z\overline{d})^{\|d}$ and $l = r^\sharp$.

Applying Theorem 5.1 it is possible to give another characterization of the inverse along an element.

THEOREM 5.3. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ with generalized inverse $\overline{d}$. Then, the following statements are equivalent.*

(i) *The element a is invertible along d.*

(ii) *There exist (necessarily unique) s, $t \in \mathcal{R}$ such that $a = \overline{d}s + t$, $s \in \mathcal{R}^\sharp$, $p_s = 1 - d\overline{d}$ and $\overline{d}dtd\overline{d} = 0$. In addition, in this case $a^{\|d} = s^\sharp d$.*

(iii) *There exist (necessarily unique) u, $v \in \mathcal{R}$ such that $a = u\overline{d} + v$, $u \in \mathcal{R}^\sharp$, $p_u = 1 - \overline{d}d$ and $\overline{d}dvd\overline{d} = 0$. Moreover, in this case $a^{\|d} = du^\sharp$.*

In particular, if $p \in \mathcal{R}^\bullet$, a necessary and sufficient condition for $a \in \mathcal{R}^p$ is that there exist (necessarily unique) s, $t \in \mathcal{R}$ such that $a = s + t$, $s \in \mathcal{R}^\sharp$, $p_s = 1 - p$ and $ptp = 0$. Furthermore, in this case $a^{\|p} = s^\sharp$.

Proof. If a is invertible along d, then according to Theorem 5.1 (i), there exist $s \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$ and $t \in (1 - \overline{d}d)\mathcal{R}d\overline{d} \oplus \mathcal{R}(1 - d\overline{d})$ such that $a = \overline{d}s + t$. Note that according to Remark 2.5 (ii) and Theorem 5.1 (i), $s \in \mathcal{R}^\sharp$, $p_s = 1 - d\overline{d}$ and $a^{\|d} = s^\sharp d$. Moreover, $\overline{d}dtd\overline{d} = 0$.

On the other hand, if statement (ii) holds, then according to Remark 2.5 (ii), $s \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$ and $t \in (1 - \overline{d}d)\mathcal{R}d\overline{d} \oplus \mathcal{R}(1 - d\overline{d})$. Thus, according to Theorem 5.1 (i), $a \in \mathcal{R}^{\|d}$.

Let s_1, s_2 and t_1 and t_2 be such that $a = \overline{d}s_1 + t_1 = \overline{d}s_2 + t_2$, s_1, $s_2 \in (d\overline{d}\mathcal{R}d\overline{d})^{-1}$ and t_1, $t_2 \in (1 - \overline{d}d)\mathcal{R}d\overline{d} \oplus \mathcal{R}(1 - d\overline{d})$. Then, $t_1 = t_2$ and $\overline{d}s_1 = \overline{d}s_2$. However, multiplying by d on the left side gives $s_1 = s_2$.

Similar arguments prove the equivalence between statements (i) and (iii), applying Theorem 5.1 (ii) instead of Theorem 5.1 (i).

The last statement can be proved using similar arguments and applying in particular Theorem 5.1 (iii). □

EXAMPLE 5.4. Under the same hypotheses in Theorem 5.3, let a be invertible along d. Then, according to [5, Lemma 3] and [5, Theorem 7] (other references are [7, Definition 1.3] and [7, Theorem 2.1]), $\overline{d} = a(da)^\sharp$ is a generalized inverse of d, and

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a can be decomposed in the following way: $a = a(da)^\sharp da + (1 - a(da)^\sharp d)a$. Note that $s = da \in \mathcal{R}^\sharp$ and $t = (1 - a(da)^\sharp d)a$ is such that $\overline{d}dtd\overline{d} = 0$, as $dt = (d - daa^{\|d})a = 0$.

Using a similar argument, it is not difficult to prove that $\overline{d} = (ad)^\sharp a$ is a generalized inverse of d, $a = ad(ad)^\sharp a + (1 - ad(ad)^\sharp)a$, $u = ad \in \mathcal{R}^\sharp$ and $v = (1 - ad(ad)^\sharp)a$ is such that $\overline{d}dvd\overline{d} = 0$.

Next the particular cases of group, (generalized) Drazin and commuting Moore-Penrose invertible elements will be considered.

THEOREM 5.5.

(a) *Consider $a \in \mathcal{R}$.*

(i) *If $a \in \mathcal{R}^\sharp$ and $n \in \mathbb{N}$, then*

$$\mathcal{R}^{\|a^n} = \mathcal{R}^{\|(a^\sharp)^n} = \mathcal{R}^{\|1-p_a}.$$

In addition, if x belongs to one of these sets, then $x^{\|a^n} = x^{\|(a^\sharp)^n} = x^{\|1-p_a}$.

(ii) *If $a \in \mathcal{R}^D$, $\mathrm{ind}(a) = k$, $n \in \mathbb{N}$, $1 \le j \le k-1$ and $m \ge k$, then*

$$\mathcal{R}^{\|(a^d)^n} = \mathcal{R}^{\|a^m} = \mathcal{R}^{\|a^j a^d a} = \mathcal{R}^{\|1-a^\pi}.$$

Moreover, if x belongs to one of these sets, then $x^{\|(a^d)^n} = x^{\|a^m} = x^{\|a^j a^d a} = x^{\|1-a^\pi}$.

(iii) *If $a \in \mathcal{R}^{gD}$ and $n \in \mathbb{N}$, then*

$$\mathcal{R}^{\|(a^D)^n} = \mathcal{R}^{\|a^n a^d a} = \mathcal{R}^{\|1-a^\pi}.$$

Furthermore, if x belongs to one of these sets, then $x^{\|(a^d)^n} = x^{\|a^n a^d a} = x^{\|1-a^\pi}$.

(b) *Suppose that $\mathcal{R}$ has an involution.*

(iv) *If $a \in \mathcal{R}$ is EP and $n \in \mathbb{N}$, then*

$$\mathcal{R}^{\|a^n} = \mathcal{R}^{\|(a^\dagger)^n} = \mathcal{R}^{\|(a^*)^n} = \mathcal{R}^{\|((a^\dagger)^*)^n} = \mathcal{R}^{\|aa^\dagger}.$$

If x belongs to one of these sets, then $x^{\|a^n} = x^{\|(a^\dagger)^n} = x^{\|(a^*)^n} = x^{\|((a^\dagger)^*)^n} = x^{\|aa^\dagger}$.

Proof. (i) According to Theorem 5.1,

$$\begin{aligned}
\mathcal{R}^{\|a} &= a^\sharp((1-p_a)\mathcal{R}(1-p_a))^{-1} + p_a\mathcal{R}(1-p_a) \oplus \mathcal{R}p_a,\\
\mathcal{R}^{\|a^\sharp} &= a((1-p_a)\mathcal{R}(1-p_a))^{-1} + p_a\mathcal{R}(1-p_a) \oplus \mathcal{R}p_a,\\
\mathcal{R}^{\|1-p_a} &= ((1-p_a)\mathcal{R}(1-p_a))^{-1} + p_a\mathcal{R}(1-p_a) \oplus \mathcal{R}p_a.
\end{aligned}$$

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However, since a and $a^\sharp \in ((1-p_a)\mathcal{R}(1-p_a))^{-1}$ (Remark 2.5 (ii)),

$$a^\sharp((1-p_a)\mathcal{R}(1-p_a))^{-1} = a((1-p_a)\mathcal{R}(1-p_a))^{-1} = ((1-p_a)\mathcal{R}(1-p_a))^{-1}.$$

Thus, $\mathcal{R}^{\|a} = \mathcal{R}^{\|a^\sharp} = \mathcal{R}^{\|1-p_a}$. In addition, since according to Remark 2.5 (iv), $a^n \in \mathcal{R}^\sharp$, $(a^n)^\sharp = (a^\sharp)^n$ and $p_{a^n} = p_a$, applying what has been proved to a^n, $\mathcal{R}^{\|a^n} = \mathcal{R}^{\|(a^\sharp)^n} = \mathcal{R}^{\|1-p_a}$ $(n \in \mathbb{N})$.

Note that to prove the remaining statement, it is enough to consider the case $n = 1$. Let $a \in \mathcal{R}^\sharp$ and $x \in \mathcal{R}^{\|a}$. According to Theorem 5.1 (i) applied to $\overline{d} = a^\sharp$, $x^{\|a} = wa$, where $w \in (aa^\sharp\mathcal{R}aa^\sharp)^{-1}$ is such that $waxaa^\sharp = axaa^\sharp w = aa^\sharp$. On the other hand, according to Theorem 5.1 (iii) applied to $aa^\sharp$, $x^{\|aa^\sharp} = v$, where $v \in (aa^\sharp\mathcal{R}aa^\sharp)^{-1}$ is such that $vaa^\sharp xaa^\sharp = aa^\sharp xaa^\sharp v = aa^\sharp$. Now well, since $a = aaa^\sharp$

$$waaa^\sharp xaa^\sharp = vaa^\sharp xaa^\sharp.$$

However, since $aa^\sharp xaa^\sharp \in (aa^\sharp\mathcal{R}aa^\sharp)^{-1}$,

$$x^{\|a} = wa = v = x^{\|aa^\sharp}.$$

(ii) Recall that according to Remark 2.5 (iii). $a^d \in \mathcal{R}^\sharp$, $(a^d)^\sharp = aa^da$ and $p_{a^d} = a^\pi$. In addition, if $1 \le j \le k-1$ and $m \ge k$ $(k = \mathrm{ind}(a))$, then it is not difficult to prove that $((a^d)^\sharp)^j = a^ja^da$ and $((a^d)^\sharp)^m = a^m$. To prove statement (ii), apply statement (i) to a^d.

(iii) As in the proof of statement (ii), according to Remark 2.5 (iii). $a^D \in \mathcal{R}^\sharp$, $(a^D)^\sharp = aa^Da$ and $p_{a^D} = a^\pi$. Moreover, if $n \in \mathbb{N}$, then $((a^D)^\sharp)^n = a^na^Da$. Then, apply statement (i) to a^D.

(iv) If $a \in \mathcal{R}$ is EP, then a, $a^\dagger$, a^* and $(a^\dagger)^* \in \mathcal{R}^\sharp$. Note that $a^\sharp = a^\dagger$ and $(a^*)^\sharp = (a^\dagger)^*$. Moreover, $p_a = p_{a^\dagger} = p_{a^*} = p_{(a^\dagger)^*} = 1 - aa^\dagger$. Then, apply statement (i). □

Next a characterization of invertible elements along a group invertible element will be considered.

Corollary 5.6. *Consider $d \in \mathcal{R}^\sharp$ and $a \in \mathcal{R}$. Then, the following are equivalent.*

(i) *The element a is invertible along d.*

(ii) *There exist (necessarily unique) s, $t \in \mathcal{R}$ such that $a = s+t$, $s \in \mathcal{R}^\sharp$, $p_s = p_d$ and $(1-p_d)t(1-p_d) = 0$. Moreover, $a^{\|d} = s^\sharp$.*

Proof. Apply Theorem 5.5 (i) and Theorem 5.3. □

Remark 5.7. Clearly, if $d \in \mathcal{R}^{-1} \cup \mathcal{R}^\bullet$, then Corollary 5.6 applies to d. In addition, according to the proof of Theorem 5.5, Corollary 5.6 applies to $d = x^n$

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($x \in \mathcal{R}^\sharp$, $n \in \mathbb{N}$), to $d = x^n x^D x$ ($x \in \mathcal{R}^{gD}$, $n \in \mathbb{N}$) or to $d = y^j$ ($y \in \mathcal{R}^D$, $j \in \mathbb{N}$, $j \geq \mathrm{ind}(y)$). Moreover, if $\mathcal{R}$ is a unital ring with an involution, according again to Theorem 5.5, Corollary 5.6 applies to $d = x^n$, $d = (x^\dagger)^n$, $d = (x^*)^n$ or $d = ((x^*)^\dagger)^n$ ($x \in \mathcal{R}$ an EP element, $n \in \mathbb{N}$).

In the following theorem, given $d \in \hat{\mathcal{R}}$, the invertibility along elements related to d will be studied.

Theorem 5.8. *Consider $d \in \hat{\mathcal{R}}$. Then, if $a \in \mathcal{R}$ and $u \in \mathcal{R}^{-1}$, the following statements are equivalent.*

(i) $a \in \mathcal{R}^{\| d}$.

(ii) $au^{-1} \in \mathcal{R}^{\| ud}$.

(iii) $u^{-1}a \in \mathcal{R}^{\| du}$.

Moreover, $\mathcal{R}^{\| ud} = \mathcal{R}^{\| d}u^{-1}$, $\mathcal{R}^{\| du} = u^{-1}\mathcal{R}^{\| d}$ and if one of the statements holds, then $(au^{-1})^{\| ud} = ua^{\| d}$ and $(u^{-1}a)^{\| du} = a^{\| d}u$.

Proof. In first place, note that if $d \in \hat{R}$, then $ud \in \hat{\mathcal{R}}$. In fact, if $\overline{d} \in \mathcal{R}$ is such that $d = d\overline{d}d$, then $ud = ud(\overline{d}u^{-1})ud$. In addition, since $\overline{d}u^{-1}ud = \overline{d}d$, $(\overline{d}u^{-1}ud\mathcal{R}\overline{d}u^{-1}ud)^{-1} = (\overline{d}d\mathcal{R}\overline{d}d)^{-1}$ and $\overline{d}u^{-1}udau^{-1}ud = \overline{d}dad$. Consequently, according to Theorem 4.1, statements (i) and (ii) are equivalent. Furthermore, applying Theorem 5.1 (ii), a direct calculation proves that $\mathcal{R}^{ud} = \mathcal{R}^{\| d}u^{-1}$.

Let $a \in \mathcal{R}^{\| d}$. Then, according to Theorem 5.1 (ii), there exist $z \in (\overline{d}d\mathcal{R}\overline{d}d)^{-1}$ and $m \in \overline{d}d\mathcal{R}(1 - d\overline{d}) \oplus (1 - \overline{d}d)\mathcal{R}$ such that $a = z\overline{d} + m$ and $a^{\| d} = (z\overline{d})^{\| d} = dw$, where $w \in \overline{d}d\mathcal{R}\overline{d}d$ is such that $zw = wz = \overline{d}d$. Thus, $au^{-1} = z\overline{d}u^{-1} + mu^{-1}$ and according to Theorem 5.1 (ii), $(au^{-1})^{\| ud} = (z\overline{d}u^{-1})^{\| ud} = udw = ua^{\| d}$.

The equivalence between statements (i) and (iii) and the remaining identities can be proved using similar arguments. □

6. The reverse order law. In this section, the reverse order law for the inverse along an element will be studied. In the first place, the definition of the notion under consideration will be given.

Given $d \in \hat{\mathcal{R}}$, a, $b \in \mathcal{R}^{\| d}$ will be said *to satisfy the reverse order law,* if $ab \in \mathcal{R}^{\| d}$ and $(ab)^{\| d} = b^{\| d}a^{\| d}$. Recall that $\mathcal{R}^{\| 0} = \mathcal{R}$ and $a^{\| 0} = 0$, $a \in \mathcal{R}$; in addition, if $d \in \mathcal{R}^{-1}$, then $\mathcal{R}^{\| d} = \mathcal{R}^{-1}$ and if $a \in \mathcal{R}^{\| d}$, then $a^{\| d} = a^{-1}$. In particular, if $d \in \mathcal{R}^{-1} \cup 0$, then any pair of elements a, $b \in \mathcal{R}^{\| d}$ satisfy the reverse order law. In what follows, the elements $d \in \hat{\mathcal{R}}$ that have this property will be characterized.

Theorem 6.1. *Consider $d \in \hat{R}$ and suppose that the reverse order law holds for any pair of elements a, $b \in \mathcal{R}^{\| d}$. Then, d is group invertible.*

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Proof. Let $\overline{d} \in \mathcal{R}$ be such that $d = d\overline{d}d$. According to Theorem 5.1 (i), $\overline{d}d\overline{d} \in \mathcal{R}^{\|d}$. Moreover, $(\overline{d}d\overline{d})^{\|d} = d$.

Consider $a = \overline{d}d\overline{d} = b$. Since the reverse order law holds for ab, $ab \in \mathcal{R}^{\|d}$ and $(ab)^{\|d} = b^{\|d}a^{\|d} = d^2$. In addition, Definition 2.3 implies that $d^2\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}d^2 = \mathcal{R}d$. Therefore, according to [2, Theorem 4], $d \in \mathcal{R}^\sharp$. $\square$

In order to prove the main result of this section, a particular case will be studied first.

Proposition 6.2. *Consider $p \in \mathcal{R}^\bullet \setminus \{0, 1\}$ and a, $b \in \mathcal{R}^{\|p}$. Then, the following statements are equivalent.*

(i) *The reverse order law holds for a and b.*

(ii) $pa(1 - p)bp = 0$.

In particular, the reverse order law holds for any pair of elements a, $b \in \mathcal{R}^{\|p}$ if and only if $p\mathcal{R}(1 - p) \cdot (1 - p)\mathcal{R}p = 0$.

Proof. According to the statement (iii) of Theorem 5.1, applying the Pierce decomposition to a and b, these elements have the following properties: $a = pap + m$, $b = pbp + n$, pap, $pbp \in (p\mathcal{R}p)^{-1}$, $m = pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p)$, $n = pb(1 - p) + (1 - p)bp + (1 - p)b(1 - p)$, $a^{\|d} = r$ and $b^{\|d} = s$, where r and s are the inverse of pap and pbp in $(p\mathcal{R}p)^{-1}$, respectively.

Now, a direct calculation proves that $ab = papbp + z$, where $z \in \mathcal{R}$. In addition, it is not difficult to prove that $z \in p\mathcal{R}(1 - p) \oplus (1 - p)\mathcal{R}p \oplus (1 - p)\mathcal{R}(1 - p)$ if and only if $pa(1 - p)bp = 0$. According to the statement (iii) of Theorem 5.1, statement (i) implies statement (ii).

On the other hand, if statement (ii) holds, $ab = papbp + z$, with $z \in p\mathcal{R}(1 - p) \oplus (1 - p)\mathcal{R}p \oplus (1 - p)\mathcal{R}(1 - p)$. However, $papbp \in (p\mathcal{R}p)^{-1}$, with inverse sr in $p\mathcal{R}p$. Therefore, according again to the statement (iii) of Theorem 5.1, $ab \in \mathcal{R}^p$ and $(ab)^{\|d} = b^{\|d}a^{\|d}$. $\square$

For example, if $p \in \mathcal{R} \setminus \{0, 1\}$ is a central idempotent, then p satisfies the hypothesis of Proposition 6.2. In fact, in this case $p\mathcal{R}(1 - p) = (1 - p)\mathcal{R}p = 0$. Next follows the main result of this section.

Theorem 6.3. *Let $d \in \mathcal{R}^\sharp$ and consider a, $b \in \mathcal{R}^{\|d}$. Then, the following statements are equivalent.*

(i) *The reverse order law for a and b holds.*

(ii) $(1 - p_d)ap_db(1 - p_d) = 0$.

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In particular, the reverse order law holds for any pair of elements a, $b \in \mathcal{R}^{\|d}$ *if and only if* $(1-p_d)\mathcal{R}p_d \cdot p_d\mathcal{R}(1-p_d) = 0$.

Proof. Apply statement (i) of Theorem 5.5 and Proposition 6.2. ∎

Note that when $d = 0$, $p_0 = 1$ and when $d \in \mathcal{R}^{-1}$, $p_d = 0$, so that the conditions in statements (ii) of Proposition 6.2 and Theorem 6.3 are trivially satisfied.

7. Commutative inverses along an element. In this section, given $d \in \hat{\mathcal{R}}$ and $a \in \mathcal{R}^{\|d}$, it will be characterized when $aa^{\|d} = a^{\|d}a$. Note that in this case, since $a^{\|d}aa^{\|d} = a^{\|d}$, $a^{\|d} \in \mathcal{R}^{\sharp}$ and $(a^{\|d})^{\sharp} = a^2a^{\|d}$ (see Remark 2.5 (iii)). Next follows the first characterization of this section.

THEOREM 7.1. *Consider* $d \in \hat{\mathcal{R}}$ *and* $a \in \mathcal{R}^{\|d}$. *Then, the following are equivalent.*

(i) $aa^{\|d} = a^{\|d}a$.
(ii) $d \in \mathcal{R}^{\sharp}$ *and* $ap_d = p_da$.
(iii) $d \in \mathcal{R}^{\sharp}$ *and* $a = x + m$, *where* $x \in ((1-p_d)\mathcal{R}(1-p_d))^{-1}$ *and* $m \in p_d\mathcal{R}p_d$.

Proof. Note that according to Corollary 5.6, statements (ii) and (iii) are equivalent.

Assume that $aa^{\|d} = a^{\|d}a$. Recall that, according to [5, Lemma 3], $a^{\|d}ad = d = daa^{\|d}$. In addition, since $a^{\|d} \in d\mathcal{R}$, there is $x \in \mathcal{R}$ such that $a^{\|d} = dx$. Now, using the hypothesis, $d = daa^{\|d} = da^{\|d}a = d^2xa \in d^2\mathcal{R}$. Similarly, since $a^{\|d}ad = d$ and $a^{\|d} = yd$, for some $y \in \mathcal{R}$ ($a^{\|d} \in \mathcal{R}d$), $d = ayd^2 \in \mathcal{R}d^2$. Therefore, according to [2, Theorem 4], $d \in \mathcal{R}^{\#}$.

Next consider $d^{\sharp}$. Note that according to the structure of $a^{\|d}$ presented in Theorem 5.1 (i) (using $\overline{d} = d^{\sharp}$), it is easy to prove that

$$a^{\|d}dd^{\sharp} = a^{\|d} = dd^{\sharp}a^{\|d},$$

equivalently,

$$a^{\|d}p_d = 0 = p_da^{\|d}.$$

Consequently, $p_daa^{\|d} = 0 = a^{\|d}ap_d$ and since $a^{\|d}ad = d$,

$$aa^{\|d} = a^{\|d}a = a^{\|d}add^{\sharp} + a^{\|d}a(1-dd^{\sharp}) = dd^{\sharp} + a^{\|d}ap_d = dd^{\sharp} = d^{\sharp}d.$$

As a result,

$$add^{\sharp} = aa^{\|d}a = dd^{\sharp}a,$$

which implies that $ap_d = p_da$.

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To prove the converse, recall that according to Theorem 5.5 (i), $\mathcal{R}^{\|d} = \mathcal{R}^{\|1-p_d}$. Moreover, since $ap_d = p_d a$, according to Theorem 5.1 (iii), there exist $x \in (dd^\sharp \mathcal{R} dd^\sharp)^{-1}$ and $m \in p_d \mathcal{R} p_d$ such that $a = x + m$. Moreover, according to Theorem 5.1 (iii), $a^{\|d} = w$, where $w \in dd^\sharp \mathcal{R} dd^\sharp$ is such that $xw = wx = dd^\sharp$. However, a straightforward calculation proves that

$$aa^{\|d} = xw = wx = a^{\|d}a = dd^\sharp. \quad \square$$

Thanks to Theorem 7.1, a characterization of group invertible elements in terms of commuting inverse along an element can be derived.

Corollary 7.2. *Consider $d \in \hat{\mathcal{R}}$. Then, the following statements are equivalent.*

(i) $d \in \mathcal{R}^\sharp$.

(ii) *There exists $a \in \mathcal{R}^{\|d}$ such that $aa^{\|d} = a^{\|d}a$.*

Proof. If $d \in \mathcal{R}^\sharp$, then consider $a = d$. In fact, according to [5, Theorem 11], $a^{\|d}$ exists, moreover, $a^{\|d} = d^\sharp$.

On the other hand, if statements (ii) holds, then apply Theorem 7.1. $\square$

Next follows the second characterization of this section.

Theorem 7.3. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ such that a is invertible along d. Then, the following statements are equivalent.*

(i) $a^{\|d}a = aa^{\|d}$.

(ii) $da \in \mathcal{R}d$ *and* $ad \in d\mathcal{R}$.

Proof. Let $\overline{d} \in \mathcal{R}$ be such that $d = d\overline{d}d$. Recall that according to [7, Theorem 3.2], $da + 1 - d\overline{d}$ and $ad + 1 - \overline{d}d$ are invertible in $\mathcal{R}$ and $a^{\|d} = (da + 1 - d\overline{d})^{-1}d = d(ad + 1 - \overline{d}d)^{-1}$. Hence,

$$\begin{aligned} a^{\|d}a = aa^{\|d} &\Leftrightarrow (da + 1 - d\overline{d})^{-1}da = ad(ad + 1 - \overline{d}d)^{-1} \\ &\Leftrightarrow da(ad + 1 - \overline{d}d) = (da + 1 - d\overline{d})ad \\ &\Leftrightarrow da(1 - \overline{d}d) = (1 - d\overline{d})ad. \end{aligned}$$

Assume that $a^{\|d}a = aa^{\|d}$. If the last equality is multiplied on the left by $d\overline{d}$, then $da(1 - d^-d) = 0$. Thus, $(1 - dd^-)ad = 0$. Therefore, $da = da\overline{d}d \in \mathcal{R}d$ and $ad = d\overline{d}ad \in d\mathcal{R}$.

To prove the converse, suppose that $da \in \mathcal{R}d$. Then there exists $u \in \mathcal{R}$ such that $da = ud$. Consequently, $da\overline{d}d = ud\overline{d}d = ud = da$. Similarly, $ad \in d\mathcal{R}$ implies

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that $d\overline{d}ad = ad$. As a result, $da(1 - \overline{d}d) = 0 = (1 - d\overline{d})ad$, which is equivalent to $a^{\|d}a = aa^{\|d}$. □

REMARK 7.4. Suppose that $\mathcal{R}$ has an involution and consider $a \in \mathcal{R}$ a Moore-Penrose invertible element. From Theorem 7.3, it follows that a necessary and sufficient condition for a to be EP is that $aa^* \in a^*\mathcal{R}$ and $a^*a \in \mathcal{R}a^*$.

8. Inner inverse. In this section, inverses along an element that are also inner inverses will be studied. Next follows a characterization of this object.

THEOREM 8.1. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ such that a is invertible along d. The following statements are equivalent:*

(i) *$a^{\|d}$ is an inner inverse of a.*
(ii) *$\mathcal{R} = d\mathcal{R} \oplus a^{-1}(0)$.*
(iii) *$\mathcal{R} = \mathcal{R}d \oplus a_{-1}(0)$.*

Proof. Since a is invertible along d, $a^{\|d}\mathcal{R} = d\mathcal{R}$ and $\mathcal{R}a^{\|d} = \mathcal{R}d$. Apply then Remark 2.2 (iv). □

In the following theorem, some special cases will be considered.

THEOREM 8.2. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ such that a is invertible along d and $a^{\|d}$ is an inner inverse of a. The following statements hold.*

(i) *If $\overline{d}$ is an inner inverse of d, then $a^{\|d}$ is an inner and outer inverse of $\overline{d}dad\overline{d}$.*
(ii) *If d is group invertible, then $a^{\|d} = (d^{\#}dadd^{\#})^{\#}$.*
(iii) *If the ring $\mathcal{R}$ has an involution and d is Moore-Penrose invertible, then $a^{\|d} = (d^{\dagger}dadd^{\dagger})^{\dagger}$.*

Proof. Note that according to Definition 2.3, $a^{\|d}\overline{d}d = a^{\|d} = d\overline{d}a^{\|d}$.

(i) From the previous observation,

$$a^{\|d}(\overline{d}dad\overline{d})a^{\|d} = (a^{\|d}\overline{d}d)a(d\overline{d}a^{\|d}) = a^{\|d}aa^{\|d} = a^{\|d}$$

and

$$(\overline{d}dad\overline{d})a^{\|d}(\overline{d}dad\overline{d}) = \overline{d}da(d\overline{d}a^{\|d}\overline{d}d)ad\overline{d} = \overline{d}daa^{\|d}ad\overline{d} = \overline{d}dad\overline{d}.$$

(ii) It remains to prove that $a^{\|d}(d^{\sharp}dadd^{\sharp}) = (d^{\sharp}dadd^{\sharp})a^{\|d}$. This follows from $a^{\|d}(d^{\sharp}dadd^{\sharp}) = (a^{\|d}d^{\sharp}d)add^{\sharp} = a^{\|d}add^{\sharp} = dd^{\sharp}$ and $(d^{\sharp}dadd^{\sharp})a^{\|d} = d^{\sharp}daa^{\|d} = d^{\sharp}d$ (Theorem 5.1 (i)–(ii)).

(iii) It is enough to prove that $a^{\|d}(d^{\dagger}dadd^{\dagger})$ and $(d^{\dagger}dadd^{\dagger})a^{\|d}$ are Hermitian. In fact, $a^{\|d}(d^{\dagger}dadd^{\dagger}) = dd^{\dagger}$ and $(d^{\dagger}dadd^{\dagger})a^{\|d} = d^{\dagger}d$. □

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In particular, when EP elements are considered, new expressions of the group and Moore-Penrose inverse can be obtained.

COROLLARY 8.3. *Suppose that $\mathcal{R}$ has an involution and let $a \in \mathcal{R}$ be EP. The following statements hold.*

(i) $a^\dagger = ((aa^\sharp)^* a(a^\sharp a)^*)^\sharp$.
(ii) $a^\sharp = (a^\dagger a^3 a^\dagger)^\dagger$.

Proof. Recall that $a^\sharp = a^\dagger$, $a^\dagger = a^{\| a^*}$ and $a^\sharp = a^{\| a}$ ([5, Theorem 11]). In addition, recall that a^* is group invertible and $(a^*)^\sharp = (a^\sharp)^*$. Apply then Theorem 8.2. $\square$

To present more expressions of the group and the Moore-Penrose inverse, the following theorem will be useful.

THEOREM 8.4. *Consider $a \in \mathcal{R}$ and $d \in \hat{\mathcal{R}}$ such that a invertible along d. If x is an inner inverse of dad, then $a^{\| d} = dxd$.*

Proof. Since $dadxdad = dad$, $d(adxdad - ad) = 0$. According to Theorem 3.3 (xiii), $a^{\| a}(adxdad - ad) = 0$, i.e., $a^{\| a}adxdad = a^{\| a}ad$. According again to Theorem 3.3 (xiii), $a^{\| a}adxdaa^{\| a} = a^{\| a}aa^{\| a} = a^{\| a}$. However, according to [5, Lemma 3], $a^{\| a}ad = d = daa^{\| a}$. Therefore, $dxd = a^{\| a}$. $\square$

COROLLARY 8.5. *Consider $a \in \mathcal{R}$. The following statements hold.*

(i) *If a is group invertible and $\overline{a}$ is an inner inverse of a, then $a\overline{a}a^\sharp = a^\sharp = a^\sharp \overline{a} a$.*
(ii) *If a is group invertible and x is an inner inverse of a^3, then $a^\sharp = axa$.*

If in addition $\mathcal{R}$ has an involution, then the following statements hold.

(iii) *If a is Moore-Penrose invertible and $\overline{a}$ is an inner inverse of a, then $a^\dagger(a\overline{a})^* = a^\dagger = (\overline{a}a)^* a^\dagger$.*
(iv) *If a is Moore-Penrose invertible and x is an inner inverse of a^*aa^*, then $a^\dagger = a^*xa^*$.*

Proof. To prove statement (i) (respectively, statement (iii)) recall that according to [5, Theorem 11] $a^\sharp = a^{\| a}$ (respectively, $a^\dagger = a^{\| a^*}$). Then apply $a^{\| d}\overline{d}d = a^{\| d} = d\overline{d}a^{\| d}$ (see the proof of Theorem 8.2) to $d = a$ and $\overline{d} = \overline{a}$ (respectively, $d = a^*$ and $\overline{d} = (\overline{a})^*$).

To prove statement (iii) (respectively, statement (iv)), use that $a^\sharp = a^{\| a}$ (respectively, $a^\dagger = a^{\| a^*}$) ([5, Theorem 11]) and then apply Theorem 8.4 with $d = a$ (respectively, $d = a^*$). $\square$

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