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POTENTIALLY EVENTUALLY POSITIVE STAR SIGN PATTERNS*

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Abstract. An n-by-n real matrix A is eventually positive if there exists a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. An n-by-n sign pattern $\mathcal{A}$ is potentially eventually positive (PEP) if there exists an eventually positive real matrix A with the same sign pattern as $\mathcal{A}$. An n-by-n sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP. Berman et al. [A. Berman, M. Catral, L.M. Dealba, A. Elhashash, F. Hall, L. Hogben, I.J. Kim, D.D. Olesky, P. Tarazaga, M.J. Tsatsomeros, and P. van den Driessche. Sign patterns that allow eventual positivity. *Electronic Journal of Linear Algebra*, 19:108–120, 2010.] established some sufficient and some necessary conditions for an n-by-n sign pattern to allow eventual positivity and classified the potentially eventually positive sign patterns of order $n \leq 3$. However, the identification and classification of PEP sign patterns of order $n \geq 4$ remain open. In this paper, all the n-by-n PEP star sign patterns are classified by identifying all the MPEP star sign patterns.

Key words. Star sign pattern, Primitive digraph, PEP sign pattern, MPEP sign pattern.

AMS subject classifications. 15A48, 05C50, 15A18.

1. Introduction.

A *sign pattern* is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in $\{+, -, 0\}$. We denote the set of all n-by-n sign patterns by Q_n. The *qualitative class* of $\mathcal{A}$ is the set of all real matrices with the same sign pattern as $\mathcal{A}$. A *permutation similarity* on a pattern $\mathcal{A} \in Q_n$ is a sign pattern of the form $\mathcal{P}^T \mathcal{A} \mathcal{P}$, where $\mathcal{P}$ is a permutation matrix. A pattern $\mathcal{A}$ is *reducible* if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is *irreducible* if it is not reducible; see, e.g. [3, 7].

A *subpattern* of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a *proper subpattern* of $\mathcal{A}$. If $\mathcal{B}$ is a subpattern

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of $\mathcal{A}$, then $\mathcal{A}$ is a *superpattern* of $\mathcal{B}$. For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, we define the *positive part* of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. The *negative part* of $\mathcal{A}$ can be defined similarly.

Next, we recall some graph theoretical concepts from [3, 7, 8] and the references therein.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is *combinatorially symmetric* if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices $1, 2, \ldots, n$ and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the *graph* of the pattern $\mathcal{A}$. A combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a *star sign pattern* if $G(\mathcal{A})$ is a star.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has *signed digraph* $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \ldots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). A (directed) *simple cycle of length* k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ such that the vertices $i_1, \ldots, i_k$ are distinct. Recall that a digraph $D = (V, E)$ is *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. It is well known that a digraph D is primitive if and only if there exists a natural number k such that for all $v_i \in V$, $v_j \in V$, there is a walk of length k from v_i to v_j. A nonnegative sign pattern $\mathcal{A}$ is *primitive* if its signed digraph $\Gamma(\mathcal{A})$ is primitive; see, e.g. [2] for more details.

A sign pattern matrix $\mathcal{A}$ *requires* property P if every real matrix $A \in Q(\mathcal{A})$ has the property P and *allows* P (or is *potentially* P) if there is some $A \in Q(\mathcal{A})$ that has property P.

The n-by-n real matrix A is *eventually positive* if there exists a nonnegative integer k_0 such that $A^k > 0$ for all $k \geq k_0$; see, e.g., [5, 9, 13]. An n-by-n sign pattern $\mathcal{A}$ is *potentially eventually positive* (PEP), if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [2, 6] and the references therein. An n-by-n sign pattern $\mathcal{A}$ is *a minimal potentially eventually positive sign pattern* (MPEP sign pattern) if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP; see, e.g. [12]. Sign patterns that allow eventual positivity were studied first in [2], where a sufficient condition and some necessary conditions for a sign pattern to be potentially eventually positive were established. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be potentially eventually positive remains open. Also open is the classification of sign patterns that are potentially eventually positive. Recently, PEP sign patterns with reducible positive part were constructed in [1]. In [4], sign patterns that require or allow power-positivity were investigated, and a connection between the PEP sign patterns and potentially power-positive sign patterns was established. The minimal potentially power-positive sign patterns were considered in [10]. More recently, the MPEP tridiagonal sign patterns were identified

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and all PEP tridiagonal sign patterns were classified in [11].

In this paper, we focus on the potential eventual positivity of star sign patterns. Our work is organized as follows. In Section 2, some preliminaries of PEP star sign patterns are established. The PEP star sign patterns with exactly one nonzero diagonal entry are identified in Section 3. In Section 4, the MPEP star sign patterns are identified, and consequently, all PEP star sign patterns are classified.

2. Preliminaries. We begin this section by restating several necessary or sufficient conditions for a sign pattern to allow eventual positivity which were established in [2]. Following [2], we denote a sign pattern consisting entirely of positive (respectively, negative) entries by $[+]$ (respectively, $[-]$).

Lemma 2.1. ([2], Theorem 2.1) *If $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP.*

Lemma 2.2. ([2], Theorem 3.1, Theorem 3.3, Lemma 4.3 and Corollary 4.5) *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then the following hold:*

(a) $\mathcal{A}$ is irreducible.

(b) Every row and column of $\mathcal{A}$ has at least one $+$ and the minimal number of $+$ entries in $\mathcal{A}$ is $n+1$.

(c) Every superpattern of $\mathcal{A}$ is PEP.

(d) If $\hat{\mathcal{A}}$ is the sign pattern obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$, then $\hat{\mathcal{A}}$ is PEP.

(e) There is an eventually positive matrix $A \in Q(\mathcal{A})$ such that $\rho(A) = 1$, $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ all ones matrix, and if $n \geq 2$, then the sum of all the off-diagonal entries of A is positive.

Lemma 2.3. ([2], Proposition 5.3) *If $\mathcal{A}$ is the checkerboard block sign pattern*

$$\begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

Recall that the n-by-n real matrix A is said to *possess the strong Perron-Frobenius property* if its spectral radius $\rho(A)$ is a simple, positive and strictly dominant eigenvalue and the corresponding eigenvector is positive; see, for instance, [2, 5, 9] for more

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details.

LEMMA 2.4. ([9], Theorem 2.2) *For an n-by-n real matrix A, the following properties are equivalent:*

(1) Both A and A^T possess the strong Perron-Forbenius property.

(2) A is an eventually positive matrix.

(3) A^T is an eventually positive matrix.

We note that the potential eventual positivity of sign patterns is preserved under permutation similar and transposition; that is, an n-by-n sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{P}^T\mathcal{A}\mathcal{P}$ or $\mathcal{P}^T\mathcal{A}^T\mathcal{P}$ is PEP, where $\mathcal{P}$ is an n-by-n permutation matrix. In this case, we say they are equivalent. Thus, without loss of generality, we may assume that an n-by-n star sign pattern $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \begin{bmatrix} ? & * & * & \cdots & * \\ * & ? & 0 & \cdots & 0 \\ * & 0 & ? & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & ? \end{bmatrix} \tag{2.1}$$

where $*$ denotes the nonzero entries, ? denotes one of $+$, $-$ and 0.

Next we turn to the necessary conditions for an n-by-n star sign pattern to be potentially eventually positive.

PROPOSITION 2.5. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern. If $\mathcal{A}$ is PEP, then $\mathcal{A}$ contains at least one positive diagonal entry.*

Proof. Suppose to the contrary that $\mathcal{A}$ is PEP and contains no positive diagonal entry, i.e., $\alpha_{ii} = 0$ or $-$ for $i = 1, 2, \ldots, n$. By (b) of Lemma 2.2, $\alpha_{i1} = \alpha_{1i} = +$ for all $i = 2, 3, \ldots, n$. Consequently, $\mathcal{A}$ is a subpattern of the sign pattern

$$\hat{\mathcal{A}} = \left[\begin{array}{c|cccc} - & + & + & \cdots & + \\ \hline + & - & - & \cdots & - \\ + & - & - & \cdots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ + & - & - & \cdots & - \end{array}\right].$$

The checkerboard block sign pattern $\hat{\mathcal{A}}$ is not PEP by Lemma 2.3. It follows that sign pattern $\mathcal{A}$ is not PEP by (c) of Lemma 2.2; a contradiction. $\square$

PROPOSITION 2.6. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern. If $\mathcal{A}$ is PEP, then $\mathcal{A}$ is symmetric.*

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Proof. Since sign pattern $\mathcal{A}$ is PEP, it follows by (e) of Lemma 2.2 that there exists an eventually positive matrix $A = [a_{ij}] \in Q(\mathcal{A})$ such that the spectral radius of $\mathcal{A}$ is 1 and the sum of entries of each row must be equal to 1. Note that $1 - a_{k1} = a_{kk}$. Let $(1, w_2, \ldots, w_n)$ be the positive left Perron-Frobenius eigenvector of A. Since $w^T A = w^T$, we have $a_{1k} + w_k(1 - a_{k1}) = w_k$ and thus $w_k = \frac{a_{1k}}{a_{k1}} > 0$. It follows that $\alpha_{k1}\alpha_{1k} > 0$ for all $k = 2, 3, \ldots, n$. □

Theorem 2.7. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern. If $\mathcal{A}$ is PEP, then $\alpha_{i1} = \alpha_{1i} = +$ for all $i = 2, 3, \ldots, n$.*

Proof. Since $\mathcal{A}$ is PEP, $\alpha_{i1} = \alpha_{1i}$ for all $i = 2, 3, \ldots, n$ by Proposition 2.6. Let m be the number of i such that $\alpha_{i1} = \alpha_{1i} = -$. To complete the proof, it suffices to show that $m = 0$. By a way of contradiction, assume that $m \geq 1$. Without loss of generality, suppose that $\alpha_{i1} = \alpha_{1i} = -$ for $i = 2, 3, \ldots, m+1$, and $\alpha_{j1} = \alpha_{1j} = +$ for $j = m+2, m+3, \ldots, n$. Let

$$\mathcal{B} = \left[\begin{array}{c|ccc|ccc}
+ & - & \cdots & - & + & \cdots & + \\ \hline
- & + & & & & & \\
\vdots & & \ddots & & & & \\
- & & & + & & & \\ \hline
+ & & & & + & & \\
\vdots & & & & & \ddots & \\
+ & & & & & & +
\end{array}\right]$$

obtained from $\mathcal{A}$ by changing all 0 and $-$ diagonal entries of $\mathcal{A}$ to $+$. By (d) of Lemma 2.2, $\mathcal{B}$ is PEP. It is clear that $\mathcal{B}$ is a subpattern of sign pattern

$$\mathcal{C} = \begin{bmatrix} + & [-] & [+] \\ [-] & [+]_{m\times m} & [-] \\ [+] & [-] & [+]_{(n-m-1)\times(n-m-1)} \end{bmatrix}.$$

It follows that $\mathcal{C}$ is also PEP. However, the checkerboard block sign pattern $\mathcal{C}$ is not PEP by Lemma 2.3. It is a contradiction. □

We end this section by classifying the PEP star sign patterns of order $n \leq 3$ which can be shown directly by considering the minimality of PEP star sign patterns and Theorem 2.7.

Proposition 2.8. *Let $\mathcal{A}$ be an n-by-n star sign pattern.*

(a) If $n = 1$, then $\mathcal{A}$ is PEP if and only if $\mathcal{A} = [+]$.

(b) If $n = 2$, then $\mathcal{A}$ is PEP if and only if $\mathcal{A}$ is equivalent to a superpattern of

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the following sign pattern

$$\begin{bmatrix} + & + \\ + & 0 \end{bmatrix}.$$

(c) If $n = 3$, then $\mathcal{A}$ is PEP if and only if $\mathcal{A}$ is equivalent to a superpattern of the following two PEP sign patterns:

$$\begin{bmatrix} + & + & + \\ + & 0 & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & + & 0 \\ + & 0 & 0 \end{bmatrix}.$$

3. PEP star sign patterns with exactly one nonzero diagonal entry. In this section, we use the previous results to identify the n-by-n PEP star sign patterns with exactly one nonzero diagonal entry.

THEOREM 3.1. *The following star sign patterns are MPEP:*

$$\mathcal{A}_1 = \begin{bmatrix} + & + & \cdots & + \\ + & 0 & & \\ \vdots & & \ddots & \\ + & & & 0 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} 0 & + & + & \cdots & + \\ + & + & 0 & & \\ + & 0 & 0 & & \\ \vdots & & & \ddots & \\ + & & & & 0 \end{bmatrix}.$$

Proof. By (b) of Lemma 2.2, no sign pattern obtained from $\mathcal{A}_1$ by changing an off diagonal entry in row or column 1 to a 0 is PEP. Since the sign pattern $\mathcal{B}$ obtained from $\mathcal{A}_1$ by replacing the (1, 1) entry with a zero is not primitive, $\mathcal{B}$ is not PEP. It follows that $\mathcal{A}_1$ is MPEP. $\mathcal{A}_2$ is MPEP can be proved similarly. □

THEOREM 3.2. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern with exactly one nonzero diagonal entry. Then $\mathcal{A}$ is PEP if and only if $\mathcal{A}$ is equivalent to either $\mathcal{A}_1$ or $\mathcal{A}_2$.*

Proof. The sufficiency is clear by Theorem 3.1. For the necessity, since $\mathcal{A}$ has exactly one nonzero diagonal entry, $G(\mathcal{A})$ is isomorphic to one of two star graphs shown in Figure 3.1.

Case 1. $G(\mathcal{A})$ has one loop on the center vertex 1. Since star sign pattern $\mathcal{A}$ is PEP, $\alpha_{i1} = \alpha_{1i} = +$ for $i = 2, 3, \ldots, n$, by Theorem 2.7. If $\alpha_{11} = 0$ or $-$, then $\mathcal{A}$ is not PEP by Proposition 2.5. Hence, $\alpha_{11} = +$. It follows that $\mathcal{A}$ is equivalent to $\mathcal{A}_1$.

Case 2. $G(\mathcal{A})$ has one loop on the vertex 2. If star sign pattern $\mathcal{A}$ is PEP, then $\alpha_{22} = +$ by Proposition 2.5, and $\alpha_{i1} = \alpha_{1i} = +$ by Theorem 2.7, for $i = 2, 3, \ldots, n$. It follows that $\mathcal{A}$ is equivalent to $\mathcal{A}_2$. □

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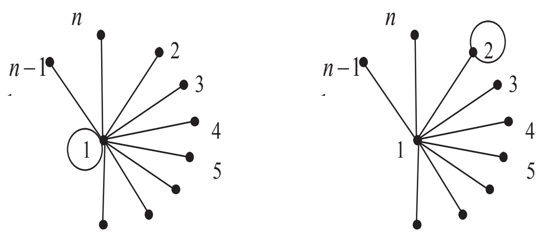

FIG. 3.1. *Star graphs with one loop.*

4. MPEP star sign patterns. Now, we turn to identify all the MPEP star sign patterns and classify all the PEP star sign patterns.

PROPOSITION 4.1. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern and $\alpha_{11} = +$. If $\mathcal{A}$ is MPEP, then $\alpha_{ii} = 0$ for all $i \geq 2$.*

Proof. Assume that sign pattern $\mathcal{A}$ is MPEP. Then by Theorem 2.7, $\mathcal{A}$ is a superpattern of the MPEP sign pattern $\mathcal{A}_1$. It follows that $\mathcal{A} = \mathcal{A}_1$. □

We note that if the condition that α_{11} is a positive diagonal entry in Proposition 4.1 is replaced by $\alpha_{11} = -$, then the conclusion that $\alpha_{ii} = 0$ for all $i \geq 2$ doesn't hold. In fact, if all the diagonal entries of $\mathcal{A}$ are 0 except $\alpha_{11} = -$, then $\mathcal{A}$ is not PEP by Proposition 2.5. Based on above discussions, the following corollary holds readily.

COROLLARY 4.2. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern. If α_{11} is a nonzero diagonal entry, then $\mathcal{A}$ is MPEP if and only if $\mathcal{A}$ is equivalent to $\mathcal{A}_1$.*

PROPOSITION 4.3. *Let $\mathcal{A} = [\alpha_{ij}]$ be an n-by-n star sign pattern, $\alpha_{11} = 0$ and $\alpha_{22} = +$. If $\mathcal{A}$ is MPEP, then $\alpha_{ii} = 0$ for all $i \geq 3$.*

Proof. Assume that sign pattern $\mathcal{A}$ is MPEP. Then by Theorem 2.7, $\mathcal{A}$ is a superpattern of the MPEP sign pattern $\mathcal{A}_2$. It follows that $\mathcal{A} = \mathcal{A}_2$. □

THEOREM 4.4. *Let $\mathcal{A}$ be an n-by-n star sign pattern. Then $\mathcal{A}$ is MPEP if and only if $\mathcal{A}$ is equivalent to one of $\mathcal{A}_1$ and $\mathcal{A}_2$ stated in Theorem 3.1.*

Proof. The sufficiency follows from Theorem 3.1. For the necessity, let $\mathcal{A} = [\alpha_{ij}]$. If $\mathcal{A}$ is MPEP, then $\mathcal{A}$ has at least one positive diagonal entry by Proposition 2.5. Up to equivalence, it suffices to show the following two cases.

Case 1. $\alpha_{11} = +$. If $\mathcal{A}$ is MPEP, then by proposition 4.1, $\alpha_{ii} = 0$ for all $i \geq 2$. If follows readily from Theorem 2.7 that $\mathcal{A}$ is equivalent to $\mathcal{A}_1$.

Case 2. $\alpha_{11} = 0$ *and* $\alpha_{22} = +$. Then by Proposition 4.3, the diagonal entries $\alpha_{ii} = 0$, for $i = 3, \ldots, n$. It follows that $\mathcal{A}$ is equivalent to $\mathcal{A}_2$ by Theorem 2.7. □

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We end this paper by classifying the n-by-n PEP star sign patterns.

COROLLARY 4.5. *Let $\mathcal{A}$ be an n-by-n star sign pattern. Then $\mathcal{A}$ is PEP if and only if $\mathcal{A}$ is a superpattern of a sign pattern that is equivalent to one of $\mathcal{A}_1$ and $\mathcal{A}_2$ stated in Theorem 3.1.*

Proof. Corollary 4.5 follows directly from Theorem 4.4, Lemma 2.2 and Theorem 2.7. □

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