

STOCHASTIC FORMS OF NON-NEGATIVE MATRICES AND PERRON-REGULARITY*

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Abstract. Given a real non-negative square matrix A , the problem of determining when two distinct constructions of stochastic matrices associated to A coincide is studied. All the constructions (or stochastic forms) that are considered are diagonal forms, i.e., the transformations act like $A \mapsto \alpha D^{(r)} A D^{(c)}$, where $D^{(r)}$ and $D^{(c)}$ are diagonal matrices with positive diagonals and $\alpha > 0$, all depending on A .

Key words. Non-negative matrix, Stochastic form, Diagonal form, Tensor product, Perron-regular.

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1. Introduction. There exist several distinct approaches to construct stochastic matrices of transition probabilities of discrete time (homogeneous) Markov chains. Usually a real non-negative square matrix A indexed by the (finite) state space is given and then a certain normalization f yields a stochastic matrix $f(A)$, a *stochastic form* of A . For example, a common standard procedure is to divide each non-zero entry of A by the sum of the entries in the corresponding row [1, 3, 9, 10, 11, 15, 20, 25, 30, 33, 39, 40, 41, 42, 43] (this method is used very often on random walks on simple directed graphs, letting A be the corresponding adjacency matrix). As another example, if *a priori* the Markov chain is known to be doubly stochastic, then one may be able to apply the well known Sinkhorn-Knopp construction to A [8, 22, 24, 27, 28, 29, 35, 37, 38]. Brualdi, Parter and Schneider [8] have studied stochastic forms that result by multiplying the matrix A from the left and the right by the same diagonal matrix D , $A \mapsto DAD$. A final important example occurs when A is irreducible [6, 12, 26, 34, 36]: If $(\lambda_A, \mathbf{p}_A)$ is a right Perron eigenpair of A , then $\lambda_A^{-1} P_A^{-1} A P_A$ is stochastic, where P_A is the diagonal matrix with \mathbf{p}_A in the main diagonal. Our main purpose here is to study when two different stochastic forms coincide. We will consider the four stochastic forms just mentioned and call them *standard*, *Sinkhorn-Knopp*, *Brualdi-Parter-Schneider* and *Perron*, respectively.

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The paper is organized as follows. Section 2 starts with basic definitions and notation. Section 3 presents the stochastic forms and briefly surveys each case, stating well known results and proving basic facts that will be used later. Section 4 contains the main results of the paper which are concerned with conditions for two distinct stochastic forms to yield the same stochastic matrix, a property that we call *stochastic regularity*. We show that stochastic regularity is preserved under tensor products (for an account of this operation on Hilbert spaces see [31]). The proofs rely on direct computations as well as on applications of previous results; in particular, we systematically apply the Perron-Frobenius Theorem. A non-negative square irreducible matrix is called *Perron-regular* if its standard and Perron stochastic forms coincide (Theorem 4.8). This case deserves special attention because if A is the adjacency matrix of a strongly connected simple directed graph, then the Perron stochastic form yields the unique measure of maximal entropy [34], whereas the standard stochastic form is widely used, particularly in random walks on simple directed graphs. We give an algebraic characterization of Perron-regularity in terms of Perron vectors (Definition 4.7). In Section 5, we identify all the (weighted) Perron-regular graphs (Theorems 5.4 and 5.5), and also introduce and study Perron-regular degrees (roughly speaking, the Perron-regular degree measures the “degrees of freedom” imposed by the algebraic characterization of Perron-regularity, and Theorem 5.9 shows that it always attains its maximum possible value). We leave the following as an open problem.

PROBLEM 1.1. *Identify all the Perron-regular directed graphs.*

Theorem 4.12 suggests that such an identification may be given as a decomposition into tensor products of “elementary” Perron-regular directed graphs (e.g. regular directed graphs). Also the colonial digraph introduced in Section 5 may be useful in an inductive approach to Problem 1.1. Although we mainly focus on Perron-regularity, i.e. when the standard and the Perron stochastic forms coincide, the results presented in this paper may be seen as a basic set of tools to address this type of “regularity” problem. Other stochastic forms can be of interest like those coming from the Metropolis-Hastings algorithm [32] as well as transition probabilities defined in terms of square roots of vertex degrees [17].

Observe that the standard, Brualdi-Parter-Schneider and Perron stochastic forms admit each both row and column versions. The Sinkhorn-Knopp doubly stochastic form results by iterating the row and column versions of the standard stochastic form. It is therefore natural to ask what happens when iterating the row and column versions of both the Brualdi-Parter-Schneider and the Perron stochastic forms. This is addressed in Remarks 3.7 and 3.11.

2. Preliminaries. Let $n \geq 1$ be a positive integer and let $[n] = \{1, \dots, n\}$. The identity matrix of size n is denoted by I_n . The column vector of size n whose entries

are all equal to 1 is denoted by J_n , i.e., $J_n^T = (1, \dots, 1) = \{1\}^n$. Given a subset $\Omega \subset \mathbb{R}$ of the real numbers, an Ω -matrix is a matrix whose entries belong to Ω (for example, \mathbb{R}^+ -matrices correspond to matrices with real non-negative entries). Let $A = (A_{i,j})_{i,j=1}^n$ be a square \mathbb{R}^+ -matrix of size n . A is *row* (resp., *column*) *stochastic* if $AJ_n = J_n$ (resp., $J_n^T A = J_n^T$). A is *doubly stochastic* if it is both row and column stochastic. A *diagonal* of A is $(A_{1,\sigma(1)}, \dots, A_{n,\sigma(n)})$ where σ is a permutation of $[n]$, and the *main diagonal* is $\text{diag}(A) = (A_{1,1}, \dots, A_{n,n})$. A has *support* if there exists a positive diagonal. A has *total support* if every non-zero entry of A belongs to a positive diagonal. A is *irreducible* if for every $i, j \in [n]$, there exists an integer $N \geq 1$ such that $A_{i,j}^N > 0$. A is *fully indecomposable* if there exist no non-empty proper subsets $X, Y \subseteq [n]$ such that $\#X + \#Y = n$ and $A_{i,j} = 0$ for every $i \in X$ and $j \in Y$ (if the restriction is relaxed by requiring $X \cap Y = \emptyset$, then a definition equivalent to being irreducible is obtained). Let

$$\begin{aligned} \mathbf{M}^r &= \{A : A \text{ is a square } \mathbb{R}^+ \text{-matrix with no zero rows}\} \\ \mathbf{M}^c &= \{A : A \text{ is a square } \mathbb{R}^+ \text{-matrix with no zero columns}\} \\ \mathbf{M} &= \mathbf{M}^r \cap \mathbf{M}^c \\ \mathbf{P}^r &= \{A \in \mathbf{M}^r : A \text{ is row stochastic}\} \\ \mathbf{P}^c &= \{A \in \mathbf{M}^c : A \text{ is column stochastic}\} \\ \mathbf{P} &= \mathbf{P}^r \cap \mathbf{P}^c \\ \mathbf{B}^r &= \{A \in \mathbf{M}^r : A \text{ satisfies the row condition 1 in Theorem 3.4}\} \\ \mathbf{B}^c &= \{A \in \mathbf{M}^c : A \text{ satisfies the column condition 1 in Theorem 3.4}\} \\ \mathbf{B} &= \mathbf{B}^r \cap \mathbf{B}^c \\ \mathbf{S} &= \{A \in \mathbf{M} : A \text{ has support}\} \\ \mathbf{TS} &= \{A \in \mathbf{S} : A \text{ has total support}\} \\ \mathbf{I} &= \{A \in \mathbf{M} : A \text{ is irreducible}\} \\ \mathbf{FI} &= \{A \in \mathbf{I} : A \text{ is fully indecomposable}\} \\ \mathbf{D} &= \{D \in \mathbf{M} : D \text{ is a diagonal matrix with a positive diagonal}\}. \end{aligned}$$

By a *row* (resp., *column*) *stochastic form*, we will mean a function $f: \mathbf{X} \rightarrow \mathbf{Y}$, where $\mathbf{X} \subseteq \mathbf{M}^r$ (resp., $\mathbf{X} \subseteq \mathbf{M}^c$) and $\mathbf{Y} \subseteq \mathbf{P}^r$ (resp., $\mathbf{Y} \subseteq \mathbf{P}^c$). If $\mathbf{Y} \subseteq \mathbf{P}$, then we say that f is a *doubly stochastic form*. Unless otherwise stated, we only consider stochastic forms f that are *diagonal forms*, i.e., f normalizes the matrix A by a rule

$$A \xrightarrow{f} \alpha D^{(r)} A D^{(c)} \quad \text{with} \quad \alpha > 0 \quad \text{and} \quad D^{(r)}, D^{(c)} \in \mathbf{D},$$

and here $D^{(r)}$, $D^{(c)}$ and α depend on A (and so they are actually functions of A). To represent a diagonal form like this we will write

$$f \leftarrow (\alpha; D^{(r)}, D^{(c)}).$$

Diagonal forms are *pattern-invariant*, that is, the original matrix and its image have the same size and the same *pattern*, i.e. the positions of their non-zero entries coincide,

since for every $i, j \in [n]$ we have

$$A_{i,j} > 0 \Leftrightarrow f(A)_{i,j} = \alpha D_{i,i}^{(r)} A_{i,j} D_{j,j}^{(c)} > 0.$$

Then A will have (total) support if and only if $f(A)$ does, and the same holds for the properties of being irreducible and fully indecomposable. A has a *doubly stochastic pattern* if it has the pattern of a doubly stochastic matrix. Also, A has a *graph* (or *symmetric*) *pattern* if it has the pattern of a symmetric matrix.

Recall the following two well known products of matrices: Let A and B be two real matrices of sizes $h \times k$ and $p \times q$ respectively. The *Kronecker* (or *tensor*) *product* of A and B is the matrix $A \otimes B$ of size $hp \times kq$ defined by

$$A \otimes B = \begin{pmatrix} A_{1,1}B & A_{1,2}B & \cdots & A_{1,k}B \\ A_{2,1}B & A_{2,2}B & \cdots & A_{2,k}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{h,1}B & A_{h,2}B & \cdots & A_{h,k}B \end{pmatrix}.$$

If A and B have the same size (i.e., $p = h$ and $q = k$), then the *Hadamard product* of A and B is the matrix $C = A \circ B$ of size $h \times k$ defined by

$$C_{i,j} = A_{i,j}B_{i,j} \quad \text{for every } i \in [h] \text{ and } j \in [k].$$

Let $\frac{1}{A}$ be the matrix defined by the rule¹

$$\left(\frac{1}{A}\right)_{i,j} = \begin{cases} \frac{1}{A_{i,j}} & \text{if } A_{i,j} \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for every $i \in [h]$ and $j \in [k]$. We will also write $A/B = \frac{A}{B} = A \circ \frac{1}{B}$.

3. Stochastic forms. Henceforth, unless otherwise stated, A and B are square matrices of size $n \geq 1$.

3.1. Standard. Let $A \in \mathbf{M}^r$ (resp., $A \in \mathbf{M}^c$). Let $S_A^{(r)} \in \mathbf{D}$ (resp., $S_A^{(c)} \in \mathbf{D}$) be the diagonal matrix with the positive diagonal

$$\text{diag}(S_A^{(r)})^T = \mathbf{r}_A = \left(r_1^{(A)}, \dots, r_n^{(A)}\right)^T = A J_n$$

¹Observe that if A is invertible and its inverse is A^{-1} , then, in general, $A^{-1} \neq \frac{1}{A}$, but in the particular case when A is a diagonal matrix with no zero entries in the diagonal, then $A^{-1} = \frac{1}{A}$.

$$\left(\text{resp., } \text{diag}(S_A^{(c)}) = \mathbf{c}_A = \left(c_1^{(A)}, \dots, c_n^{(A)} \right) = J_n^T B \right).$$

The *standard row* (resp., *column*) *stochastic form* $\mathcal{S}^r: \mathbf{M}^r \rightarrow \mathbf{P}^r$ (resp., $\mathcal{S}^c: \mathbf{M}^c \rightarrow \mathbf{P}^c$) is defined by

$$\mathcal{S}^r \leftarrow \left(1; \frac{1}{S_A^{(r)}}, I_n \right) \quad \left(\text{resp., } \mathcal{S}^c \leftarrow \left(1; I_n, \frac{1}{S_A^{(c)}} \right) \right).$$

PROPOSITION 3.1. *Let $A \in \mathbf{M}^r$ (resp., $A \in \mathbf{M}^c$). Suppose that $D \in \mathbf{D}$ is such that $D^{-1}A \in \mathbf{P}^r$ (resp., $AD^{-1} \in \mathbf{P}^c$), or in other words, that A admits a stochastic form $(1; D^{-1}, I_n)$ (resp., $(1; I_n, D^{-1})$). Then $\mathcal{S}^r \leftarrow (1; D^{-1}, I_n)$ (resp., $\mathcal{S}^c \leftarrow (1; I_n, D^{-1})$), or more precisely, $D = S_A^{(r)}$ (resp., $D = S_A^{(c)}$).*

Proof. It is straightforward. \square

PROPOSITION 3.2. *Let $A, B \in \mathbf{M}^r$ (resp., $A, B \in \mathbf{M}^c$). Then $\mathcal{S}^r(A) = \mathcal{S}^r(B)$ (resp., $\mathcal{S}^c(A) = \mathcal{S}^c(B)$) if and only if there exists a unique diagonal matrix $D \in \mathbf{D}$ such that $A = DB$ (resp., $A = BD$). The matrix D is given by*

$$\text{diag}(D) = \frac{\mathbf{r}_A}{\mathbf{r}_B} \quad \left(\text{resp., } \text{diag}(D) = \frac{\mathbf{c}_A}{\mathbf{c}_B} \right).$$

Proof. $\mathcal{S}^r(A) = \mathcal{S}^r(B)$ (resp., $\mathcal{S}^c(A) = \mathcal{S}^c(B)$) if and only if the diagonal matrix $D \in \mathbf{D}$ defined by $\text{diag}(D)^T = \frac{\mathbf{r}_A}{\mathbf{r}_B}$ $\left(\text{resp., } \text{diag}(D) = \frac{\mathbf{c}_A}{\mathbf{c}_B} \right)$ yields $A = DB$ (resp., $A = BD$). Uniqueness follows from Proposition 3.1. \square

The standard stochastic form is very common and occurs in a large number of works. There are surveys with references on the subject, starting with Lovasz's [25], also [9], and books [15, 43]. Contributions vary among the many application models [1]. They include many aspects like limiting distributions [10], hitting times [3, 33], results under positive recurrence [30], entropy [41], graph reconstruction [42], cuts [11], networks [39, 40] and applications to genomic data [20].

3.2. Sinkhorn-Knopp. The *Sinkhorn-Knopp* doubly stochastic form $\mathcal{S}: \mathbf{S} \rightarrow \mathbf{P}$ is defined for a matrix $A \in \mathbf{S}$ by²

$$\mathcal{S}(A) := \lim_{k \rightarrow \infty} (\mathcal{S}^c \circ \mathcal{S}^r)^k(A). \quad (3.1)$$

In [37] Sinkhorn showed that if A is a positive matrix, then $\mathcal{S}(A)$ is well defined and converges to a doubly stochastic matrix. Moreover, he showed that, in the domain

²Here \circ denotes composition, not Hadamard product, henceforth it will be clear from context.

of positive square matrices, \mathcal{S} is a diagonal form, i.e. there exist $D^{(r)}, D^{(c)} \in \mathbf{D}$ (depending on A) so that $\mathcal{S} \leftarrow (1; D^{(r)}, D^{(c)})$. He even proved that if $E^{(r)}, E^{(c)} \in \mathbf{D}$ are such that $E^{(r)}AE^{(c)}$ is doubly stochastic, then there exists $p > 0$ such that $E^{(r)} = pD^{(r)}$ and $E^{(c)} = p^{-1}D^{(c)}$, i.e., $D^{(r)}$ and $D^{(c)}$ are *unique up to a scalar factor* (see Theorem 3.3 below). This setting was studied by other authors too: In [27], Marcus and Newman proved the existence of doubly stochastic diagonal forms for positive square and symmetric matrices, whereas Menon shows in [29] the existence of doubly stochastic diagonal forms for positive square matrices using the Brouwer fixed point theorem (Maxfield and Minc work on the problem too in [28]).

If A is only non-negative (not necessarily positive), then the limit in (3.1) may not exist, and if it does, then it may not be a diagonal form. Sinkhorn and Knopp studied this more general case and the following is the main result in [38].

THEOREM 3.3 (Sinkhorn-Knopp [38]). *Let $A \in \mathbf{M}$.*

1. $A \in \mathbf{S}$ if and only if $\lim_{k \rightarrow \infty} (\mathcal{S}^c \circ \mathcal{S}^r)^k(A)$ converges to an element in \mathbf{P} .
2. $A \in \mathbf{TS}$ if and only if there exist $B \in \mathbf{P}$ and $D^{(r)}, D^{(c)} \in \mathbf{D}$ such that $B = D^{(r)}AD^{(c)}$.
3. If $A \in \mathbf{TS}$ and $B, D^{(r)}$ and $D^{(c)}$ are as in 2, then B is unique, but it may be that $D^{(r)}$ and $D^{(c)}$ are not unique (up to a scalar factor).
4. If $A \in \mathbf{TS}$, then $D^{(r)}$ and $D^{(c)}$ in 2 are unique (up to a scalar factor) if and only if $A \in \mathbf{FI}$.
5. If $A \in \mathbf{TS}$, then $\lim_{k \rightarrow \infty} (\mathcal{S}^c \circ \mathcal{S}^r)^k(A) = D^{(r)}AD^{(c)}$ for some $D^{(r)}, D^{(c)} \in \mathbf{D}$.
6. If $A \in \mathbf{S} \setminus \mathbf{TS}$, then there are no $D^{(r)}, D^{(c)} \in \mathbf{D}$ such that $\lim_{k \rightarrow \infty} (\mathcal{S}^c \circ \mathcal{S}^r)^k(A) = D^{(r)}AD^{(c)}$.

This more general situation in which the domain is further from positive matrices has also been studied by other authors: In [35], Perfect and Mirsky show that $A \in \mathbf{TS}$ if and only if³ A has a doubly stochastic pattern (compare with 2 in Theorem 3.3). London [24] as well as Letac [22] showed that A has a doubly stochastic pattern if and only if there exists B and $D^{(r)}, D^{(c)} \in \mathbf{D}$ as in 2 in Theorem 3.3. Based on the nonlinear operator introduced by Menon in [29], Brualdi, Parter and Schneider proved in [8] parts of Theorem 3.3, namely 2 and one direction of 4 (the assumption: full indecomposability).

3.3. Brualdi-Parter-Schneider. In [8], stochastic forms $(1; D, D)$ with $D \in \mathbf{D}$ are introduced and studied. We refer to such stochastic forms as *Brualdi-Parter-*

³In particular, as pointed out in [8], if $A \in \mathbf{FI}$, then A has a doubly stochastic pattern. This is because $A \in \mathbf{TS}$ if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of fully indecomposable matrices, hence $\mathbf{FI} \subset \mathbf{TS}$ (see [7, 8] for more on total support and full indecomposability, also [13]).

Schneider.

THEOREM 3.4 (Brualdi, Parter and Schneider [8]). *Let $A \in \mathbf{M}^r$ (resp., $A \in \mathbf{M}^c$). Then for any non-negative matrix B with the same pattern as A , there exists $D \in \mathbf{D}$ such that $DBD \in \mathbf{P}^r$ (resp., $DBD \in \mathbf{P}^c$) if and only if the following condition is satisfied:*

1. *If the rows and columns of A are permuted simultaneously so that $A_{i,j} = 0$ for $1 \leq i, j \leq s$, then there exists k and l with $s < k \leq n$ and $1 \leq l \leq s$ such that $A_{k,j} = 0$ (resp., $A_{j,k} = 0$) for $j = 1, \dots, s$ and $A_{l,k} > 0$ (resp., $A_{k,l} > 0$).*

In this case, the matrix D is unique.

COROLLARY 3.5 (Brualdi, Parter and Schneider [8]). *If $A \in \mathbf{M}$ has a positive main diagonal, then condition 1 in Theorem 3.4 holds.*

COROLLARY 3.6 (Brualdi, Parter and Schneider [8]). *If $A \in \mathbf{M}$ is symmetric, then condition 1 in Theorem 3.4 holds if and only if the main diagonal of A is positive.*

We let $\mathcal{B}^r: \mathbf{B}^r \rightarrow \mathbf{P}^r$ (resp., $\mathcal{B}^c: \mathbf{B}^c \rightarrow \mathbf{P}^c$) denote the Brualdi-Parter-Schneider row (resp., column) stochastic form. For $A \in \mathbf{B}^r$ (resp., $A \in \mathbf{B}^c$), let $D_A^{(r)}$ (resp., $D_A^{(c)}$) be the unique matrix in \mathbf{D} such that $\mathcal{B}^r(A) = D_A^{(r)} A D_A^{(r)}$ (resp., $\mathcal{B}^c(A) = D_A^{(c)} A D_A^{(c)}$).

REMARK 3.7. If $A \in \mathbf{B}$, then $\lim_{k \rightarrow \infty} (\mathcal{B}^c \circ \mathcal{B}^r)^k(A)$ converges to a doubly stochastic matrix and there exists $D \in \mathbf{D}$ such that $\lim_{k \rightarrow \infty} (\mathcal{B}^c \circ \mathcal{B}^r)^k(A) = DAD$. By the uniqueness property in Theorem 3.4,

$$\mathcal{B}^r(A) = \lim_{k \rightarrow \infty} (\mathcal{B}^c \circ \mathcal{B}^r)^k(A) = \mathcal{B}^c(A).$$

(See Theorem 4.13).

PROPOSITION 3.8. *Let $A, B \in \mathbf{B}^r$ (resp., $A, B \in \mathbf{B}^c$). Then $\mathcal{B}^r(A) = \mathcal{B}^r(B)$ (resp., $\mathcal{B}^c(A) = \mathcal{B}^c(B)$) if and only if there exists a unique diagonal matrix $D \in \mathbf{D}$ such that $A = DBD$. In this case, D is given by*

$$\text{diag}(D) = \frac{\text{diag}(D_B^{(r)})}{\text{diag}(D_A^{(r)})} \quad \left(\text{resp., } \text{diag}(D) = \frac{\text{diag}(D_B^{(c)})}{\text{diag}(D_A^{(c)})} \right).$$

Proof. $\mathcal{B}^r(A) = \mathcal{B}^r(B)$ if and only if $A = (D_A^{(r)})^{-1} D_B^{(r)} B D_B^{(r)} (D_A^{(r)})^{-1}$. Then the result follows from the fact that $DE = ED$ for every $D, E \in \mathbf{D}$ and from the uniqueness property in Theorem 3.4.

The rest of the proof is analogous. \square

3.4. Perron. The Perron stochastic form is defined for an irreducible matrix $A \in \mathbf{I}$. Let $\lambda_A > 0$ be the Perron value of A and let $\mathbf{p}_A = (p_1^{(A)}, \dots, p_n^{(A)})^T$ and $\mathbf{q}_A = (q_1^{(A)}, \dots, q_n^{(A)})$ be right and left Perron eigenvectors of A respectively (that is, $\lambda_A > 0$ is the spectral radius of A , $A\mathbf{p}_A = \lambda_A\mathbf{p}_A$, $\mathbf{q}_AA = \lambda_A\mathbf{q}_A$ and both \mathbf{p}_A and \mathbf{q}_A are positive). Let $P_A, Q_A \in \mathbf{D}$ be defined by $\text{diag}(P_A) = \mathbf{p}_A^T$ and $\text{diag}(Q_A) = \mathbf{q}_A$. The *Perron row* (resp., *column*) *stochastic form* $\mathcal{P}^r: \mathbf{I} \rightarrow \mathbf{P}^r$ (resp., $\mathcal{P}^c: \mathbf{I} \rightarrow \mathbf{P}^c$) is defined by

$$\mathcal{P}^r \leftarrow \left(\frac{1}{\lambda_A}; \frac{1}{P_A}, P_A \right) \quad \left(\text{resp., } \mathcal{P}^c \leftarrow \left(\frac{1}{\lambda_A}; Q_A, \frac{1}{Q_A} \right) \right). \quad (3.2)$$

PROPOSITION 3.9 ([26]). *Let $A \in \mathbf{I}$. Let $D \in \mathbf{D}$ and $\alpha > 0$, and define $B = \alpha D^{-1}AD$ (resp., $B = \alpha DAD^{-1}$). If $B \in \mathbf{P}^r$ (resp., $B \in \mathbf{P}^c$), then $B = \mathcal{P}^r(A)$ (resp., $B = \mathcal{P}^c(A)$), $\alpha = \lambda_A^{-1}$ and D is equal to P_A (resp., Q_A) modulo a positive scalar multiple.*

We include a proof for completeness.

Proof. Let $\mathbf{d} = \text{diag}(D)^T$ and suppose that $\alpha D^{-1}AD \in \mathbf{P}^r$ (resp., $\alpha DAD^{-1} \in \mathbf{P}^c$). Then $A\mathbf{d} = \frac{1}{\alpha}\mathbf{d}$ (resp., $\mathbf{d}^T A = \frac{1}{\alpha}\mathbf{d}^T$), and then the Perron-Frobenius Theorem implies the result. \square

COROLLARY 3.10. *If $A \in \mathbf{P}^r \cap \mathbf{I}$, then $\mathcal{P}^r \circ \mathcal{P}^c(A) = A$ and $\mathcal{P}^c \circ \mathcal{P}^r(A^T) = A^T$.*

Proof. Since $\mathcal{P}^r \circ \mathcal{P}^c(A) = (Q_A^{-1}P_{\mathcal{P}^c(A)})^{-1}A(Q_A^{-1}P_{\mathcal{P}^c(A)})$ is stochastic, Proposition 3.9 implies that $Q_A^{-1}P_{\mathcal{P}^c(A)}$ is a scalar multiple of the identity matrix I_n (since $\mathcal{P}^r(A) = A = I_n^{-1}AI_n$). Then

$$\frac{p_i^{(\mathcal{P}^c(A))}}{p_j^{(\mathcal{P}^c(A))}} = \frac{q_i^{(A)}}{q_j^{(A)}} \quad \text{for every } i, j \in [n]$$

and the first claim follows. The second one is analogous. \square

REMARK 3.11. Corollary 3.10 implies that if $A \in \mathbf{I}$, then

$$\lim_{k \rightarrow \infty} (\mathcal{P}^c \circ \mathcal{P}^r)^k(A)$$

converges if and only if $\mathcal{P}^r(A) \in \mathbf{P}$, and in this case it converges to $\mathcal{P}^r(A)$. (See Theorem 4.15.)

PROPOSITION 3.12 ([26]). *Let $A, B \in \mathbf{I}$. Then $\mathcal{P}^r(A) = \mathcal{P}^r(B)$ (resp., $\mathcal{P}^c(A) = \mathcal{P}^c(B)$) if and only if there exists $D \in \mathbf{D}$ and a real number $\rho > 0$ such that*

$$A = \frac{D^{-1}BD}{\rho} \quad \left(\text{resp., } A = \frac{DBD^{-1}}{\rho} \right). \quad (3.3)$$

The matrix D is, up to a scalar multiple, given by

$$\text{diag}(D) = \frac{\mathbf{p}_B}{\mathbf{p}_A} \quad \left(\text{resp., } \text{diag}(D) = \frac{\mathbf{q}_B}{\mathbf{q}_A} \right).$$

Compare with Propositions 3.2 and 3.8. We include a proof for completeness.

Proof. Suppose that $\mathcal{P}^r(A) = \mathcal{P}^r(B)$ (resp., $\mathcal{P}^c(A) = \mathcal{P}^c(A)$). If $D \in \mathbf{D}$ is defined by $\text{diag}(D)^T = \frac{\mathbf{p}_B}{\mathbf{p}_A}$ (resp., $\text{diag}(D) = \frac{\mathbf{q}_B}{\mathbf{q}_A}$) and $\rho = \frac{\lambda_B}{\lambda_A}$, then it is straightforward to check that (3.3) holds.

Conversely, suppose that (3.3) holds. Then

$$\mathcal{P}^r(A) = \frac{(DP_A)^{-1}B(DP_A)}{\rho\lambda_A} \quad \left(\text{resp., } \mathcal{P}^c(A) = \frac{(DQ_A)B(DQ_A)^{-1}}{\rho\lambda_A} \right).$$

is stochastic. Therefore, $\lambda_B = \rho\lambda_A$ and $\mathbf{p}_B = (DP_A I_n)^T$ (resp., $\mathbf{q}_B = I_n DQ_A$) (by uniqueness, up to a positive scale factor, of Perron vectors). Then $\mathcal{P}^r(B) = \mathcal{P}^r(A)$ (resp., $\mathcal{P}^c(B) = \mathcal{P}^c(A)$). \square

The Perron stochastic form appears in the proof of the Perron-Frobenius Theorem (see e.g. [6, 12, 36], or also [16] for a historical approach in relationship with continued fractions). It has been used in symbolic dynamics (the books [21, 23] are classic introductions to this subject): With it Marcus and Tuncel [26] study classification problems of Markov chains presented by matrices over positive integral semirings of exponential functions. Parry discovered that the Perron stochastic form restricted to irreducible $\{0, 1\}$ -matrices induces the unique probability measures of maximal entropy on the (vertex) shift spaces associated to these type of matrices (see [34] for details).

4. Intersections. In this section, we address the problem of determining when two stochastic forms of a given matrix coincide. Observe that determining when a row stochastic form coincides with a column stochastic form corresponds to considering when either stochastic form coincides with the Sinkhorn-Knopp doubly stochastic form.

4.1. Standard and Sinkhorn-Knopp. In this first case, we look at matrices $A \in \mathbf{M}$ such that $\mathcal{S}^r(A)$ (and $\mathcal{S}^c(A)$) are doubly stochastic.

PROPOSITION 4.1. *Let $A \in \mathbf{M}$. If $\mathcal{S}^r(A) \in \mathbf{P}$ (resp., $\mathcal{S}^c(A) \in \mathbf{P}$), then $A \in \mathbf{TS}$ and $\mathcal{S}^r(A) = \mathcal{S}(A) = \mathcal{S}^c(A)$. Moreover, if $A \in \mathbf{FI}$, then $\mathcal{S}^r(A) \in \mathbf{P}$ (resp., $\mathcal{S}^c(A) \in \mathbf{P}$) if and only if $\mathcal{S}^r(A) = \mathcal{S}^c(A)$.*

Proof. Suppose that $\mathcal{S}^r(A) = \left(S_A^{(r)}\right)^{-1} A = \left(S_A^{(r)}\right)^{-1} A I_n \in \mathbf{P}$. Then 2 in Theorem 3.3 implies that $A \in \mathbf{TS}$. It follows from 5 and 3 in Theorem 3.3 that $\mathcal{S}^r(A) = \mathcal{S}(A)$. If, in addition, $A \in \mathbf{FI}$, then 4 in Theorem 3.3 implies that there exists a constant $\kappa > 0$ such that $S_A^{(r)} = \kappa S_A^{(c)}$, but then $\mathbf{r}_A \cdot J_n^T = \kappa \mathbf{c}_A \cdot J_n$, and hence, $\kappa = 1$. Clearly, if $\mathcal{S}^r(A) = \mathcal{S}^c(A)$, then $\mathcal{S}^r(A) \in \mathbf{P}$.

The rest of the proof is analogous. \square

The following is a particular instance of Proposition 4.1.

PROPOSITION 4.2. *Let $A \in \mathbf{M}$. Then $\mathcal{S}^r(A) = \mathcal{S}(A) = \mathcal{S}^c(A)$ if and only if $r_i^{(A)} = c_j^{(A)}$ for every $i, j \in [n]$ such that $A_{i,j} \neq 0$.*

Proof. $\mathcal{S}^r(A) = \mathcal{S}^c(A)$ if and only if $\left(S_A^{(r)}\right)^{-1} A = A \left(S_A^{(c)}\right)^{-1}$ and this holds if and only if

$$\frac{r_i^{(A)}}{c_j^{(A)}} A_{i,j} = A_{i,j} \quad \text{for every } i, j \in [n]. \quad \square$$

The condition in Proposition 4.2 can be compared with that of *reversible systems* (see e.g. [2, 14, 15]), i.e., with symmetric doubly stochastic matrices, in particular if the matrix has positive entries.

EXAMPLE 4.3. The matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 10 & 5 \\ 1 & 0 & 0 \end{pmatrix}$ has total support but is not fully indecomposable, $\mathcal{S}^r(A) = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{pmatrix} \in \mathbf{P}$ but $\mathcal{S}^r(A) \neq \mathcal{S}^c(A)$ (e.g. $A_{1,2} \neq 0$ and $r_1^{(A)} = 3 \neq 11 = c_2^{(A)}$), and two distinct representations (not equal modulo a positive scalar multiple) of $\mathcal{S}^r(A)$ as a diagonal form are

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{15} & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{15} & 0 \\ 0 & 0 & \frac{1}{15} \end{pmatrix},$$

the last being the form of the limit process of iterating $\mathcal{S}^c \circ \mathcal{S}^r$ on A .

PROPOSITION 4.4. *Let $A, B \in \mathbf{M}$ and suppose that $\mathcal{S}^r(A) = \mathcal{S}(A) = \mathcal{S}^c(A)$ and $\mathcal{S}^r(B) = \mathcal{S}(B) = \mathcal{S}^c(B)$. Then $\mathcal{S}^r(A \otimes B) = \mathcal{S}(A \otimes B) = \mathcal{S}^c(A \otimes B)$.*

Proof. Let the sizes of A and B be $n \geq 1$ and $m \geq 1$ respectively. If $i, j \in [n]$ and

$h, k \in [m]$ are such that $(A \otimes B)_{m(i-1)+h, m(j-1)+k} = A_{i,j} B_{h,k} \neq 0$, then

$$r_{m(i-1)+h}^{(A \otimes B)} = \sum_{x=1}^n \sum_{y=1}^m A_{i,x} B_{h,y} = r_i^{(A)} r_h^{(B)} = c_j^{(A)} c_k^{(B)} = \sum_{x=1}^n \sum_{y=1}^m A_{x,j} B_{y,k} = c_{m(j-1)+k}^{(A \otimes B)}.$$

The third equality above, and hence, the result follows from Proposition 4.2. \square

4.2. Standard and Brualdi-Parter-Schneider. Here we look at the case when the standard and the Brualdi-Parter-Schneider stochastic forms coincide.

PROPOSITION 4.5. *Let $A \in \mathbf{B}^r$ (resp., $A \in \mathbf{B}^c$). Then $\mathcal{S}^r(A) = \mathcal{B}^r(A)$ (resp., $\mathcal{S}^c(A) = \mathcal{B}^c(A)$) if and only if there exists a constant $\alpha > 0$ such that $S_A^{(r)} = \alpha I_n$ (resp., $S_A^{(c)} = \alpha I_n$).*

Proof. Let $D = D_A^{(r)}$. Then $\mathcal{S}^r(A) = \mathcal{B}^r(A)$ if and only if $A = S_A^{(r)} D A D$, and this holds if and only if for every $i, j \in [n]$ we have $r_i^{(A)} D_{i,i} D_{j,j} = 1$. In particular $D_{i,i} = \frac{1}{\sqrt{r_i^{(A)}}}$ and then $r_i^{(A)} = D_{j,j}^{-2}$ for every $i, j \in [n]$. Fix $j_0 \in [n]$. Then for every $i, i' \in [n]$, $r_i^{(A)} = D_{j_0, j_0}^{-2} = r_{i'}^{(A)}$.

The rest of the proof is analogous. \square

PROPOSITION 4.6. *Let $A, B \in \mathbf{B}^r$ (resp., $A, B \in \mathbf{B}^c$) and suppose that*

$$\mathcal{S}^r(A) = \mathcal{B}^r(A) \text{ and } \mathcal{S}^r(B) = \mathcal{B}^r(B)$$

$$(\text{resp., } \mathcal{S}^c(A) = \mathcal{B}^c(A) \text{ and } \mathcal{S}^c(B) = \mathcal{B}^c(B)).$$

Then $\mathcal{S}^r(A \otimes B) = \mathcal{B}^r(A \otimes B)$ (resp., $\mathcal{S}^c(A \otimes B) = \mathcal{B}^c(A \otimes B)$).

Proof. Let the sizes of A and B be $n \geq 1$ and $m \geq 1$ respectively. By Proposition 4.5, there exist $\alpha, \beta > 0$ such that $S_A^{(r)} = \alpha I_n$ and $S_B^{(r)} = \beta I_m$. It is straightforward to verify that $S_{A \otimes B}^{(r)} = \alpha \beta I_{nm}$, and hence, the result for rows follows from Proposition 4.5.

The rest of the proof is analogous. \square

4.3. Standard and Perron. In this case, we are considering sums of rows (and columns) of matrices together with (Perron) eigenpairs. Similar contexts have been previously considered, e.g. to study sums of columns of row stochastic matrices together with their spectra as well as their stationary distributions [18, 19].

To start, we first define “locally constant” vectors and “Perron-regularity”. For each $i \in [n]$, let the *row* (resp., *column*) *neighborhood* of i be

$$N_A^r(i) = \{j \in [n] : A_{i,j} \neq 0\} \quad (\text{resp., } N_A^c(i) = \{j \in [n] : A_{j,i} \neq 0\}).$$

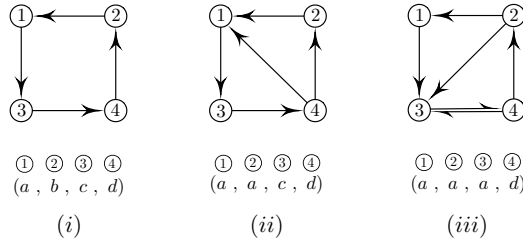


FIG. 4.1. Three different examples of directed graphs on four vertices and the form of their corresponding locally row constant vectors. (i) Any vector is locally row constant. (ii) A locally row constant vector must have the first two entries equal. (iii) A locally row constant vector must have the first three entries equal.

A column (resp., row) n -vector $\mathbf{v} = (v_1, \dots, v_n)^T$ (resp., $\mathbf{v} = (v_1, \dots, v_n)$) is *locally row* (resp., *column*) *constant with respect to* A if for each $i \in [n]$ we have $v_j = v_k$ for every $j, k \in N_A^r(i)$ (resp., $N_A^c(i)$). In this case, let

$$\mathbf{v}^r = \left(v_1^{(r)}, \dots, v_n^{(r)} \right)^T \quad \left(\text{resp., } \mathbf{v}^c = \left(v_1^{(c)}, \dots, v_n^{(c)} \right) \right),$$

where $v_i^{(r)} = v_j$ (resp., $v_i^{(c)} = v_j$) for some (every) $j \in N_A^r(i)$ (resp., $j \in N_A^c(i)$). If a vector is locally row (resp., column) constant with respect to a matrix A , then it is locally row (resp., column) constant with respect to any matrix B with the same pattern as A . Henceforth we will simply refer to a vector as locally (row or column) constant as long as it is clear in the context with respect to which matrix pattern. Figure 4.1 illustrates with directed graphs examples of locally row constant vectors (see Section 5 for more on directed graphs associated to matrices).

DEFINITION 4.7. $A \in \mathbf{I}$ is *row* (resp., *column*) *Perron-regular* if its right (resp., left) Perron vector \mathbf{p}_A (resp., \mathbf{q}_A) is locally row (resp., column) constant (with respect to A itself).

THEOREM 4.8. An irreducible matrix $A \in \mathbf{I}$ is row (resp., column) Perron-regular if and only if $\mathcal{S}^r(A) = \mathcal{P}^r(A)$ (resp., $\mathcal{S}^c(A) = \mathcal{P}^c(A)$).

Proof. If $\mathcal{S}^r(A) = \mathcal{P}^r(A)$, then

$$\frac{A_{i,j}}{r_i^{(A)}} = \frac{p_j^{(A)}}{\lambda_A p_i^{(A)}} A_{i,j} \quad \text{for every } i, j \in [n]. \quad (4.1)$$

Then for every $j \in N_A^r(i)$ we have $p_j^{(A)} = \frac{\lambda_A p_i^{(A)}}{r_i^{(A)}}$, so that the value of $p_j^{(A)}$ only depends on i . Then \mathbf{p}_A is locally row constant and hence A is row Perron-regular.

Conversely, suppose that A is row Perron-regular. If $i \in [n]$ and $j \in N_A^r(i)$, then

$$\lambda_A p_i^{(A)} - r_i^{(A)} p_j^{(A)} = \sum_{k=1}^n A_{i,k} (p_k^{(A)} - p_j^{(A)}) = \sum_{k \in N_A^r(i)} A_{i,k} (p_k^{(A)} - p_j^{(A)}) = 0$$

because \mathbf{p}_A is locally row constant. Therefore, (4.1) holds, and hence, $\mathcal{S}^r(A) = \mathcal{P}^r(A)$.

The rest of the proof is analogous. \square

PROPOSITION 4.9. *If $A \in \mathbf{I}$ is row (resp., column) Perron-regular, then*

$$\lambda_A \mathbf{p}_A = \mathbf{r}_A \circ \mathbf{p}_A^r \quad (\text{resp., } \lambda_A \mathbf{q}_A = \mathbf{c}_A \circ \mathbf{q}_A^c)$$

Proof. If A is row Perron-regular, then

$$\lambda_A p_i^{(A)} = \sum_{j \in N_A^r(i)} A_{i,j} p_j^{(A)} = \left(p_i^{(A)}\right)^{(r)} \sum_{j \in N_A^r(i)} A_{i,j} = \left(p_i^{(A)}\right)^{(r)} c_i^{(A)}$$

for every $i \in [n]$.

The rest of the proof is analogous. \square

REMARK 4.10. Proposition 3.12 characterizes when two irreducible matrices A and B yield the same Perron stochastic form. This can occur even if A is Perron-regular whereas B is not since for any $D \in \mathbf{D}$ and $\rho > 0$, if $B = \frac{D^{-1}AD}{\rho}$ (resp., $B = \frac{D^{-1}AD}{\rho}$), then $\lambda_B = \frac{\lambda_A}{\rho}$ and $\mathbf{p}_B = \frac{\mathbf{p}_A}{\text{diag}(D)}$ (resp., $\mathbf{q}_B = \frac{\mathbf{q}_A}{\text{diag}(D)}$).

COROLLARY 4.11. *Let $A, B \in \mathbf{I}$ be such that $\mathcal{P}^r(A) = \mathcal{P}^r(B)$ (resp., $\mathcal{P}^c(A) = \mathcal{P}^c(B)$). If $\frac{\mathbf{p}_A}{\mathbf{p}_B}$ (resp., $\frac{\mathbf{q}_A}{\mathbf{q}_B}$) is locally row (resp., column) constant, then A is row (resp., column) Perron-regular if and only if B is row (resp., column) Perron-regular.*

Proof. It follows from Remark 4.10. \square

THEOREM 4.12. *Let $A, B \in \mathbf{I}$ be row (resp., column) Perron-regular matrices. Then $A \otimes B$ is row (resp., column) Perron-regular.*

Proof. Clearly, $A \otimes B \in \mathbf{I}$. Suppose that the sizes of A and B are $n \geq 1$ and $m \geq 1$ respectively. We only prove the case for rows since the case for columns is analogous.

Let $\{(\lambda_i, \mathbf{v}^{(i)})\}_{i \in [n]}$ and $\{(\mu_j, \mathbf{w}^{(j)})\}_{j \in [m]}$ be the right eigenpairs of A and B respectively. Then $\{(\lambda_i \mu_j, \mathbf{v}^{(i)} \otimes \mathbf{w}^{(j)})\}_{i \in [n], j \in [m]}$ are the right eigenpairs of $A \otimes B$. In

particular, if (λ, \mathbf{v}) and (μ, \mathbf{w}) are the right Perron eigenpairs of A and B respectively, then the Perron-Frobenius Theorem implies that $(\lambda\mu, \mathbf{v} \otimes \mathbf{w})$ is the right Perron eigenpair of $A \otimes B$. Given $k \in [nm]$, let $i, j \in [nm]$ be such that $(A \otimes B)_{k,i}, (A \otimes B)_{k,j} > 0$. Let $a, b, c \in \{0, \dots, n-1\}$ and $r, s, t \in [m]$ be such that $k = an + r$, $i = bn + s$ and $j = cn + t$. Since

$$(A \otimes B)_{k,i} = A_{a+1,b+1}B_{r,s} > 0 \quad \text{and} \quad (A \otimes B)_{k,j} = A_{a+1,c+1}B_{r,t} > 0,$$

the row Perron-regularity of A and B imply that $\mathbf{v}_{b+1} = \mathbf{v}_{c+1}$ and $\mathbf{w}_s = \mathbf{w}_t$, hence

$$(\mathbf{v} \otimes \mathbf{w})_{k,i} = \mathbf{v}_{b+1}\mathbf{w}_s = \mathbf{v}_{c+1}\mathbf{w}_t = (\mathbf{v} \otimes \mathbf{w})_{k,j}.$$

Therefore, the result follows from Theorem 4.8. \square

4.4. Sinkhorn-Knopp and Brualdi-Parter-Schneider. Here we look at the case when the Brualdi-Parter-Schneider normalization is doubly stochastic.

THEOREM 4.13. *Let $A \in \mathbf{B}^r$ (resp., $A \in \mathbf{B}^c$). Then $\mathcal{B}^r(A) \in \mathbf{P}$ (resp., $\mathcal{B}^c(A) \in \mathbf{P}$) if and only if there exists a (unique) positive solution $\mathbf{x} = (x_1, \dots, x_n)$ to the homogeneous linear system*

$$\mathbf{x}(A - A^T) = \mathbf{0} \tag{4.2}$$

(in particular $\det(A - A^T) = 0$) which satisfies

$$x_j \sum_{i=1}^n x_i A_{i,j} = x_j \sum_{i=1}^n x_i A_{j,i} = 1 \quad \text{for every } j \in [n]. \tag{4.3}$$

In this case, $A \in \mathbf{TS}$ and $\mathcal{B}^r(A) = \mathcal{B}^c(A) = \mathcal{S}(A)$.

Proof. Let $D = D_A^{(r)}$. Then $\mathcal{B}^r(D) \in \mathbf{P}$ if and only if

$$\sum_{i=1}^n D_{i,i} D_{j,j} A_{i,j} = 1 \quad \text{and} \quad \sum_{i=1}^n D_{i,i} D_{j,j} A_{j,i} = 1$$

for every $j \in [n]$, in particular

$$\sum_{i=1}^n D_{i,i} (A_{i,j} - A_{j,i}) = 0 \quad \text{for every } j \in [n],$$

i.e., $\mathbf{x} = \text{diag}(D)$ is a positive solution to (4.2) which satisfies (4.3) for every $j \in [n]$ (with $\alpha = 1$). Conversely, if \mathbf{x} is a positive solution to (4.2) satisfying (4.3), then it follows that $D \in \mathbf{D}$ defined by $\text{diag}(D) = \mathbf{x}$ is such that $DAD \in \mathbf{P}$, and the unicity property in Theorem 3.4 implies that $\mathcal{B}^r(A) = DAD$, and therefore, also $\mathcal{B}^r(A) = \mathcal{B}^c(A)$.

If $\mathcal{B}^r(D) \in \mathbf{P}$, then 2 in Theorem 3.3 implies that $A \in \mathbf{TS}$, and hence 5 and 3 in Theorem 3.3 imply that $\mathcal{B}^r(A) = \mathcal{S}(A)$. \square

THEOREM 4.14. *Let $A, B \in \mathbf{B}$ and suppose that $\mathcal{B}^r(A) = \mathcal{S}(A) = \mathcal{B}^c(A)$ and $\mathcal{B}^r(B) = \mathcal{S}(B) = \mathcal{B}^c(B)$. Then $\mathcal{B}^r(A \otimes B) = \mathcal{S}(A \otimes B) = \mathcal{B}^c(A \otimes B)$.*

Proof. Let the sizes of A and B be $n \geq 1$ and $m \geq 1$ respectively. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$ be the solutions to (4.2) that satisfy (4.3) for A and B respectively. For every $i \in [n]$ and $j \in [m]$ we have

$$\begin{aligned} (\mathbf{x} \otimes \mathbf{y}(A \otimes B - (A \otimes B)^T))_{(i-1)m+j} &= \sum_{h=1}^n \sum_{k=1}^m x_h y_k (A_{h,i} B_{k,j} - A_{i,h} B_{j,k}) \\ &= \sum_{h=1}^n x_h A_{h,i} \sum_{k=1}^m y_k B_{k,j} - \sum_{h=1}^n x_h A_{i,h} \sum_{k=1}^m y_k B_{j,k} \\ &= \left(\sum_{h=1}^n (x_h A_{h,i} - x_h A_{i,h}) \right) \left(\sum_{k=1}^m (y_k B_{k,j} - y_k B_{j,k}) \right) = 0. \end{aligned}$$

(the third and last equalities hold by hypothesis). Then $\mathbf{x} \otimes \mathbf{y}$ is a solution to (4.2). To finish,

$$x_i y_j \sum_{h=1}^n \sum_{k=1}^m x_h y_k A_{h,i} B_{k,j} = x_i \sum_{h=1}^n x_h A_{h,i} \left(y_j \sum_{k=1}^m y_k B_{k,j} \right) = 1.$$

(the last equality holds by hypothesis). Then $\mathbf{x} \otimes \mathbf{y}$ satisfies (4.3) for $A \otimes B$. The result now follows from Theorem 4.13. \square

4.5. Sinkhorn-Knopp and Perron. Here we look at the case when the Perron stochastic form is doubly stochastic.

THEOREM 4.15. *Let $A \in \mathbf{I}$. The following are equivalent:*

1. $\mathcal{P}^r(A) \in \mathbf{P}$.
2. $\mathcal{P}^c(A) \in \mathbf{P}$.
3. $\mathcal{P}^r(A) = \mathcal{P}^c(A)$.
4. $(\mathbf{p}_A^{-1})^T$ is a left eigenvector of A .
5. $(\mathbf{q}_A^{-1})^T$ is a right eigenvector of A .
6. $\mathcal{P}^r(A) = \mathcal{S}(A)$.
7. $\mathcal{P}^c(A) = \mathcal{S}(A)$.

Proof. $\mathcal{P}^r(A) \in \mathbf{P}$ if and only if

$$J_n^T \frac{P_A^{-1} A P_A}{\lambda_A} = J_n^T,$$

and this holds if and only if $(\mathbf{p}_A^{-1})^T A = \lambda_A (\mathbf{p}_A^{-1})^T$. Then $1 \Leftrightarrow 4$. Similarly, $2 \Leftrightarrow 5$.

Clearly $3 \Rightarrow 1, 2$. On the other hand, $4 \Rightarrow 3$ since the Perron-Frobenius Theorem implies that

$$\mathcal{P}^c(A) = \frac{P_A^{-1} A P_A}{\lambda_A} = \mathcal{P}^r(A).$$

Similarly, $5 \Rightarrow 3$.

Clearly $6 \Rightarrow 1$ and $7 \Rightarrow 2$. Conversely, if $\mathcal{P}(A) \in \mathbf{P}$, then 2 and 5 in Theorem 3.3 implies that $A \in \mathbf{TS}$ and $\mathcal{P}^r(A) = \mathcal{S}(A)$, in particular $1 \Rightarrow 6$, and similarly $2 \Rightarrow 7$. \square

THEOREM 4.16. *Let $A, B \in \mathbf{I}$ and suppose that $\mathcal{P}^r(A) = \mathcal{S}(A) = \mathcal{P}^c(A)$ and $\mathcal{P}^r(B) = \mathcal{S}(B) = \mathcal{P}^c(B)$. Then $\mathcal{P}^r(A \otimes B) = \mathcal{S}(A \otimes B) = \mathcal{P}^c(A \otimes B)$.*

Proof. We already know that $\mathbf{p}_{A \otimes B} = \mathbf{p}_A \otimes \mathbf{p}_B$ (from the proof of Theorem 4.12), hence $(\mathbf{p}_{A \otimes B}^{-1})^T = (\mathbf{p}_A^{-1})^T \otimes (\mathbf{p}_B^{-1})^T$ is a left eigenvector of $A \otimes B$ (again from the proof of Theorem 4.12). The result now follows from Theorem 4.15. \square

4.6. Brualdi-Parter-Schneider and Perron. Here we look at the case when the Brualdi-Parter-Schneider and the Perron normalizations coincide.

PROPOSITION 4.17. *Let $A \in \mathbf{I} \cap \mathbf{B}^r$ (resp., $A \in \mathbf{I} \cap \mathbf{B}^c$). Let $D = D_A^{(r)}$ (resp., $D = D_A^{(c)}$) and let $\mathbf{x} = (x_1, \dots, x_n)$ be $\mathbf{x} = \mathbf{p}_A$ (resp., $\mathbf{x} = \mathbf{q}_A$). Then $\mathcal{B}^r(A) = \mathcal{P}^r(A)$ (resp., $\mathcal{B}^r(A) = \mathcal{P}^c(A)$) if and only if*

$$\lambda_A D_{i,i} D_{j,j} = \frac{x_j}{x_i} \text{ for every } i, j \in [n] \text{ such that } A_{i,j} \neq 0. \quad (4.4)$$

If this is the case and, in addition, A has a positive diagonal (in particular, if A is symmetric, see Corollary 3.6), then A is a positive scalar multiple of a doubly stochastic matrix.

Proof. Clearly $\mathcal{B}^r(A) = \mathcal{P}^r(A)$ (resp., $\mathcal{B}^c(A) = \mathcal{P}^c(A)$) if and only if (4.4) holds. If in addition A has a positive diagonal, then $D_{i,i} = \frac{1}{\sqrt{\lambda_A}}$ for every $i \in [n]$, in particular $x_i = x_j$ for every $i, j \in [n]$ such that $A_{i,j} \neq 0$, but then irreducibility implies that $x_i = x_j$ for every $i, j \in [n]$. \square

PROPOSITION 4.18. *Let $A, B \in \mathbf{I} \cap \mathbf{B}^r$ (resp., $A, B \in \mathbf{I} \cap \mathbf{B}^c$) and suppose that*

$$\mathcal{B}^r(A) = \mathcal{P}^r(A) \text{ and } \mathcal{B}^r(B) = \mathcal{P}^r(B)$$

$$(\text{resp., } \mathcal{B}^c(A) = \mathcal{P}^c(A) \text{ and } \mathcal{B}^c(B) = \mathcal{P}^c(B)).$$

Then $\mathcal{B}^r(A \otimes B) = \mathcal{P}^r(A \otimes B)$ (resp., $\mathcal{B}^c(A \otimes B) = \mathcal{P}^c(A \otimes B)$).

Proof. Let the sizes of A and B be $n \geq 1$ and $m \geq 1$ respectively. Let $D = D_A^{(r)}$, $E = D_B^{(r)}$, $F = D_{A \otimes B}^{(r)}$, $\mathbf{x} = \mathbf{p}_A$, $\mathbf{y} = \mathbf{p}_B$ and $\mathbf{z} = \mathbf{p}_{A \otimes B} = \mathbf{p}_A \otimes \mathbf{p}_B$. It is easy to show that $F = D \otimes E$. For every $i, j \in [n]$ and $h, k \in [m]$ for which $(A \otimes B)_{m(i-1)+h, m(j-1)+k} = A_{i,j} B_{h,k} \neq 0$, we have

$$\begin{aligned} \lambda_{A \otimes B} F_{m(i-1)+h, m(j-1)+k} &= \lambda_A \lambda_B D_{i,i} E_{h,h} D_{j,j} E_{k,k} \\ &= \frac{x_j y_k}{x_i y_h} = \frac{z_{m(j-1)+k, m(j-1)+k}}{z_{m(i-1)+h, m(i-1)+h}}. \end{aligned}$$

Then (4.4) holds for $A \otimes B$, and hence, Proposition 4.17 implies the result for rows.

The rest of the proof is analogous. \square

5. Directed graphs and Perron-regularity. We need to give more definitions and notation. For a complete reference to graphs and matrices see [5].

5.1. Graphs and matrices. Given a square \mathbb{R}^+ -matrix A , let $A^\#$ be the *zero-nonzero pattern* of the matrix A , i.e., the $\{0, 1\}$ -matrix defined by

$$A_{i,j}^\# = A_{i,j}^0 \quad \text{for every } i, j \in [n], \text{ with } 0^0 = 0$$

(then two matrices A and B have the same pattern if and only if $A^\# = B^\#$). Let $G = G(A^\#)$ be the (*simple*) *directed graph* with *adjacency matrix* $A^\#$, that is, G consists of the *vertex set* $V = V(G) = [n]$ and *edge set* $E = E(G) = \{(i, j) \in V(G) \times V(G) : A_{i,j}^\# = 1\}$. There is also the corresponding *weight function* $\omega = \omega_A : E \rightarrow \mathbb{R}^+ \setminus \{0\}$ defined for every $(i, j) \in E$ by $\omega(i, j) = A_{i,j}$. (G, ω) is the *weighted directed graph* associate with A . A and (G, ω) determine each other; in particular, $A^\#$ and G do too. Henceforth all definitions about A are translated to (G, ω) (or simply to G if the definition depends solely on the pattern $A^\#$). For instance:

- A *graph* is a directed graph with symmetric adjacency matrix.
- A weighted directed graph (G, ω) is *row* (resp., *column*) *Perron-regular* if its associated \mathbb{R}^+ -matrix $A = A(G, \omega)$ is row (resp., column) Perron-regular. In particular, a directed graph G is row (resp., column) Perron-regular if its adjacency matrix $A^\# = A^\#(G)$ is row (resp., column) Perron-regular.
- Etc.

(We can also extend the notation given so far for matrices to directed graphs so that we can consistently exchange symbols, e.g. $N_G^r(v)$ will denote the row neighborhood of $v \in V$. Also, the forthcoming definitions and notation for weighted directed graphs are henceforth translated to matrices.) The *weighted row* (resp., *column*) *degree* of $v \in V$ is

$$\delta_\omega^r(v) = r_v^{(A)} \quad \left(\text{resp., } \delta_\omega^c(v) = c_v^{(A)} \right),$$

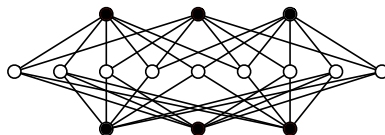


FIG. 5.1. A connected bipartite biregular graph.

and the (unweighted) row (resp., column) degree of v is

$$\delta_G^r(v) = r_v^{(A^\#)} = |N_G^r(v)| \quad \left(\text{resp., } \delta_G^c(v) = c_v^{(A^\#)} = |N_G^c(v)| \right).$$

(G, ω) is weight row (resp., column) regular if for some real number $\alpha > 0$ we have

$$\delta_\omega^r(v) = \alpha \quad (\text{resp., } \delta_\omega^c(v) = \alpha) \quad \text{for every } v \in V.$$

In particular, G is row (resp., column) regular if for some integer $d > 0$ we have

$$\delta_G^r(v) = d \quad (\text{resp., } \delta_G^c(v) = d) \quad \text{for every } v \in V.$$

G is bipartite if there exists a partition of the vertex set such that if $(u, v) \in E$, then u and v belong to distinct parts, i.e., $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and if $(u, v) \in E$, then $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$. If G is bipartite, then (G, ω) is row (resp., column) weight biregular if $\delta_\omega^r(v) = \delta_\omega^r(u)$ (resp., $\delta_\omega^c(v) = \delta_\omega^c(u)$) for every $v, u \in V_i$, $i = 1, 2$. In particular, a directed graph G is row (resp., column) biregular if it is weight row (column) biregular for $\omega: E \rightarrow \{1\}$, i.e., if $\delta_G^r(v) = \delta_G^r(u)$ (resp., $\delta_G^c(v) = \delta_G^c(u)$) for every $v, u \in V_i$, $i = 1, 2$.

A path in G of length $\ell \geq 1$ is a sequence of vertices $\gamma = (x_0, x_1, \dots, x_\ell) \in V^{\ell+1}$ such that $(x_{i-1}, x_i) \in E$ for every $i = 1, \dots, \ell$. In this case, we say that γ is a x_0x_ℓ -path. In terms of paths, G is strongly connected if for every pair of vertices $u, v \in V$, there exists a uv -path in G (equivalently, $A^\#(G)$ is irreducible). The path γ is:

- closed if $x_0 = x_\ell$,
- a loop if it is closed and $\ell = 1$ and
- reversible if $\gamma^{-1} = (x_\ell, \dots, x_1, x_0)$ is also a path in G .

We say that G is connected by reversible paths if for every $u, v \in V$, there exists a reversible uv -path. Clearly, directed graphs which are connected by reversible paths are strongly connected.

If G is actually a graph, then $\delta_G^r(v) = \delta_G^c(v)$ for every $v \in V$, and hence, we can define $\delta_G(v)$ the degree of v as its row (or column) degree. Also, the graph G is regular if it is row (equivalently column) regular. G is a bipartite biregular graph if as a directed graph is bipartite and row (or column) biregular (see Figure 5.1). Finally, G is connected if as a directed graph is strongly connected.

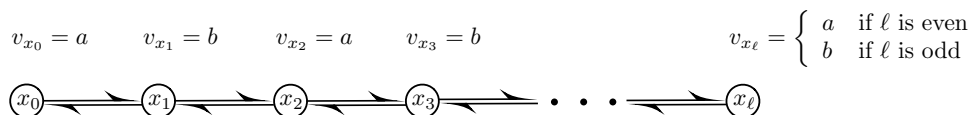


FIG. 5.2. If $v = (v_i)_{i \in [n]}$ is a locally row (or column) constant vector, then the entries that correspond to the vertices of a reversible path can take at most two possible values a and b . If the path is closed (i.e., $x_\ell = x_0$), then $a = b$ if ℓ is odd.

5.2. Perron-regular graphs. We will identify all the Perron-regular (weighted) graphs.

PROPOSITION 5.1. *Let $G = (V, E)$ be a strongly connected directed graph and let $\omega: E \rightarrow (0, \infty)$ be a weight function. If (G, ω) is weight row (resp., column) regular, then it is row (resp., column) Perron-regular.*

Proof. Since $(d, J_{|V|})$ (resp., $(d, J_{|V|}^T)$) is a right (resp., left) Perron eigenpair, the result follows from Theorem 4.8 for a column (resp., row) vector with all its entries equal to a given constant is always locally row (resp., column) constant (with respect to any square \mathbb{R}^+ -matrix with the appropriate size). \square

LEMMA 5.2. *Let $G = (V, E)$ be a directed graph which is connected by reversible paths and let $\omega: E \rightarrow (0, \infty)$ be a weight function. If (G, ω) is row (resp., column) Perron-regular, then the entries of a Perron vector of $A(G, \omega)$ take at most two distinct possible values. If in addition there exists a cycle of odd length which is a reversible path, then (G, ω) is weight row (resp., column) regular.*

Proof. Again this is a direct consequence of Theorem 4.8 (see Figure 5.2). \square

COROLLARY 5.3. *Suppose that G is a graph which is not bipartite. If G is (row or column) Perron-regular, then it is regular.*

Proof. A graph is bipartite if and only if it possesses no odd cycles (see e.g. [4]), and then Lemma 5.2 implies the result since connected graphs are connected by reversible paths. \square

THEOREM 5.4. *A connected graph is row (resp., column) Perron-regular if and only if it is regular or bipartite biregular.*

Proof. Connected regular graphs are both row and column Perron-regular by Proposition 5.1. Theorem 4.8 implies that connected bipartite biregular graphs are both row and column Perron-regular.

Conversely, suppose that a connected row (resp., column) Perron-regular graph $G = (V, E)$ is not regular. By Corollary 5.3, G is bipartite. Let $V = V_1 \cup V_2$ be the

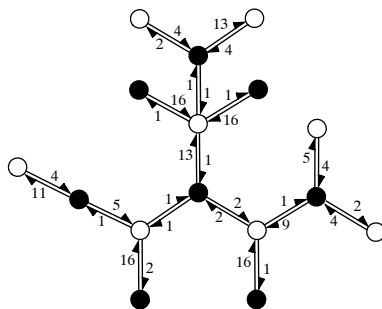


FIG. 5.3. A row Perron-regular weighted tree. The vertices are partitioned according to height parity (once a root is chosen, say a black vertex). It is row Perron-regular because $\delta_\omega^r(u) = 16$ and $\delta_\omega^r(v) = 4$ for every pair of vertices u and v of even and odd heights, respectively. According to the proof of Theorem 5.5, the corresponding Perron value is 8.

partition of the vertex set such that $E \cap V_i^2 = \emptyset$ for $i = 1, 2$. By Lemma 5.2, the right (resp., left) Perron vector of $A^\#(G)$ is constant on the entries corresponding to V_i for each $i = 1, 2$. Hence, G is a bipartite biregular graph. \square

THEOREM 5.5. *Let $G = (V, E)$ be a connected graph and $\omega: E \rightarrow (0, \infty)$ a weight function. If G is not bipartite, then (G, ω) is row (resp., column) Perron-regular if and only if it is weight row (resp., column) regular.*

Otherwise, if $V = (V_1, V_2)$ is the partition of the vertices of G as a bipartite graph, then the weight functions $\omega: E \rightarrow (0, \infty)$ such that (G, ω) is Perron-regular are precisely those that make it a row (resp., column) weight biregular, i.e., $\delta_\omega^r(v) = \alpha$ and $\delta_\omega^r(u) = \beta$ (resp., $\delta_\omega^c(v) = \alpha$ and $\delta_\omega^c(u) = \beta$) for every $v \in V_1$ and $u \in V_2$, for a given pair of real numbers $\alpha, \beta \in (0, \infty)$.

Proof. We give a proof for rows. $A = A(G, \omega)$ is row Perron-regular if and only if the entries in the right Perron vector corresponding to each part V_1 and V_2 are equal. Assume that $p_v^{(A)} = 1$ for every $v \in V_1$. Let $x = p_u^{(A)}$ for some (every) $u \in V_2$. Then $x\alpha = \lambda_A$ and $\beta = \lambda_A x$. Therefore, $\lambda_A = \sqrt{\alpha\beta}$ and $x = \sqrt{\beta/\alpha}$. \square

A row Perron-regular bipartite weighted graph is shown in Figure 5.3.

5.3. Trees. A *tree* is a connected graph $G = (V, E)$ such that any cycle has length two. A vertex $v \in V$ of a tree is *terminal* if $\delta_G(v) = 1$. Choose a distinguished vertex $R \in V$, called the *root*, and refer to G together with R as a *rooted tree*. A vertex which is not terminal nor the root is called *internal*. In a rooted tree we can define the *height* of a vertex $v \in V$ as the length of the shortest (unique) path between v and the root R (the root has height zero). The *height of a rooted tree* is the maximum height among the vertices. A *star* is a tree for which there exists a root R , called the

center, that makes it a rooted tree of height one.

PROPOSITION 5.6. *A tree is row (resp., column) Perron-regular if and only if it is a single vertex or a star. Moreover, if A is the adjacency matrix of a star with $n + 1$ vertices, then*

$$\lambda_A = \sqrt{n} \quad \text{and} \quad \mathbf{p}_A = \left(\sqrt{n} \mid J_n^T \right)^T \quad \left(\text{resp., } \mathbf{q}_A = \left(\sqrt{n} \mid J_n^T \right) \right). \quad (5.1)$$

Proof. Single vertices and stars are the only trees which are either regular or bipartite biregular. Therefore, the first claim follows from Theorem 5.4. (5.1) follows from the proof of Theorem 5.5. \square

5.4. Perron-regular degree. Any strongly connected directed graph $G = (V, E)$ admits a weight function $\omega: E \rightarrow (0, \infty)$ such that (G, ω) is row (resp., column) Perron-regular, e.g. any weight function ω such that (G, ω) is weight row (resp., column) regular for in this case $J_{|V|}^T$ (resp., $J_{|V|}$) is a right (resp., left) Perron vector. But there may exist less trivial weight functions giving rise to Perron-regular directed graphs, probably with Perron vectors having some entries distinct. The *weighted row (resp., column) Perron-regular degree* of G is

$$\rho^r(G) = \max \left\{ |\{p_i^{(A(G, \omega))} : i \in V\}| : \begin{array}{l} \omega: E \rightarrow (0, \infty) \text{ and } A(G, \omega) \\ \text{is row Perron-regular} \end{array} \right\}$$

$$\left(\text{resp., } \rho^c(G) = \max \left\{ |\{q_i^{(A(G, \omega))} : i \in V\}| : \begin{array}{l} \omega: E \rightarrow (0, \infty) \text{ and } A(G, \omega) \\ \text{is column Perron-regular} \end{array} \right\} \right).$$

Also, if G is row (resp., column) Perron-regular, let the (*unweighted*) *row (resp., column) Perron-regular degree* of G be

$$\rho_0^r(G) = |\{p_i^{(A^\#(G))} : i \in V\}| \quad \left(\text{resp., } \rho_0^c(G) = |\{q_i^{(A^\#(G))} : i \in V\}| \right).$$

If G is a graph, then $\rho^r(G) = \rho^c(G)$ and $\rho_0^r(G) = \rho_0^c(G)$ and then we simply let $\rho(G) = \rho^r(G)$ and $\rho_0(G) = \rho_0^r(G)$.

5.5. Colonies. Perron-regular degrees are bounded above by what we call the “colonial order”, i.e., by the number of “colonies”. A “colony” is a set of vertices that correspond to entries in a locally constant vector that are forced to be equal. To be precise, let $G = (V, E)$ be a strongly connected directed graph. For each $u \in V$ let

$$K_G^{r,0}(u) = N_G^r(u) \quad \left(\text{resp., } K_G^{c,0}(u) = N_G^c(u) \right)$$

and define $K_G^{r,k}(v)$ (resp., $K_G^{r,k}(v)$) inductively for every $k \geq 1$ by the rule

$$K_G^{r,k}(u) = K_G^{r,k-1}(u) \cup \bigcup_{v \in V} \left\{ N_G^r(v) : K_G^{r,k-1}(u) \cap N_G^r(v) \neq \emptyset \right\}$$

$$\left(\text{resp., } K_G^{c,k}(u) = K_G^{c,k-1}(u) \cup \bigcup_{v \in V} \left\{ N_G^c(v) : K_G^{c,k-1}(u) \cap N_G^c(v) \neq \emptyset \right\} \right).$$

Then $K_G^{r,k}(u)$ (resp., $K_G^{c,k}(u)$) eventually stabilizes as $k \rightarrow \infty$ (because $|V| < \infty$), i.e., there exists $k_0 \geq 0$ such that $K_G^{r,k}(u) = K_G^{r,k_0}(u)$ (resp., $K_G^{c,k}(u) = K_G^{c,k_0}(u)$) for every $k \geq k_0$ (and for every $u \in V$ too, e.g. $k_0 \geq n$). Define $K_G^r(u) = K_G^{r,k_0}(u)$ (resp., $K_G^c(u) = K_G^{c,k_0}(u)$) and call it the *row* (resp., *column*) *colony*⁴ of u . By definition, for every $u, v \in V$, either $K_G^r(u) = K_G^r(v)$ (resp., $K_G^c(u) = K_G^c(v)$) or $K_G^r(u) \cap K_G^r(v) = \emptyset$ (resp., $K_G^c(u) \cap K_G^c(v) = \emptyset$), that is, the colonies of two distinct vertices are either equal or disjoint. Let V^r (resp., V^c) be the set of row (resp., column) colonies of all the vertices in V , i.e.

$$V^r = \{K_G^r(u)\}_{u \in V} = \{V_1^r, \dots, V_{h_r}^r\} \quad (\text{resp., } V^c = \{K_G^c(u)\}_{u \in V} = \{V_1^c, \dots, V_{h_c}^c\}).$$

The row (resp., column) colonies form a partition of the vertex set. The *row* (resp., *column*) *colonial order* of G is the number of distinct row (resp., column) colonies $h_r = h_r(G)$ (resp., $h_c = h_c(G)$). For example, if G is a graph, then h_r and h_c are both equal to two if G is bipartite, otherwise they are both equal to one (Theorem 5.5).

PROPOSITION 5.7. *Let $(G = (V, E), \omega)$ be a weighted directed graph and $A = A(G, \omega) \in \mathbf{I}$. Then A is row (resp., column) Perron-regular if and only if for every $i = 1, \dots, h_r$ (resp., $i = 1, \dots, h_c$),*

$$p_u^{(A)} = p_v^{(A)} \quad \left(\text{resp., } q_u^{(A)} = q_v^{(A)} \right)$$

for every $u, v \in V_i^r$ (resp., $u, v \in V_i^c$).

Proof. This is a direct consequence of Theorem 4.8 and the definition of colonies. \square

For any strongly connected directed graph G we have

$$\rho^r(G) \leq h_r(G) \quad \text{and} \quad \rho^c(G) \leq h_c(G).$$

First we observe that on graphs the bound is attained. More precisely:

COROLLARY 5.8. *Let $G = (V, E)$ be a connected graph. Then*

$$\rho(G) = \begin{cases} 2 & \text{if } G \text{ is bipartite,} \\ 1 & \text{otherwise.} \end{cases}$$

⁴Hence, a colony consists of sets of neighborhoods.

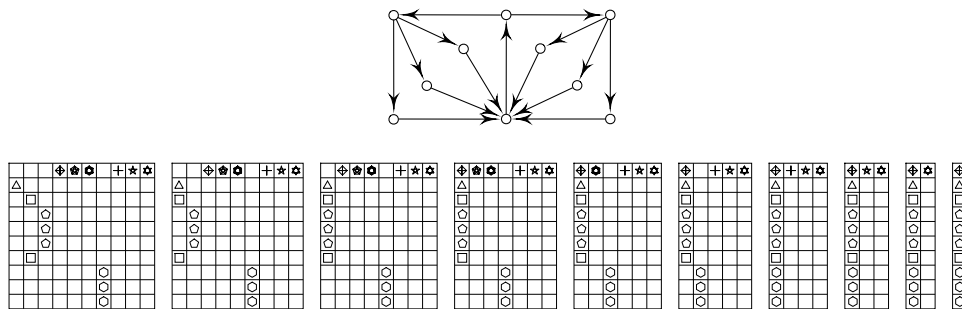


FIG. 5.4. A row (non-regular) Perron-regular directed graph G . The row neighborhood structure of the digraph implies that the form of the locally row constant vectors is $(a, b, c, d, d, d, d, c, f, f)^T$, i.e., the row colonial order is five. The row colonies can be found with the following procedure: Write a “symbolic” matrix so that the columns are distinguished symbols located according to the pattern of G (the transpose of $A^\#(G)$). Inductively merge two columns, leaving the symbols intact if there is no intersection, otherwise, choose one of the symbols and replace all the occurrences of the other symbols in the intersection by the chosen symbol. At the end a row colony is a set of vertices sharing the same symbol, and hence, the number of distinct symbols is the row colonial order. The right Perron vector of $A^\#(G)$ is $(\lambda^6, \lambda^3, \lambda, 1, 1, 1, \lambda, 1, 1, 1)^T$, where $\lambda = \sqrt[4]{6}$ is the corresponding Perron value. Then $\rho_0^r(G) = 4 < 6 = h_r(G)$.

In particular, if G is a single vertex or a star, then

$$\rho_0(G) = \begin{cases} 2 & \text{if } G \text{ has at least three vertices,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 5.5 and Proposition 5.6 \square

Hence, for any connected graph G we have $\rho(G) = h_r(G)$. Also if G is regular or bipartite biregular, then $\rho_0(G) = h_r(G)$ unless G is a single edge. An example of a row Perron-regular directed graph G for which $\rho_0(G) < h_r(G)$ is illustrated in Figure 5.4. Still the weighted Perron-regular degree always attains the upper bound for any strongly connected directed graph.

THEOREM 5.9. Let $G = (V, E)$ be a strongly connected directed graph. Then

$$\rho^r(G) = h_r(G) \quad (\text{resp., } \rho^c(G) = h_c(G)).$$

More precisely, for any row (resp., column) Perron-regular matrix A , there exists a row (resp., column) Perron-regular matrix B such that $\mathcal{P}^r(A) = \mathcal{P}^r(B)$ (resp., $\mathcal{P}^c(A) = \mathcal{P}^c(B)$) and the number of distinct entries in \mathbf{p}_B (resp., \mathbf{q}_B) is $h_r(G(A^\#))$ (resp., $h_c(G(A^\#))$).

Proof. Let $\omega: E \rightarrow X$ be a weight function such that (G, ω) is row Perron-regular. Let $A = A(G, \omega)$. Then \mathbf{p}_A is locally row constant. For any other locally

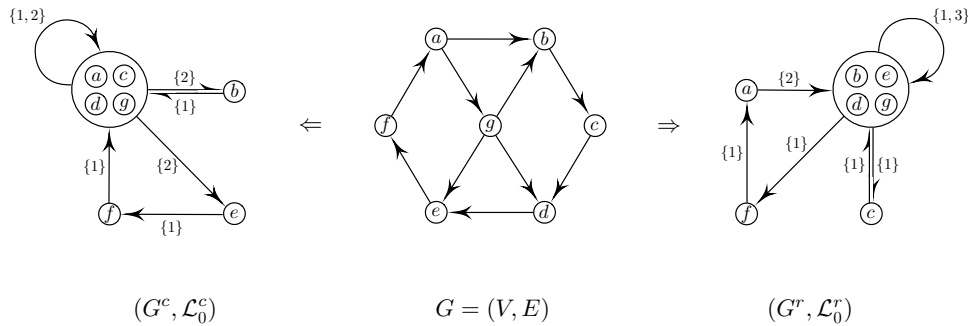


FIG. 5.5. A directed graph and the corresponding row and column colonial directed graphs together with the row and column degree structure. By Proposition 5.10, the original directed graph is not row nor column Perron-regular since the labelings of the loops are not singletons.

row constant vector \mathbf{v} , let $D \in \mathbf{D}$ be defined by $\text{diag}(D) = \mathbf{v}$. By Corollary 4.11, the matrix $B = D^{-1}AD$ is row Perron-regular and yields the same Perron row stochastic form as A . Since $\mathbf{p}_B = \frac{\mathbf{p}_A}{\mathbf{v}}$, we can find \mathbf{v} so that the claim holds.

The rest of the proof is analogous. \square

5.6. Colonial directed graph. By definition, there are surjective functions $\sigma^r, \tau^r: V \rightarrow \{1, \dots, h_r\}$ (resp., $\sigma^c, \tau^c: V \rightarrow \{1, \dots, h_c\}$) such that $u \in V_{\sigma^r(u)}$ and $N_G^r(u) \subset V_{\tau^r(u)}$ (resp., $u \in V_{\sigma^c(u)}$ and $N_G^c(u) \subset V_{\tau^c(u)}$) for every $u \in V$. Let

$$E^r = \{(V_{\sigma^r(v)}, V_{\tau^r(v)})\}_{v \in V} \quad (\text{resp., } E^c = \{(V_{\sigma^c(v)}, V_{\tau^c(v)})\}_{v \in V}).$$

$G^r = (V^r, E^r)$ (resp., $G^c = (V^c, E^c)$) is the row (resp., column) colonial directed graph of G . It is strongly connected since G is assumed to be strongly connected. Finally, given a weight function $\omega: E \rightarrow (0, \infty)$, let the *weighted row* (resp., *column*) *degree structure* be the labelling⁵ $\mathcal{L}_\omega^r: E^r \rightarrow \text{Pow}^*((0, \infty))$ (resp., $\mathcal{L}_\omega^c: E^c \rightarrow \text{Pow}^*((0, \infty))$) defined for every $e \in E^r$ (resp., $e \in E^c$) by

$$\mathcal{L}_\omega^r(e) = \{\delta_\omega^r(v) : v \in V \text{ is such that } (\sigma^r(v), \tau^r(v)) = e\}.$$

$$(\text{resp., } \mathcal{L}_\omega^c(e) = \{\delta_\omega^c(v) : v \in V \text{ is such that } (\tau^c(v), \sigma^c(v)) = e\}).$$

The (*unweighted*) row (resp., column) *degree structure* is the labelling $\mathcal{L}_0^r = \mathcal{L}_\omega^r$ (resp., $\mathcal{L}_0^c = \mathcal{L}_\omega^c$) for the weight function $\omega(e) = 1$ for every $e \in E$. An example is depicted in Figure 5.5.

PROPOSITION 5.10. *Let $G = (V, E)$ be a directed graph and $\omega: E \rightarrow (0, \infty)$ a weight function. If $|\mathcal{L}_\omega^r(e)| > 1$ (resp., $|\mathcal{L}_\omega^c(e)| > 1$) for some $e \in E^r$ (resp., $e \in E^c$), then (G, ω) is not row (resp., column) Perron-regular.*

⁵ $\text{Pow}^*(Y)$ denotes the set of non-empty finite subsets of the set Y .

Proof. It follows from Proposition 5.7. \square

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