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HOMEOMORPHIC IMAGES OF ORTHOGONAL BASES*

M. KEBRYAEE† AND M. RADJABALIPOUR†

Abstract. Necessary and sufficient conditions are obtained for a sequence $\{x_j : j \in \mathbb{J}\}$ in a Hilbert space to be, up to the elimination of a finite subset of $\mathbb{J}$, the linear homeomorphic image of an orthogonal basis of some Hilbert space K. This extends a similar result for orthonormal bases due to Holub [J.R. Holub. Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces. *Proc. Amer. Math. Soc.*, 122(3):779–785, 1994]. The proofs given here are based on simple linear algebra techniques.

Key words. Separable Hilbert space, Riesz basis, Orthogonal basis, Analysis operator, Cofinite-rank operator.

AMS subject classifications. 15A03, 46A35, 40A05.

1. Introduction. Throughout the paper, let H be a fixed separable Hilbert space and, to avoid triviality, assume without loss of generality that $\dim(H) \neq 0$. The class of all bounded linear operators from a Banach space X into a Banach space Y is denoted by $B(X, Y)$ and by $B(X)$ in case $X = Y$. In the present paper, we study (finite or infinite) sequences $\{x_j\}_{j\in\mathbb{J}}$ in H which generate H and are (linear) homeomorphic images of orthogonal bases; more precisely, there exists a bijective Hilbert space operator $U \in B(K, H)$ such that

$$\begin{aligned} UK = H,\ x_j = U\phi_j,\ \langle \phi_i, \phi_j\rangle = \delta_{ij}\|\phi_j\|^2,\ \ \forall\, i, j \in \mathbb{J},\ \text{ and} \\ K = \overline{\text{span}}\{\phi_j :\ j \in \mathbb{J}\}. \end{aligned} \tag{1.1}$$

(By $\overline{\text{span}}(\Delta)$, we mean the closure of the linear subspace spanned by the subset Δ of a given Banach space.) If $\{\phi_j\}_{j\in\mathbb{J}}$ is orthonormal, then $\{x_j\}_{j\in\mathbb{J}}$ is called a Riesz basis. We may and shall assume without loss of generality that $\mathbb{J} = \mathbb{N}$ or $\mathbb{J} = \{1, 2, \dots, n\}$ with $n = \dim(K)$.

J.R. Holub [4] shows that, for a sequence $\{x_j\}_{j\in\mathbb{J}}$ in H, the following assertions (a)-(c) are equivalent.

(a) A cofinite subset of $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis for H.

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†Department of Mathematics and Kerman Science and Research Branch, Islamic Azad University, Kerman, Iran (mradjabalipour@gmail.com). The research is supported by Allameh Tabatabaee's grant from Iranian National Elite Foundation.

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(b) $0 < \inf_{||x||=1} \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 \le \sup_{||x||=1} \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 < \infty$; moreover, if $\sum_{j\in\mathbb{J}} c_j x_j$ is a convergent series in H, then $(c_j)_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$.

(c) $0 < \inf_{||x||=1} \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 \le \sup_{||x||=1} \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 < \infty$; moreover, the closure of the set $\mathfrak{R} := (\langle x, x_j\rangle)_{j\in\mathbb{J}} :\ x \in H\}$ is a cofinite-dimensional subspace of $\ell^2(\mathbb{J})$.

Note that each of the conditions $(a) - (c)$ imply that $H = \overline{\text{span}}\{x_j :\ j \in \mathbb{J}\}$ and $x_j \neq 0$ for all but finitely many $j \in \mathbb{J}$. If $\mathbb{J}_1 = \{j \in \mathbb{J} :\ x_j \neq 0\}$, then $\ell^2(\mathbb{J}) = \ell^2(\mathbb{J}_1) \oplus \ell^2(\mathbb{J}\backslash\mathbb{J}_1)$, where for our purposes the second summand is completely useless. For these reasons, we assume without loss of generality that

$$H = \overline{\text{span}}\{x_j :\ j \in \mathbb{J}\}, \text{ and } x_j \neq 0, \ \forall\, j \in \mathbb{J}. \tag{1.2}$$

Also, the part $0 < \inf_{||x||=1} \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 \le \sup_{||x||=1} \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 < \infty$ of Conditions $(b) - (c)$ implies that the so-called analysis mapping $x \mapsto (\langle x, x_j\rangle)$ is a continuous linear transformation with domain $\mathfrak{D}$ and range $\mathfrak{R}$ satisfying

$$\mathfrak{D} = H \text{ and } \mathfrak{R} = \overline{\mathfrak{R}} \subset \ell^2(\mathbb{J}). \tag{1.3}$$

We will, thus, make use of these weaker conditions in our generalizations of Holub's result to be explained below.

Let $\ell^2(w_j)$ denote the Hilbert space $L^2(\mathbb{J}, 2^{\mathbb{J}}, \mu)$ in which μ is a positive measure defined by $\mu(\{j\}) = w_j > 0$ for all $j \in \mathbb{J}$; if $0 < \inf_j w_j \le \sup w_j < \infty$, then $\ell^2(\mathbb{J}) \equiv \ell^2(w_j)$. The analysis operator corresponding to a *general* sequence $\{x_j\}$ in H is defined as the linear transformation $T : H \to \mathbb{C}^{\mathbb{J}}$ by $Tx = (\langle x, x_j\rangle)_j$.

As we mentioned earlier, Condition (1.2) makes T injective and makes it possible to define $\mathfrak{R} := (TH) \cap \ell^2(||x_j||^{-2})$ equipped with the norm inherited from $\ell^2(||x_j||^{-2})$ and to define $\mathfrak{D} := T^{-1}(\mathfrak{R})$. The first part of Condition (1.2) is an immediate consequence of each of (1.1), (a), (b) or (c); the second part is imposed to avoid redundant vectors which can occur at most finitely many times under Conditions $(a) - (c)$. The restriction $T|_{\mathfrak{D}}$ of the analysis operator T is said to be bounded if

$$||Tx||^2 = \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 ||x_j||^{-2} \le b||x||^2, \ \forall\, x \in \mathfrak{D}. \tag{1.4}$$

We tried to mimic Holub's proof to extend his results to the case that $\{x_j\}_{j\in\mathbb{J}}$ has a cofinite subset which is a homeomorphic image of a general orthogonal basis $\{\phi_j\}_{j\in\mathbb{J}}$. However, we ended up with a new proof for the original results as well as their extensions which seems to be of interest to linear algebraists. The proof involves a kind of row echelon form techniques in the infinite dimensions; for this reason, we avoid the terminologies from frame theory or wavelets.

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2. Main results. The following is the main result of the paper.

THEOREM 2.1. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence in H satisfying* (1.2). *If $T|_{\mathfrak{D}}$ is bounded, then $\mathfrak{D}=\overline{\mathfrak{D}}$. Moreover, the following assertions $(a')-(c')$ are equivalent.*

(a') Up to a reordering of $\mathbb{J}$, there exists $N\in\mathbb{N}$ such that (1.1) *holds with $\mathbb{J}$ replaced by $\{j\in\mathbb{J}:\ j\geq N\}$.*
(b') $\mathfrak{D}=H$, $T|_{\mathfrak{D}}$ is bounded, $\mathfrak{R}=\overline{\mathfrak{R}}$ and, if $\sum_{j\in\mathbb{J}}c_jx_j$ is a convergent series in H, then $(c_j||x_j||)_{j\in\mathbb{J}}\in\ell^2(\mathbb{J})$.
(c') $\mathfrak{D}=H$, $T|_{\mathfrak{D}}$ is bounded and $\mathfrak{R}=\overline{\mathfrak{R}}$ with $\dim(\mathfrak{R}^\perp)=m$ for some integer m.

Proof. The proof of $\mathfrak{D}=\overline{\mathfrak{D}}$ follows from the closability of T proven by Antonie and Balasz [1]; however, to avoid the inconveniency of the superficial differences, we brief the proof here. Let $y_n\in\mathfrak{D}$ converge to $y\in\overline{\mathfrak{D}}$. Then $(\langle y_n,x_j\rangle)_j$ converges to $Ty=(c_j)_j\in\ell^2(||x_j||^{-2})$ as $n\to\infty$. Hence, for each $j\in\mathbb{J}$, $\langle y_n,x_j\rangle\to c_j$ as $n\to\infty$. Thus, $c_j=\langle y,x_j\rangle\ \forall j\in\mathbb{J}$, and $y\in\mathfrak{D}$. This establishes the first part of the theorem. Now, we continue the proof in three steps.

$(a')\Rightarrow(b')$: It is sufficient to prove (b') for $J=\mathbb{J}\cap\{N,N+1,N+2,\ldots\}$. Since $||x_j||\leq||U||\ ||\phi_j||\leq||U||\ ||U^{-1}||\ ||x_j||$ for all $j\in J$, it follows that, for any unit vector $x\in H$,

$$\begin{aligned}0<||U^{-1}||^{-2}||U||^{-2}&\leq||U||^{-2}||U^*x||^2=||U||^{-2}\sum_{j\in J}|\langle U^*x,\phi_j/||\phi_j||\rangle|^2\\&\leq||Tx||^2=\sum_{j\in J}|\langle x,x_j\rangle|^2||x_j||^{-2}=\sum_{j\in J}|\langle U^*x,\phi_j/||\phi_j||\rangle|^2(||\phi_j||/||x_j||)^2\\&\leq\sum_{j\in J}|\langle U^*x,\phi_j/||\phi_j||\rangle|^2=||U^{-1}||^2||U^*x||^2\leq||U^{-1}||^2||U||^2<\infty.\end{aligned}$$

This shows that $\mathfrak{D}=H$, the linear transformations $T_{\mathfrak{D}}:H\to\mathfrak{R}$ and $T_{\mathfrak{D}}^{-1}:\mathfrak{R}\to H$ are bounded and, hence, the subspace $\mathfrak{R}$ is closed.

Next, assume $\sum_{j\in\mathbb{J}}c_jx_j$ is a convergent series in H for some sequence of complex numbers $(c_j)_{j\in\mathbb{J}}$. Then $\sum_{j\in\mathbb{J}\setminus J}|c_j|^2||x_j||^2<\infty$ and

$$||U||^{-2}\sum_{j\in J}|c_j|^2||x_j||^2\leq\sum_{j\in J}|c_j|^2||\phi_j||^2=||\sum_{j\in J}c_j\phi_j||^2=||U^{-1}\sum_{j\in J}c_jx_j||^2<\infty.$$

$(b')\Rightarrow(c')$: Trivially, $\mathfrak{D}=H$, $T|_{\mathfrak{D}}$ is bounded and $\mathfrak{R}=\overline{\mathfrak{R}}$. Assume, if possible, $\dim(\mathfrak{R}^\perp)=\infty$; in this case, $\mathbb{J}=\mathbb{N}$. Let $\{\psi_1,\psi_2,\ldots,\psi_n,\ldots\}$ be any infinite sequence of linearly independent vectors in $\ker(T^*)$. Let $\{e_j:\ j\in\mathbb{J}\}$ be the standard $\{0,1\}$-basis of $\ell^2(||x_j||^{-2})$. Define $\xi_j=||x_j||e_j$ for all $j\in\mathbb{N}$ and observe that $\{\xi_j:\ j\in\mathbb{N}\}$ is an orthonormal basis of $\ell^2(||x_j||^{-2})$. Write $\psi_n=\sum_{j=1}^\infty c_{nj}\xi_j$ for $n\in\mathbb{N}$. Applying the row echelon form techniques to infinite square arrays, we can restrict ourselves to

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an infinite subsequence of $\{\psi_n\}_{n\in\mathbb{N}}$ to assume without loss of generality that

$$||\psi_n|| = 1, \ \ \psi_n = \sum_{j=k_n}^{\infty} c_{nj}\xi_j \ \text{ and } \ c_{n,k_n} \neq 0 \tag{2.1}$$

for all $n \in \mathbb{N}$ and for some positive integers $k_1 < k_2 < k_3 < \cdots$. Since $T^*\psi_n = 0$ $\forall n \in \mathbb{N}$, we can procede by induction on n to replace the sequence $k_1, k_2, k_3, \ldots$ by a subsequence to assume without loss of generality that

$$||T^*(\sum_{j=k_n}^{m} c_{nj}\xi_j)|| < 2^{-n} \ \text{ and } \ 1 - 2^{-n} \leq \sum_{j=k_n}^{m} |c_{nj}|^2 \leq 1 \ \forall m \geq k_n.$$

It is easy to see that $T^*\xi_j = ||x_j||^{-1}x_j$ for all $j \in \mathbb{N}$ and, letting $m_n = k_{n+1} - 1$, the series

$$\sum_{n=1}^{\infty} n^{-1/2} \sum_{j=k_n}^{m_n} (c_{nj}||x_j||^{-1})x_j \quad \left(\text{or } \ T^* \sum_{n=1}^{\infty} n^{-1/2} \sum_{j=k_n}^{m_n} c_{nj}\xi_j\right)$$

converges. On the other hand, the series

$$\sum_{n=1}^{\infty} n^{-1} \sum_{j=k_n}^{m_n} |(c_{nj}||x_j||^{-1})|^2 ||x_j||^2$$

diverges; a contradiction.

$(c') \Rightarrow (a')$: Let $\xi_j = ||x_j||e_j$ be as in the proof of $(b') \Rightarrow (c')$ and recall that $T^*e_j = ||x_j||^{-2}x_j$ for all $j \in \mathbb{J}$. The proof will be complete if we can eliminate a finite subset of $\mathbb{J}$ to arrive at a subset J such that the restriction of T^* to $K := \overline{\text{span}}\{e_j : \ j \in J\}$ is a homeomorphism onto H. Let $m = \dim(\mathfrak{R}^\perp)$. If $m \geq 1$, choose a (not necessarily orthogonal) basis $\{\psi_1, \psi_2, \ldots, \psi_m\}$ for $\ker(T^*) = \mathfrak{R}^\perp$ and write $\psi_i = \sum_{j\in\mathbb{J}} c_{ij}\xi_j$ ($j \in \mathbb{J}, \ i = 1, 2, \ldots, m$). Applying the row echelon form and relabeling a finite number of ξ_j's, one can assume without loss of generality that

$$\begin{aligned}
\psi_1 &= \xi_1 + c_{1,m+1}\xi_{m+1} + c_{1,m+2}\xi_{m+2} + \cdots \\
\psi_2 &= \xi_2 + c_{2,m+1}\xi_{m+1} + c_{2,m+2}\xi_{m+2} + \cdots \\
\psi_3 &= \xi_3 + c_{3,m+1}\xi_{m+1} + c_{3,m+2}\xi_{m+2} + \cdots \\
&\vdots \qquad\qquad \vdots \\
\psi_m &= \xi_m + c_{m,m+1}\xi_{m+1} + c_{m,m+2}\xi_{m+2} + \cdots.
\end{aligned} \tag{2.2}$$

The desired J and K can be now defined as $J = \mathbb{J} \cap \{m+1, m+2, \ldots\}$ and $K := \overline{\text{span}}(\{e_j : \ j \in J\}$. Define $U \in B(K, H)$ by $U = T^*|_K$. Then $Ue_j = ||x_j||^{-2}x_j$ or, equivalently, $U\xi_j = ||x_j||^{-1}x_j$ for all $j \in J$. Let $Ux = 0$ for some $x \in K$ and write

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$x = u \oplus v$ with $u \in \mathfrak{R}$ and $v \in \ker(T^*) = \mathfrak{R}^\perp$. Then $T^*u = T^*v = 0$ and, hence, $u = 0$. Consequently, $x \in \ker(T^*)$ and

$$x = \alpha_1\psi_1 + \cdots + \alpha_m\psi_m = \alpha_1\xi_1 + \cdots + \alpha_m\xi_m + \sum_{j\in J}\alpha_j\xi_j \in K$$

for some complex numbers α_i, $i \in \mathbb{J}$. Thus, $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ and, hence, $x = 0$. Thus, $U : K \to H$ is a bounded injective operator and it remains to show that it is surjective.

Consider $T : H \to \mathfrak{R}\oplus\mathfrak{R}^\perp$ and $T^* : \mathfrak{R}\oplus\mathfrak{R}^\perp \to H$ with the following block matrix representations:

$$T = \begin{bmatrix} S \\ 0 \end{bmatrix} \text{ and } T^* = \begin{bmatrix} S^* & 0 \end{bmatrix},$$

in which $S : H \to \mathfrak{R}$ and $S^* : \mathfrak{R} \to H$ are linear homeomorphisms. For arbitrary $y \in H$, choose $z \in \mathfrak{R}$ such that $y = S^*z = T^*z$. There exist complex numbers c_j such that

$$z = \sum_{j\in\mathbb{J}} c_j\xi_j = \sum_{j=1}^{m} c_j\xi_j + \sum_{j\in J} c_j\xi_j = \sum_{j=1}^{m} c_j\psi_j + x,$$

where $x = -\sum_{j=1}^m c_j \sum_{i\in J} c_{ji}\xi_i + \sum_{j\in J} c_j\xi_j \in K$ and $Ux = T^*x = T^*z = y$. Thus, $U : K \to H$ is a linear homeomorphism mapping the orthogonal basis $\{||x_j||^2 e_j\}_{j\in J}$ onto the sequence $\{x_j\}_{j\in J}$. $\square$

Corollary 2.2. *If N and m make (a') and (c') equivalent, it follows necessarily that $m = N$.*

Proof. Note that $m = \dim(\mathfrak{R}^\perp)$ is unique and, in view of the proof of $(c') \Rightarrow (a')$, the integer $N = m$ establishes (a'). It remains to show that N is unique, too. For each $i = 1, 2$, let J_i be a cofinite subset of $\mathbb{J}$ such that $x_j = U_i\phi_{ji}$ for $j \in J_i$ for some linear homeomorphism $U_i \in B(K_i, H)$ and some orthogonal basis $\{\phi_{ji}\}$ of a Hilbert space K_i. Let $J = J_1 \cap J_2$ and define $H_0 = \overline{\operatorname{span}}(\{x_j : j \in J\})$. Let y_j be the projection of x_j on $H_0^\perp$. Since $\{x_j/||\phi_{ji}|| : j \in J_i\}$ is a Riesz basis for H, it follows that $\{y_j/||\phi_{ji}|| : j \in J_i\backslash J\}$ is a basis for the finite dimensional space $H_0^\perp$ for $i = 1, 2$. Thus, the sets $J_1\backslash J$ and $J_2\backslash J$ have the same cardinality and so do the sets $\mathbb{J}\backslash J_1$ and $\mathbb{J}\backslash J_2$. $\square$

Remark. It is interesting to extend Theorem 2.1 when the sequence x_j is replaced by a function $x(t)$ with values in H as t runs in a measure space $\mathfrak{T}$ equipped with an arbitrary positive measure τ. The analysis operator $T : H \to \mathbb{C}^{\mathfrak{T}}$ is defined as $(Tx)(t) = \langle x, x(t)\rangle$ for $t \in \mathfrak{T}$. Here, again, we define $\mathfrak{R} := (TH) \cap L^2(\tau)$ and $\mathfrak{D} = T^{-1}(\mathfrak{R})$. Again, here, if $T|_{\mathfrak{D}}$ is continuous and if y_n is a sequence in $\mathfrak{D}$ converging

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to $\overline{\mathfrak{D}}$, it follows that Ty_n converges pointwise to some $f \in L^2(\tau)$ such that $f(t) = \langle y, x(t)\rangle$ a.e.$[\tau]$. Therefore, we can assume without loss of generality that $f = Ty$ and, hence, $y \in \mathfrak{D}$. Thus, $\mathfrak{D}$ is closed. Since H is separable, we have no counterpart of Condition (a'). Regarding the counterpart of (b'), we run into difficulty with the type of convergence of the integral of the vector-valued functions. However, Condition (c') can be easily interpreted as Condition $(c")$ given below.

(c") $\mathfrak{D} = H$, $T|_{\mathfrak{D}}$ is bounded and $\mathfrak{R}$ is a cofinite-dimensional closed subspace of $L^2(\tau)$.

Strange to say, it turns out that the measure space $\mathfrak{T}$ of Condition $(c")$ is necessarily a countable union of atoms of τ and, hence, $x(t)$ is essentially a sequence. This can be deduced from results obtained by Askari-Hemmat, Dehghan and Radjabalipour [2] and Giv and Radjabalipour [3]. Here, we present a clear short proof of it.

Let $m = \dim(\mathfrak{R}^\perp)$ and define $K = H \oplus \mathbb{C}^m$. Let $g_1, g_2, \ldots, g_m$ be an orthonormal set in $\mathfrak{R}^\perp$ and assume they are defined everywhere on $\mathfrak{T}$. Define $y(t) = x(t) \oplus [\bar{g}_1(t)\phi_1 + \cdots + \bar{g}_m(t)\phi_m]$, where $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is the standard $\{0,1\}$-basis of $\mathbb{C}^m$. Now, if $t \in \mathfrak{T}$ and $k = h \oplus [c_1\phi_1 + \cdots + c_m\phi_m] \in K$ is arbitrary, then $y(t) \in K$, $||k||^2 = ||h||^2 + \sum_i |c_i|^2$ and $\langle k, y(\cdot)\rangle = \langle h, x(\cdot)\rangle + c_1g_1(\cdot) + c_2g_2(\cdot) + \cdots + c_mg_m(\cdot) \in L^2(\tau)$. Thus, letting T_y, $\mathfrak{R}_y$ and $\mathfrak{D}_y$ denote the analysis operator and the other associated parameters of $y(t)$, it follows that $||T_yk||^2 = ||T_xh||^2 + |c_1|^2 + \cdots + |c_m|^2 \le (||T_x||^2 + 1)||k||^2$. Therefore, $(c")$ holds when $x(t) \in H$ is replaced by $y(t) \in K$ and, in this case, $\mathfrak{R}_y \supset \mathfrak{R}_x \cup \{g_1 = T_y\phi_1, g_2 = T_y\phi_2, \ldots, g_m = T_y\phi_m\}$; i.e., $\mathfrak{R}_y = L^2(\tau)$.

Next, replace $y(t)$ by $z(t) = y(t)/||y(t)||$ and $d\tau$ by $d\nu = ||y(t)||^2 d\tau$. Again, here,

$$||T_zw||^2 = \int |\langle w, z(t)\rangle|^2 d\nu = \int |\langle w, y(t)\rangle|^2 d\tau = ||T_yw||^2 \le ||T_y||^2||w||^2 < \infty \ \forall w \in K$$

which implies that $\mathfrak{D}_z = K$ and T_z is bounded. Moreover, the mapping $W : L^2(\tau) \to L^2(\nu)$ defined by $(Wg)(t) = g(t)||y(t)||^{-1}$ is a unitary operator with inverse $(W^{-1}h)(t) = h(t)||y(t)||$ for all $g \in L^2(\tau)$ and all $h \in L^2(\nu)$. In particular, $W(T_yw) = T_zw$ for all $w \in K$, which implies that $\mathfrak{R}_z = W\mathfrak{R}_y = L^2(\nu)$. Therefore, $(c")$ holds for $z(t)$ with the extra conditions that $||z(t)|| \equiv 1$ and $\mathfrak{R}_z = L^2(\nu)$.

Finally, let E be an arbitrary set of positive ν-measure. Then, for all $t \in E$,

$$\begin{aligned}\nu(E)^{-1/2} &= \nu(E)^{-1/2}\chi_E(t) = \langle T_z^{-1}(\nu(E)^{-1/2}\chi_E), z(t)\rangle \\ &\le ||T_z^{-1}|| \cdot ||\nu(E)^{-1/2}\chi_E|| \cdot ||z(t)|| = ||T_z^{-1}|| < \infty,\end{aligned}$$

and, hence,

$$\nu(E) \ge ||T_z^{-1}||^{-2} > 0. \tag{2.3}$$

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This shows that ν and, consequently, τ are supported on the union $\cup_{j\in\mathbb{J}}E_j$ of atomic sets. Now, identifying E_j as $\{j\}$ reveals that F is a w-frame.

Summing up, we have proven the following corollary.

COROLLARY 2.3. *Let $x : \mathfrak{T} \to H$ and define $T : H \to \mathbb{C}^{\mathfrak{T}}$ by $Ty = (\langle y, x(t)\rangle)$ for $t \in \mathfrak{T}$, where $\mathfrak{T}$ is a set equipped with a positive measure τ. Let $\mathfrak{R} = (TH) \cap L^2(\tau)$ and let $\mathfrak{D} = T^{-1}(\mathfrak{R})$. Assume $T|_{\mathfrak{D}}$ is a bounded linear transformation. Then $\overline{\mathfrak{D}} = \mathfrak{D}$. Moreover, if $\mathfrak{D} = H$ and $\mathfrak{R}$ is a cofinite-dimensional closed subspace of $L^2(\tau)$, then $\mathfrak{T}$ is the disjoint union of τ-atoms $\{E_j : j \in \mathbb{J}\}$ for some countable set $\mathbb{J}$. Identifying E_j with $\{j\}$ yields a sequence $\{x_j\}$ satisfying (c'), where $x_j = x(t)$ for almost all $t \in E_j$.*

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