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ON THE MATRIX EQUATIONS $U_I X V_J = W_{I\,J}$ FOR $1 \leqslant I, J < K$ WITH $I + J \leqslant K$ *

JACOB VAN DER WOUDE†

Abstract. Conditions for the existence of a common solution X for the linear matrix equations $U_i X V_j = W_{i\,j}$ for $1 \leqslant i, j < k$ with $i + j \leqslant k$, where the given matrices $U_i, V_j, W_{i\,j}$ and the unknown matrix X have suitable dimensions, are derived. Verifiable necessary and sufficient solvability conditions, stated directly in terms of the given matrices and not using Kronecker products, are also presented. As an application, a version of the almost triangular decoupling problem is studied, and conditions for its solvability in transfer matrix and state space terms are presented.

Key words. Linear matrix equations, Common solution, Rational matrix equations, Triangular decoupling.

AMS subject classifications. 15A24.

1. Introduction. In this paper, we study a set of linear matrix equations of the form

$$U_i X V_j = W_{i\,j} \quad \text{for} \quad 1 \leqslant i, j < k, \quad \text{with} \quad i + j \leqslant k, \tag{1.1}$$

where $k \geqslant 2$ is some given integer, U_i, V_j and $W_{i\,j}$ are given matrices of suitable dimensions over a field $\mathcal{F}$, and X is the unknown matrix of suitable dimensions over the same field.

The main result of this paper are necessary and sufficient conditions for the existence of the common solution X for all the equations (1.1) directly given in terms of (matrices made up of) the matrices U_i, V_j and $W_{i\,j}$. This implies that the conditions can be used in situations where it is relevant to maintain the context of the problem.

An example of such a situation is a version of the problem of almost triangular decoupling, where the field involved is the set of rational functions, and the context is to find conditions for the solvability of the problem in terms of a state space description of the underlying system.

The results in this paper will be stated in terms of the notions of “image” and “kernel” of a matrix. Of course, the results can equivalently be expressed in terms the

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†Department of Applied Mathematics, Faculty Electrical Engineering, Mathematics an Computer Science, Delft University of Technology, The Netherlands (J.W.vanderWoude@tudelft.nl).

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so-called "column space" and "row space" of a matrix. However, in line of the system theory application, where the notions of "image" and "kernel" are most common, we have chosen to use the latter notions.

The present paper is largely based on an unpublished chapter of [6].

2. Main result.

Let $\mathcal{F}$ be an appropriate field. We say that a matrix is injective if it has full column rank. A matrix is called surjective if it has full row rank.

2.1. Some preliminaries.

The following results are well-known and/or easy to prove, cf. [4].

LEMMA 2.1. *All matrices below have suitable dimensions.*

1. *If U is a surjective matrix and V is an injective matrix, then for every matrix W, there exists a matrix X such that $UXV = W$.*
2. *More general, given matrices U, V and W, there exists a matrix X such that $UXV = W$ if and only if $im\ U \supseteq im\ W$ and $ker\ V \subseteq ker\ W$.*

LEMMA 2.2. *Let A, U, B and V be $a \times b, c \times b, a \times d$ and $c \times d$ matrices, respectively. Further, let matrix X be a $b \times d$ matrix such that $B = AX$.*

1. *Then, $ker\ [A, B] \subseteq ker\ [U, V] \iff ker\ [A, 0] \subseteq ker\ [U, V - UX]$.*
2. *Moreover, $ker\ [A, 0] \subseteq ker\ [U, V - UX] \implies V = UX$.*

2.2. General formulation.

For $k \in \mathbb{N}$, define $\underline{k} := \{1, 2, \ldots, k\}$. Hence, $\underline{k-1} = \{1, 2, \ldots, k-1\}$.

Let $U_i \in \mathcal{F}^{a_i \times b}, V_j \in \mathcal{F}^{c \times d_j}$ and $W_{ij} \in \mathcal{F}^{a_i \times d_j}$ with $i, j \in \underline{k-1}$ be given matrices with entries in $\mathcal{F}$. For all $i, j \in \underline{k-1}$, denote

$$\Lambda_i := \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_i \end{bmatrix}, \quad \Delta_j := \begin{bmatrix} V_1 & V_2 & \cdots & V_j \end{bmatrix}, \tag{2.1}$$

and

$$\Gamma_{ij} := \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1j} \\ W_{21} & W_{22} & \cdots & W_{2j} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{ij} \end{bmatrix}. \tag{2.2}$$

Then the main result of this paper is the following.

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THEOREM 2.3. *Given the matrices in (1.1), (2.1) and (2.2), the following statements are equivalent.*

1. *There exists a matrix $X \in \mathcal{F}^{b\times c}$ such that $U_iXV_j = W_{ij}$ for all $i, j \in \underline{k-1}$ with $i + j \leqslant k$.*
2. *For all $i \in \underline{k-1}$, there exists a matrix $X \in \mathcal{F}^{b\times c}$ such that $\Gamma_{i\,k-i} = \Lambda_i X\Delta_{k-i}$.*
3. *For all $i \in \underline{k-1}$, im $\Lambda_i \supseteq$ im $\Gamma_{i\,k-i}$ and ker $\Delta_i \subseteq$ ker $\Gamma_{k-i\,i}$.*

Proof. From Lemma 2.1:2, it follows that statements 2 and 3 are equivalent. Statement 1 is equivalent to the existence of a matrix $X \in \mathcal{F}^{b\times c}$ such that $\Gamma_{i\,k-i} = \Lambda_i X\Delta_{k-i}$ for all $i \in \underline{k-1}$. So, statement 1 implies statement 2. Hence, the proof of the theorem is complete if we can prove that statement 2 or 3 implies statement 1.

To prove that statement 3 implies statement 1, observe that $\Delta_j = [\Delta_{j-1}, V_j]$ for all $j \in \underline{k-1}$, where we define Δ_0 to be void (a matrix with zero columns/ rows) and im $\Delta_0 = 0$. Then it follows that im $\Delta_{j-1} \subseteq$ im $\Delta_j = [\Delta_{j-1}, V_j]$ for all $j \in \underline{k-1}$. Therefore, for all $j \in \underline{k-1}$ there exists an injective matrix $\widetilde{V}_j$ such that im $\widetilde{V}_j \subseteq$ im V_j, im $\Delta_j =$ im $\Delta_{j-1}+$ im $\widetilde{V}_j$ and im $\Delta_{j-1} \cap$ im $\widetilde{V}_j = 0$. Furthermore, for all $j \in \underline{k-1}$ there exists a square invertible matrix T_j such that $V_jT_j = [\widetilde{V}_j, \widehat{V}_j]$ with im $\widehat{V}_j \subseteq$ im Δ_{j-1}. By induction it follows that im $\Delta_j =$ im $\widetilde{V}_1+$ im $\widetilde{V}_2 + \cdots +$ im $\widetilde{V}_j$ and $[\widetilde{V}_1, \widetilde{V}_2, \ldots, \widetilde{V}_j]$ is an injective matrix for all $j \in \underline{k-1}$. Therefore, it follows that for all $j \in \underline{k-1}$ there exist $j-1$ matrices $\widetilde{T}_{1j}, \widetilde{T}_{2j}, \ldots, \widetilde{T}_{j-1\,j}$ such that

$$\widehat{V}_j = \sum_{l=1}^{j-1} \widetilde{V}_l \widetilde{T}_{lj}.$$

Also note that $[\widetilde{V}_1, \widetilde{V}_2, \ldots, \widetilde{V}_{k-1}]$ is an injective matrix.

Analogously, we can conclude the existence of $k-1$ square invertible matrices $S_1, S_2, \ldots, S_{k-1}$, and for each $i \in \underline{k-1}$, the existence of $i-1$ matrices $\widetilde{S}_{i1}, \widetilde{S}_{i2}, \ldots,$ $\widetilde{S}_{i\,i-1}$, such that for all $i \in \underline{k-1}$

$$S_iU_i = \begin{bmatrix} \widetilde{U}_i \\ \overline{U}_i \end{bmatrix}, \quad \overline{U}_i = \sum_{l=1}^{i-1} \widetilde{S}_{il}\widetilde{U}_l, \quad \ker \Lambda_i = \ker \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \\ \vdots \\ \widetilde{U}_i \end{bmatrix},$$

and

$$\begin{bmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \\ \vdots \\ \widetilde{U}_{k-1} \end{bmatrix} \text{ is a surjective matrix.}$$

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Next, define for $i, j \in \underline{k-1}$

$$S_i W_{i\,j} T_j := \begin{bmatrix} \widetilde{W}_{i\,j} & \widehat{W}_{i\,j} \\ \overline{W}_{i\,j} & W'_{i\,j} \end{bmatrix}.$$

Furthermore, for all $i, j \in \underline{k-1}$, define

$$\Lambda'_i := \left[\begin{array}{c} \widetilde{U}_1 \\ \overline{U}_1 \\ \hline \widetilde{U}_2 \\ \overline{U}_2 \\ \hline \vdots \\ \hline \widetilde{U}_i \\ \overline{U}_i \end{array}\right], \quad \Omega'_{ij} := \left[\begin{array}{cc} \widetilde{W}_{1j} & \widehat{W}_{1j} \\ \overline{W}_{1j} & W'_{1j} \\ \hline \widetilde{W}_{2j} & \widehat{W}_{2j} \\ \overline{W}_{2j} & W'_{2j} \\ \hline \vdots & \vdots \\ \hline \widetilde{W}_{ij} & \widehat{W}_{ij} \\ \overline{W}_{ij} & W'_{ij} \end{array}\right],$$

$$\Delta'_j := \left[\begin{array}{cc|cc|c|cc} \widetilde{V}_1 & \hat{V}_1 & \widetilde{V}_2 & \hat{V}_2 & \cdots & \widetilde{V}_j & \hat{V}_j \end{array}\right],$$

and

$$\Gamma'_{ij} := \left[\begin{array}{cc|cc|c|cc} \widetilde{W}_{11} & \widehat{W}_{11} & \widetilde{W}_{12} & \widehat{W}_{12} & \cdots & \widetilde{W}_{1j} & \widehat{W}_{1j} \\ \overline{W}_{11} & W'_{11} & \overline{W}_{12} & W'_{12} & \cdots & \overline{W}_{1j} & W'_{1j} \\ \hline \widetilde{W}_{21} & \widehat{W}_{21} & \widetilde{W}_{22} & \widehat{W}_{22} & \cdots & \widetilde{W}_{2j} & \widehat{W}_{2j} \\ \overline{W}_{21} & W'_{21} & \overline{W}_{22} & W'_{22} & \cdots & \overline{W}_{2j} & W'_{2j} \\ \hline \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \hline \widetilde{W}_{i1} & \widehat{W}_{i1} & \widetilde{W}_{i2} & \widehat{W}_{i2} & \cdots & \widetilde{W}_{ij} & \widehat{W}_{ij} \\ \overline{W}_{i1} & W'_{i1} & \overline{W}_{i2} & W'_{i2} & \cdots & \overline{W}_{ij} & W'_{ij} \end{array}\right].$$

Note that for all $i, j \in \underline{k-1}$

$$\Delta'_j = [\Delta'_{j-1} | \widetilde{V}_j, \hat{V}_j] \quad \text{and} \quad \Gamma'_{ij} = [\Gamma'_{i\,j-1} | \Omega'_{ij}].$$

CLAIM 2.4. *Under the conditions of statement 3, we have for all $i, j \in \underline{k-1}$ with $i + j \leqslant k$, that*

$$\widehat{W}_{ij} = \sum_{l=1}^{j-1} \widetilde{W}_{il} \widetilde{T}_{lj}, \quad \overline{W}_{ij} = \sum_{l=1}^{i-1} \widetilde{S}_{il} \widetilde{W}_{lj}, \quad W'_{ij} = \sum_{l=1}^{i-1} \sum_{t=1}^{j-1} \widetilde{S}_{il} \widetilde{W}_{lt} \widetilde{T}_{tj}.$$

Proof. Let $j \in \underline{k-1}$ be fixed and recall that $\ker \Delta_j \subseteq \ker \Gamma_{k-j\,j}$. Because the matrices $S_1, S_2, \ldots, S_{k-1}$ and $T_1, T_2, \ldots, T_{k-1}$ are square and invertible, it follows

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that ker $\Delta'_j \subseteq$ ker $\Gamma'_{k-j\,j}$. Now recall that

$$\Delta'_j = [\Delta'_{j-1}|\widetilde{V}_j, \widehat{V}_j] \text{ with } \widehat{V}_j - \sum_{l=1}^{j-1} \widetilde{V}_l \widetilde{T}_{lj} = 0$$

and

$$\Gamma'_{k-jj} = [\Gamma'_{k-j\,j-1}|\Omega'_{k-j\,j}].$$

Next define

$$\Omega''_{k-j\,j} := \left[\begin{array}{cc} \widetilde{W}_{1j} & \widehat{Z}_{1j} \\ \overline{W}_{1j} & Z'_{1j} \\ \hline \widetilde{W}_{2j} & \widehat{Z}_{2j} \\ \overline{W}_{2j} & Z'_{2j} \\ \hline \vdots & \vdots \\ \hline \widetilde{W}_{k-j\,j} & \widehat{Z}_{k-j\,j} \\ \overline{W}_{k-j\,j} & Z'_{k-j\,j} \end{array}\right],$$

where for all $i \in \underline{k-j}$

$$\widehat{Z}_{ij} = \widehat{W}_{ij} - \sum_{l=1}^{j-1} \widetilde{W}_{il}\widetilde{T}_{lj} \quad \text{and} \quad Z'_{ij} = W'_{ij} - \sum_{l=1}^{j-1} \overline{W}_{il}\widetilde{T}_{lj}.$$

It follows that (see Lemma 2.2:1)

$$(\ker \Delta'_j =) \ker [\Delta'_{j-1}, \widetilde{V}_j, \widehat{V}_j] \subseteq \ker [\Gamma'_{k-j\,j-1}, \Omega'_{k-j\,j}] (= \ker \Gamma'_{k-j\,j})$$

if and only if

$$\ker [\Delta'_j, \widetilde{V}_{j-1}, 0] \subseteq \ker [\Gamma'_{k-j\,j-1}, \Omega''_{k-j\,j}].$$

From Lemma 2.2:2, it is clear that $\widehat{Z}_{ij} = 0$ and $Z'_{ij} = 0$ for all $i \in \underline{k-j}$. Hence, for all $i \in \underline{k-j}$

$$\widehat{W}_{ij} = \sum_{l=1}^{j-1} \widetilde{W}_{il}\widetilde{T}_{lj} \quad \text{and} \quad W'_{ij} = \sum_{l=1}^{j-1} \overline{W}_{il}\widetilde{T}_{lj}.$$

By a dual reasoning we can prove that for all $i \in \underline{k-1}$ and $j \in \underline{k-i}$,

$$\overline{W}_{ij} = \sum_{l=1}^{i-1} \widetilde{S}_{il}\widetilde{W}_{lj}.$$

By combining the latter results, the proof of the claim can be completed. ☐

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Now define

$$U := \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 \\ \vdots \\ \widetilde{U}_{k-1} \end{bmatrix}, \quad V := \begin{bmatrix} \widetilde{V}_1 & \widetilde{V}_2 & \cdots & \widetilde{V}_{k-1} \end{bmatrix},$$

and

$$W := \begin{bmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} & \cdots & \widetilde{W}_{1\,k-1} \\ \widetilde{W}_{21} & \widetilde{W}_{22} & \cdots & \widetilde{W}_{2\,k-1} \\ \vdots & \vdots & & \vdots \\ \widetilde{W}_{k-1\,1} & \widetilde{W}_{k-1\,2} & \cdots & \widetilde{W}_{k-1\,k-1} \end{bmatrix}.$$

Recall that U is a surjective matrix and V is an injective matrix. Therefore, there exists a matrix X such that $UXV = W$ (see Lemma 2.1:1). We claim that the matrix X satisfies the following:

Claim 2.5.

$$U_i X V_j = W_{i\,j} \quad \textit{for all } i, j \in \underline{k-1} \quad \textit{with} \quad i + j \leqslant k.$$

Proof. Let $i, j \in \underline{k-1}$ be such that $i + j \leqslant k$. With the invertible matrices S_i and T_j introduced before, note that

$$S_i U_i X V_j T_j = \begin{bmatrix} \widetilde{U}_i \\ \overline{U}_i \end{bmatrix} X \begin{bmatrix} \widetilde{V}_j & \hat{V}_j \end{bmatrix} =$$

$$\begin{bmatrix} \widetilde{U}_i \\ \sum\limits_{l=1}^{i-1} \widetilde{S}_{il} \widetilde{U}_l \end{bmatrix} X \begin{bmatrix} \widetilde{V}_j & \sum\limits_{l=1}^{j-1} \widetilde{V}_l \widetilde{T}_{lj} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & I & 0 & \cdots & 0 \\ \widetilde{S}_{i1} & \widetilde{S}_{i2} & \cdots & \widetilde{S}_{i\,i-1} & 0 & 0 & \cdots & 0 \end{bmatrix} UXV \begin{bmatrix} 0 & \widetilde{T}_{1j} \\ 0 & \widetilde{T}_{2j} \\ \vdots & \vdots \\ 0 & \widetilde{T}_{j-1\,j} \\ I & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} =$$

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$$\begin{bmatrix} 0 & 0 & \cdots & 0 & I \\ \widetilde{S}_{i1} & \widetilde{S}_{i2} & \cdots & \widetilde{S}_{i\,i-1} & 0 \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} & \cdots & \widetilde{W}_{1\,j} \\ \widetilde{W}_{21} & \widetilde{W}_{22} & \cdots & \widetilde{W}_{2\,j} \\ \vdots & \vdots & & \vdots \\ \widetilde{W}_{i\,1} & \widetilde{W}_{i-1\,2} & \cdots & \widetilde{W}_{i\,j} \end{bmatrix} \begin{bmatrix} 0 & \widetilde{T}_{1j} \\ 0 & \widetilde{T}_{2j} \\ \vdots & \vdots \\ 0 & \widetilde{T}_{j-1\,j} \\ I & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \widetilde{W}_{i\,j} & \sum_{l=1}^{j-1} \widetilde{W}_{il}\widetilde{T}_{lj} \\ \sum_{l=1}^{i-1} \widetilde{S}_{il}\widetilde{W}_{lj} & \sum_{l=1}^{i-1}\sum_{t=1}^{j-1} \widetilde{S}_{il}\widetilde{W}_{lt}\widetilde{T}_{tj} \end{bmatrix} = \begin{bmatrix} \widetilde{W}_{i\,j} & \widehat{W}_{i\,j} \\ \overline{W}_{i\,j} & W'_{i\,j} \end{bmatrix} = S_iW_{i\,j}T_j.$$

Because S_i and T_j are invertible matrices, claim 2.5 is now immediate. ☐

In fact, we have proved statement 3 implies statement 1 and, consequently, we have completed the proof of the theorem. ☐

From the proof of Theorem 2.3 the following corollary is immediate.

COROLLARY 2.6. *Given the matrices in (1.1), (2.1) and (2.2), the following statements are equivalent.*

1. *There is a matrix* $X \in \mathcal{F}^{b\times c}$ *such that for all* $i \in \underline{k-1}$*:* $\Gamma_{i\,k-i} = \Lambda_i X \Delta_{k-i}$.
2. *For all* $i \in \underline{k-1}$*, there is a matrix* $X \in \mathcal{F}^{b\times c}$ *such that* $\Gamma_{i\,k-i} = \Lambda_i X \Delta_{k-i}$.

3. Application. To present an application of the previous result we consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + \Sigma_{i\in\underline{\mu}} G_i v_i(t), \tag{3.1}$$

$$y(t) = Cx(t), \tag{3.2}$$

$$z_i(t) = H_i x(t), \quad i \in \underline{\mu}, \tag{3.3}$$

where $\mu \in \mathbb{N}, \mu > 1$. In the above, $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^m$ the (control) input, $y(t) \in \mathbb{R}^p$ the (measurement) output, and A, B and C are matrices of suitable dimensions. Further, $v_i(t) \in \mathbb{R}^{q_i}$ denotes the i-th exogenous input and $z_i(t) \in \mathbb{R}^{r_i}$ the i-th exogenous output, where $i \in \underline{\mu}$. The matrices G_i and H_i, for $i \in \underline{\mu}$, are matrices of suitable dimensions.

We assume that the system (3.1–3.3) is controlled by means of a compensator

$$\dot{w}(t) = Kw(t) + Ly(t), \tag{3.4}$$

$$u(t) = Mx(t) + Ny(t), \tag{3.5}$$

where $w(t) \in \mathbb{R}^k$ denotes the state of the compensator, and K, L, M and N are matrices of suitable dimensions.

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The interconnection of the system (3.1–3.3) and compensator (3.4–3.5) yields a closed loop system with μ exogenous inputs and μ exogenous outputs, described by

$$\dot{x}_e(t) = A_e x_e(t) + \Sigma_{i\in\underline{\mu}} G_{i,e} v_i(t), \tag{3.6}$$

$$z_i(t) = H_{i,e} x_e(t), \quad i \in \underline{\mu}, \tag{3.7}$$

where

$$x_e(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \quad A_e = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix},$$

and

$$G_{i,e} = \begin{bmatrix} G_i \\ 0 \end{bmatrix}, \quad H_{i,e} = \begin{bmatrix} H_i & 0 \end{bmatrix}, \quad i \in \underline{\mu}.$$

Let $T(s)$ denote the transfer matrix of the closed loop system (3.6–3.7). Then $T(s)$ can be partitioned according to the dimensions of the exogenous inputs and outputs as $T(s) = (T_{i\,j}(s)),\ i, j \in \underline{\mu}$, where $T_{i\,j}(s) = H_{i,e}(sI - A_e)^{-1} G_{j,e}$ denotes the transfer matrix between the j-th exogenous input and the i-th exogenous output in the closed loop system (3.6–3.7).

We denote the transfer matrices in the open loop system (3.1–3.3) by

$$\begin{array}{rclcrcl} P(s) & = & C(sI - A)^{-1}B, & \qquad & M_j(s) & = & C(sI - A)^{-1}G_j, \\ L_j(s) & = & H_i(sI - A)^{-1}B, & & K_{i\,j}(s) & = & H_i(sI - A)^{-1}G_j, \end{array}$$

where $i, j \in \underline{\mu}$, and the transfer matrix of the compensator(3.4–3.5) by

$$F(s) = N + M(sI - K)^{-1}L.$$

An easy calculation shows that in the closed loop system (3.6–3.7)

$$T_{i\,j}(s) = K_{i\,j}(s) + L_i(s)X(s)M_j(s), \quad i, j \in \underline{\mu},$$

where $X(s) = (I - F(s)P(s))^{-1}F(s)$.

Note that the inverse in the latter expression exists as a rational matrix because $I - F(s)P(s)$ is a *bicausal* rational matrix (cf. [2]). A bicausal rational matrix is a proper rational matrix with a proper rational inverse.[1] A proper rational matrix is bicausal if and only if its determinant does not vanish at infinity. It is clear that here $X(s)$ is a proper rational matrix and that $F(s) = X(s)(I + P(s)X(s))^{-1}$.

[1] We call a matrix a (proper) rational if all its entries are in the set of (proper) rational functions, and similarly for vectors.

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In this section, we study the following problem, see [3] for a related problem, where $||\cdot||_\infty$ denotes the H_∞ norm for stable (proper) rational matrices.

DEFINITION 3.1. The *almost triangular decoupling problem by measurement feedback*, denoted ATDPM$_\mu$, for system (3.1–3.3) consists of finding, for all $\varepsilon > 0$, a measurement feedback compensator (3.4–3.5) such that in the closed loop system (3.6–3.7) there holds $||T_{ij}(s)||_\infty \leqslant \varepsilon$ for all $i, j \in \underline{\mu}$ with $i < j$.

The following corollary is immediate.

COROLLARY 3.2. *The ATDPM$_\mu$ is solvable if and only if for all $\varepsilon > 0$ there exists a proper rational matrix $X(s)$ such that $||K_{ij}(s) + L_i(s)X(s)M_j(s)||_\infty \leqslant \varepsilon$ for all $i, j \in \underline{\mu}$ with $i < j$.*

Hence, the ATDPM$_\mu$ is solvable if $K_{ij}(s) + L_i(s)X(s)M_j(s) = 0$ for all $i, j \in \underline{\mu}$ with $i < j$ are approximately solvable over the proper rational matrices.

From [4] it is known that solvability of a linear rational matrix equation over the *rational* matrices is equivalent to approximate solvability of the same equation over the *proper rational* matrices, see also [6] or [7]. Therefore, the following corollary is obvious.

COROLLARY 3.3. *The ATDPM$_\mu$ is solvable if and only if there exists a rational matrix $X(s)$ such that $K_{ij}(s) + L_i(s)X(s)M_j(s) = 0$ for all $i, j \in \underline{\mu}$ with $i < j$.*

Taking $\mathcal{F}$ equal to the field of rational functions, Theorem 2.3 can be used to express the solvability of the ATDPM$_\mu$ in terms of easily verifiable conditions using transfer matrices, or, as it turns out, certain specific subspaces in state space.

To see how these conditions look like, we introduce some terminology and subspaces, see [1], [4], [5] and [6], formulated for a system given by $\dot{x} = Ax + Bu$.

- We say that x_0 has a (ξ, ω)-representation if there are rational vectors $\xi(s)$ and $\omega(s)$ of appropriate sizes such that $x_0 = (sI - A)\xi(s) - B\omega(s)$.
- In this section, we will use the following largest almost controlled invariant subspace related to ker H, defined as $\mathcal{V}_b^*(\ker H; A, B) := \{x_0 \in \mathbb{R}^n | x_0$ has a (ξ, ω)-representation such that $H\xi(s) = 0\}$.
- Also we will use the following smallest almost conditioned invariant subspace related to im G, defined as $\mathcal{S}_b^*(\operatorname{im} G; A, C) := (\mathcal{V}_b^*(\ker G^\top; A^\top, C^\top))^\perp$, where $^\perp$ means the orthogonal complement.
- We recall that both subspaces can be computed by means of recursive algorithms only requiring a finite number of iterations.

Using the notation

$$K(s) = H(sI - A)^{-1}G, \quad L(s) = H(sI - A)^{-1}B, \quad M(s) = C(sI - A)^{-1}G.$$

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we recall the following important theorem from [5].

THEOREM 3.4. *There is a rational matrix $X(s)$ such that $K(s)+L(s)X(s)M(s) = 0$ if and only if im $G \subseteq \mathcal{V}_b^*(\text{ker } H; A, B)$ and $\mathcal{S}_b^*(\text{im } G; A, C) \subseteq \text{ker } H$.*

The theorem is crucial in the translation of the solvability conditions in terms of transfer matrices into certain subspace inclusions in state space terms.

To present the solvability conditions for the ATDPM$_\mu$, we define for $i \in \underline{\mu - 1}$

$$\check{\Delta}_i(s) := \begin{bmatrix} L_1(s) \\ L_2(s) \\ \vdots \\ L_i(s) \end{bmatrix}, \quad \check{\Lambda}_i(s) := \begin{bmatrix} M_{i+1}(s) & M_{i+2}(s) & \cdots & M_\mu(s) \end{bmatrix},$$

$$\check{\Gamma}_i(s) := \begin{bmatrix} K_{1\,i+1}(s) & K_{1\,i+2}(s) & \cdots & K_{1\,\mu}(s) \\ K_{2\,i+1}(s) & K_{2\,i+2}(s) & \cdots & K_{1\,\mu}(s) \\ \vdots & \vdots & & \vdots \\ K_{i\;i+1}(s) & K_{i\,i+2}(s) & \cdots & K_{i\,\mu}(s) \end{bmatrix}.$$

Note that for $i \in \underline{\mu - 1}$

$$\check{\Delta}_i(s) = \check{H}_i(sI - A)^{-1}B, \quad \check{\Lambda}_i(s) = C(sI - A)^{-1}\check{H}_i, \quad \check{\Gamma}_i(s) = \check{H}_i(sI - A)^{-1}\check{G}_i,$$

where

$$\check{H}_i = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_i \end{bmatrix}, \quad \check{G}_i = \begin{bmatrix} G_{i+1} & G_{i+2} & \cdots & G_\mu \end{bmatrix}.$$

With this notation it follows from Corollary 3.3 that the ATDPM$_\mu$ is solvable if and only if there exists a rational matrix $X(s)$ such that $\check{\Gamma}_i(s) + \check{\Delta}_i(s)X(s)\check{\Lambda}_i(s) = 0$ for all $i \in \underline{\mu - 1}$.

From Corollary 2.6 (and some renumbering) it follows that that the ATDPM$_\mu$ is solvable if and only if for all $i \in \underline{\mu - 1}$ there exists a rational matrix $X(s)$ such that $\check{\Gamma}_i(s) + \check{\Delta}_i(s)X(s)\check{\Lambda}_i(s) = 0$.

By Theorem 3.4, the latter can be rewritten in state space terms as follows.

THEOREM 3.5. *The ATDPM$_\mu$ is solvable if and only if for all $i \in \underline{\mu - 1}$ there holds im $\check{G}_i \subseteq \mathcal{V}_b^*(\text{ker } \check{H}_i; A, B)$ and $\mathcal{S}_b^*(\text{im } \check{G}_i; A, C) \subseteq$ ker $\check{H}_i$.*

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Hence, the solvability of the ATDPM$_\mu$ can be checked by means of state space inclusions requiring subspaces that computed iteratively by means a finite number of iterations.

4. Concluding remarks. In this paper, we have derived necessary and sufficient conditions for the existence of a common solution of the set of linear matrix equations $U_i X V_j = W_{i\,j}$, where $1 \leqslant i, j < k$ with $i + j \leqslant k$. The conditions are easily verifiable and are formulated in terms of images and kernels of (matrices of) the matrices themselves. We have illustrated the solvability conditions by means of a version of the almost triangular decoupling problem and obtained, starting from a transfer matrix (rational matrix) description, solvability conditions in state space terms that would not be easily derived directly using a state space setting only.

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