

NORM RETRIEVABLE FRAMES IN $\mathbb{R}^{N*}$

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Abstract. This paper is concerned with norm retrievable frames in $\mathbb{R}^n$. Some equivalent conditions for the norm retrievable property in $\mathbb{R}^n$ are presented. It is also shown that the property of norm retrievability is stable under small enough perturbations of the frame set only for phase retrievable frames.

Key words. Frame, Phase retrievable frame, Norm retrievable frame.

AMS subject classifications. 32C15, 42C15.

1. Introduction. The concept of frames in a Hilbert space was originally introduced by Duffin and Schaeffer in the context of non-harmonic Fourier series [11]. Frames are redundant sets of vectors in a Hilbert space, which yield one natural representation of each vector in the space, but may have infinitely many different representations for any given vector. It is this redundancy that makes frames useful in applications. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. From the last decade, various generalizations of the frames have been proposed such as frame of subspaces, pseudo-frames, oblique frames, continuous frames, fusion frames, g-frames and so on. The concept of equal norm Parseval frames on finite dimensional Hilbert spaces was first introduced by Casazza and Leonhard in [8] and it has been developed very fast over the last ten years, especially in the context of wavelets and Gabor systems.

A sequence $\{f_k\}_{k=1}^m$ is called a frame for n -dimensional Hilbert space $\mathcal{H}_n$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}_n$,

$$A\|f\|^2 \leq \sum_{k=1}^m |\langle f, f_k \rangle|^2 \leq B\|f\|^2.$$

The constant A is called a lower frame bound and B is called an upper frame bound. If only an upper bound B exists, then $\{f_k\}_{k=1}^m$ is called a B -Bessel sequence or simply Bessel when the constant is implicit. If $A = B$, $\{f_k\}_{k=1}^m$ is called a tight frame and if $A = B = 1$, it is called a Parseval frame. The values $\{\langle f, f_k \rangle\}_{k=1}^m$ are the frame coef-

*Received by the editors on March 26, 2016. Accepted for publication on June 1, 2016. Handling Editor: Froilan Dopico.

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ficients of the vector $f \in \mathcal{H}_n$ with respect to the frame $\{f_k\}_{k=1}^m$. The frame operator is the bounded linear operator $S : \mathcal{H}_n \rightarrow \mathcal{H}_n$ defined by $Sf = \sum_{k=1}^m \langle f, f_k \rangle f_k$. The frame operator S is known to be positive, self-adjoint and invertible operator.

Two frames $\{f_k\}_{k=1}^m$ and $\{g_k\}_{k=1}^m$ are dual frames for $\mathcal{H}_n$ if

$$f = \sum_{k=1}^m \langle f, f_k \rangle g_k, \quad \forall f \in \mathcal{H}_n.$$

The frame $\{\tilde{f}_k\}_{k=1}^m$ defined by $\tilde{f}_k = S^{-1}f_k$ is a dual frame of frame $\{f_k\}_{k=1}^m$ that is called canonical dual frame of $\{f_k\}_{k=1}^m$. For more information concerning frames refer to [7, 10, 13, 15].

Signal reconstruction by frames has been an important and interesting problem in physics and engineering. Some classes of tight frames for a Hilbert space which allow signal reconstruction from the absolute value of the frame coefficients were constructed in [4]. As a consequence, signal reconstruction can be done without using phase. The process of reconstruction a signal without the use of phase is known as phase retrieval. When we say that $\{f_k\}_{k=1}^m$ is a phase retrievable frame for $\mathcal{H}_n$, it means that any vector in $\mathcal{H}_n$ can be reconstructed up to a constant phase factor from the modulus of its frame coefficients. Phase retrievable frames have important applications in X-ray crystallography, electron microscopy, speech recognition and many other areas [5, 6, 12, 14].

The concept of norm retrievable frame is introduced in [1]. The authors showed that this is what is necessary for passing phase retrieval to complements. They also obtained the relationship between phase retrievable frames and norm retrievable frames.

In this article, we present some important properties of norm retrievable frames in $\mathbb{R}^n$. We obtain equivalent conditions to norm retrievable frames in $\mathbb{R}^n$. Also we study the stability of norm retrievable and phase retrievable frames under small perturbation of the frame set.

2. Main results. Phase retrievable frames have many applications in physics and engineering. The mathematical study of phase retrievable frames was started in 2006 in a landmark paper of Balan, Casazza and Edidin [4]. Also in [9], the authors highlighted recently the major advances in phase retrievable frames.

DEFINITION 2.1. A frame $\mathcal{F} = \{f_k\}_{k=1}^m$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is called phase retrievable frame if for all $f, g \in \mathbb{R}^n$ ($f, g \in \mathbb{C}^n$) satisfying $|\langle f, f_k \rangle| = |\langle g, f_k \rangle|$ for all $k = 1, 2, \dots, m$, it holds that $f = cg$ for some $c \in \{-1, 1\}$ ($c \in \{z \in \mathbb{C} : |z| = 1\}$).

Bahmanpour, Cahill, Casazza, Jasper and Woodland [1] defined norm retrievable

frames:

DEFINITION 2.2. A frame $\mathcal{F} = \{f_k\}_{k=1}^m$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is called norm retrievable frame if for all $f, g \in \mathbb{R}^n$ ($f, g \in \mathbb{C}^n$) satisfying $|\langle f, f_k \rangle| = |\langle g, f_k \rangle|$ for all $k = 1, 2, \dots, m$, it holds that $\|f\| = \|g\|$.

They showed that a frame $\mathcal{F} = \{f_k\}_{k=1}^m$ is phase retrievable frame if and only if $\mathcal{F} = \{Tf_k\}_{k=1}^m$ is norm retrievable frame for every invertible operator T .

In this section, we will concentrate on norm retrievable frames in $\mathbb{R}^n$. In the next theorem, using the span of the frame elements we give a characterization of norm retrievable frames in $\mathbb{R}^n$. The related theorem for phase retrievable frames can be found in [4, Theorem 2.8].

THEOREM 2.3. A frame $\mathcal{F} = \{f_k\}_{k=1}^m \subset \mathbb{R}^n$ is norm retrievable frame if and only if for any partition $\{I_j\}_{j=1}^2$ of $[m] := \{1, 2, \dots, m\}$, $\text{Span}\{f_k\}_{k \in I_1}^\perp \perp \text{Span}\{f_k\}_{k \in I_2}^\perp$.

Proof. Let $\text{Span}\{f_k\}_{k \in I_1}^\perp \perp \text{Span}\{f_k\}_{k \in I_2}^\perp$, for any partition $\{I_j\}_{j=1}^2$ of $[m]$ and $|\langle f, f_k \rangle| = |\langle g, f_k \rangle|$ for all $k = 1, 2, \dots, m$. Then $\langle f, f_k \rangle = \mp \langle g, f_k \rangle$ for all $k \in [m]$.

Let $I_1 = \{k \in [m] : \langle f, f_k \rangle = -\langle g, f_k \rangle\}$ and $I_2 = [m] \setminus I_1$. Thus, $f + g \in \text{Span}\{f_k\}_{k \in I_1}^\perp$ and $f - g \in \text{Span}\{f_k\}_{k \in I_2}^\perp$. Now by assumption, we have

$$\begin{aligned} 0 &= \langle f + g, f - g \rangle \\ &= \|f\|^2 - \|g\|^2 - \langle f, g \rangle + \langle g, f \rangle \\ &= \|f\|^2 - \|g\|^2, \end{aligned}$$

and hence, $\|f\| = \|g\|$. Note in the case that $I_1 = \emptyset$ or $I_2 = \emptyset$, we have $f = \mp g$ and hence $\|f\| = \|g\|$.

Conversely, let $\mathcal{F} = \{f_k\}_{k=1}^m \subset \mathbb{R}^n$ be norm retrievable frame and let $\{I_j\}_{j=1}^2$ be a partition of $[m]$. Now for any $f \in \text{Span}\{f_k\}_{k \in I_1}^\perp$ and $g \in \text{Span}\{f_k\}_{k \in I_2}^\perp$ we have $\langle f, f_k \rangle = 0$ for all $k \in I_1$ and $\langle g, f_k \rangle = 0$ for all $k \in I_2$. Thus, $\langle f + g, f_k \rangle = -\langle f - g, f_k \rangle$ for all $k \in I_1$, and $\langle f + g, f_k \rangle = \langle f - g, f_k \rangle$ for all $k \in I_2$.

Now $|\langle f + g, f_k \rangle| = |\langle f - g, f_k \rangle|$ for all $k \in [m]$, and hence, $\|f + g\| = \|f - g\|$ by the norm retrievability of $\{f_k\}_{k=1}^m$. Thus,

$$\begin{aligned} \|f\|^2 + \|g\|^2 + 2\langle f, g \rangle &= \|f + g\|^2 \\ &= \|f - g\|^2 \\ &= \|f\|^2 + \|g\|^2 - 2\langle f, g \rangle. \end{aligned}$$

This implies that $\langle f, g \rangle = 0$, and hence, $\text{Span}\{f_k\}_{k \in I_1}^\perp \perp \text{Span}\{f_k\}_{k \in I_2}^\perp$. $\square$

Two equivalent conditions for norm retrievable frames are given in the next theorem. The related theorem for phase retrievable frames can be found in [3, Theorem

2.4].

THEOREM 2.4. *Let $\mathcal{F} = \{f_k\}_{k=1}^m$ be a frame for $\mathbb{R}^n$. Then the following are equivalent.*

- (1) $\{f_k\}_{k=1}^m$ is a norm retrievable frame.
- (2) For any non orthogonal vectors $f, g \in \mathbb{R}^n$, there exist $k \in [m]$ such that $\langle f, f_k \rangle \langle g, f_k \rangle \neq 0$.
- (3) For all nonzero real numbers $\epsilon \neq 0$, there is a positive real number $C_0 > 0$ so that for all $f, g \in \mathbb{R}^n$, we have

$$(2.1) \quad R_\epsilon(f, g) := |\langle f, g \rangle - \epsilon| \|f\| \|g\|^2 + \sum_{k=1}^m |\langle f, f_k \rangle|^2 |\langle g, f_k \rangle|^2 \geq C_0 \|f\|^2 \|g\|^2.$$

Proof. (1) $\Rightarrow$ (2). If there exist non orthogonal vectors $f, g \in \mathbb{R}^n$ such that $\langle f, f_k \rangle \langle g, f_k \rangle = 0$ for all $k \in [m]$, then $f \in \text{Span}\{f_k\}_{k \in I_1}^\perp$ and $g \in \text{Span}\{f_k\}_{k \in I_2}^\perp$, where $I_1 := \{k \in [m] : \langle f, f_k \rangle = 0\}$ and $I_2 := [m] \setminus I_1$ but $\langle f, g \rangle \neq 0$. Thus, $\{f_k\}_{k=1}^m$ is not a norm retrievable frame by Theorem 2.3.

(2) $\Rightarrow$ (3), Let $\epsilon \neq 0$ be given. The map F_ϵ defined by $F_\epsilon(f, g) := |\langle f, g \rangle - \epsilon|^2 + \sum_{k=1}^m |\langle f, f_k \rangle|^2 |\langle g, f_k \rangle|^2$, for all $f, g \in \mathbb{R}^n$ is a continuous map on $\mathbb{R}^n \times \mathbb{R}^n$.

Let $C_0 := \min_{(f, g) \in N \times N} F_\epsilon(f, g)$, where N is the compact set $N := \{f \in \mathbb{R}^n : \|f\| = 1\}$. Now $C_0 > 0$ by (2). Thus, for all nonzero vectors $f, g \in \mathbb{R}^n$, we have

$$R_\epsilon(f, g) = \|f\|^2 \|g\|^2 F_\epsilon\left(\frac{f}{\|f\|}, \frac{g}{\|g\|}\right) \geq C_0 \|f\|^2 \|g\|^2.$$

If either $f = 0$ or $g = 0$, then 2.1 holds true.

(3) $\Rightarrow$ (1). If $\{f_k\}_{k=1}^m$ is not a norm retrievable frame, then there exist partition $\{I_j\}_{j=1}^2$ of $[m]$ and nonzero vectors $f \in \text{Span}\{f_k\}_{k \in I_1}^\perp$, $g \in \text{Span}\{f_k\}_{k \in I_2}^\perp$ such that $\langle f, g \rangle \neq 0$. Now for nonzero real number $\epsilon := \frac{\langle f, g \rangle}{\|f\| \|g\|}$ we have $R_\epsilon(f, g) = 0$. Thus, $C_0 = 0$ by 2.1 and this is a contradiction. $\square$

The stability of the phase retrieval property under perturbations of the frame set is given in the next theorem. This result was proved in [2] for the complex case.

THEOREM 2.5. *If $\{f_k\}_{k=1}^m$ is a phase retrievable frame for $\mathbb{R}^n$, then there exists a $\lambda > 0$ such that any set $\{f'_k\}_{k=1}^m$ satisfying $\max_{1 \leq k \leq m} \|f_k - f'_k\| < \lambda$ is also a phase retrievable frame for $\mathbb{R}^n$.*

Proof. Since $\{f_k\}_{k=1}^m$ is a phase retrievable frame for $\mathbb{R}^n$, there is a constant $C_0 > 0$ such that $R(f, g) \geq C_0 \|f\|^2 \|g\|^2$, for all $f, g \in \mathbb{R}^n$ by [3, Theorem 2.4], where

$R(f, g) := \sum_{k=1}^m |\langle f, f_k \rangle|^2 |\langle g, f_k \rangle|^2$. Let

$$\lambda < \min \left\{ \frac{A}{2\sqrt{mB}}, \frac{C_0}{\frac{A^2+6B^2}{2B} \left(\sqrt{B} + \sqrt{\frac{A^2+4B^2}{2B}} \right)} \right\},$$

where A, B are respectively the lower and upper frame bounds of frame $\{f_k\}_{k=1}^m$. Now we show that the sequence $\{f'_k\}_{k=1}^m$ satisfying $\max_{1 \leq k \leq m} \|f_k - f'_k\| < \lambda$ is also a phase retrievable frame for $\mathbb{R}^n$. It is known that each frame set is stable under small perturbations by [10, Corollary 15.1.5]. However for all $f \in \mathbb{R}^n$ we have

$$\begin{aligned} \sum_{k=1}^m |\langle f, f'_k \rangle|^2 &= \sum_{k=1}^m |\langle f, f_k \rangle - \langle f, f_k - f'_k \rangle|^2 \\ &\geq \sum_{k=1}^m |\langle f, f_k \rangle|^2 - 2 \sum_{k=1}^m |\langle f, f_k \rangle| |\langle f, f_k - f'_k \rangle| \\ &\geq \sum_{k=1}^m |\langle f, f_k \rangle|^2 - 2 \left(\sum_{k=1}^m |\langle f, f_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^m |\langle f, f_k - f'_k \rangle|^2 \right)^{\frac{1}{2}} \\ &\geq \sum_{k=1}^m |\langle f, f_k \rangle|^2 - 2 \left(\sum_{k=1}^m |\langle f, f_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^m \|f_k - f'_k\|^2 \|f\|^2 \right)^{\frac{1}{2}} \\ &\geq A\|f\|^2 - 2\lambda\sqrt{m}\|f\| \left(\sum_{k=1}^m |\langle f, f_k \rangle|^2 \right)^{\frac{1}{2}} \\ &\geq (A - 2\lambda\sqrt{mB})\|f\|^2, \end{aligned}$$

and hence, $\{f'_k\}_{k=1}^m$ is also a frame. Also for all $f \in \mathbb{R}^n$ we have

$$\begin{aligned} \sum_{k=1}^m |\langle f, f'_k \rangle|^2 &= \sum_{k=1}^m |\langle f, f_k \rangle + \langle f, f'_k - f_k \rangle|^2 \\ &\leq 2 \left(\sum_{k=1}^m |\langle f, f_k \rangle|^2 + \sum_{k=1}^m |\langle f, f'_k - f_k \rangle|^2 \right) \\ &\leq 2(B + m\lambda^2)\|f\|^2 \\ &\leq \frac{A^2 + 4B^2}{2B}\|f\|^2, \end{aligned}$$

and hence, $B' \leq \frac{A^2+4B^2}{2B}$, where $B' = \sup_{\|f\|=1} \sum_{k=1}^m |\langle f, f'_k \rangle|^2$ is the optimal upper frame bound of frame $\{f'_k\}_{k=1}^m$.

Let $R'(f, g) := \sum_{k=1}^m |\langle f, f'_k \rangle|^2 |\langle g, f'_k \rangle|^2$, for all $f, g \in \mathbb{R}^n$. Thus, for any $f, g \in$

$\mathbb{R}^n$, we have

$$\begin{aligned} |R(f, g) - R'(f, g)| &= \left| \sum_{k=1}^m |\langle f, f_k \rangle|^2 |\langle g, f_k \rangle|^2 - \sum_{k=1}^m |\langle f, f'_k \rangle|^2 |\langle g, f'_k \rangle|^2 \right| \\ &\leq \sum_{k=1}^m \left| |\langle f, f_k \rangle|^2 |\langle g, f_k \rangle|^2 - |\langle f, f'_k \rangle|^2 |\langle g, f'_k \rangle|^2 \right| \\ &= \sum_{k=1}^m \left| |\langle f, f_k \rangle| |\langle g, f_k \rangle| + |\langle f, f'_k \rangle| |\langle g, f'_k \rangle| \right. \\ &\quad \left. \times \left| |\langle f, f_k \rangle| |\langle g, f_k \rangle| - |\langle f, f'_k \rangle| |\langle g, f'_k \rangle| \right| \right|. \end{aligned}$$

Now

$$\begin{aligned} \left| |\langle f, f_k \rangle| |\langle g, f_k \rangle| - |\langle f, f'_k \rangle| |\langle g, f'_k \rangle| \right| &\leq |\langle f, f_k \rangle \langle g, f_k \rangle - \langle f, f'_k \rangle \langle g, f'_k \rangle| \\ &= |\langle f, f_k \rangle \langle g, f_k - f'_k \rangle + \langle f, f_k - f'_k \rangle \langle g, f'_k \rangle| \\ &\leq \|f\| \|g\| \|f_k - f'_k\| + \|f\| \|g\| \|f'_k - f_k\|. \end{aligned}$$

Hence, for any $f, g \in \mathbb{R}^n$, we have

$$\begin{aligned} |R(f, g) - R'(f, g)| &\leq \lambda(\sqrt{B} + \sqrt{B'}) \|f\| \|g\| \left(\sum_{k=1}^m |\langle f, f_k \rangle \langle g, f_k \rangle| + \sum_{k=1}^m |\langle f, f'_k \rangle \langle g, f'_k \rangle| \right) \\ &\leq \lambda(\sqrt{B} + \sqrt{B'}) \|f\| \|g\| \left[\left(\sum_{k=1}^m |\langle f, f_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^m |\langle g, f_k \rangle|^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\sum_{k=1}^m |\langle f, f'_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^m |\langle g, f'_k \rangle|^2 \right)^{\frac{1}{2}} \right] \\ &\leq \lambda(\sqrt{B} + \sqrt{B'}) (B + B') \|f\|^2 \|g\|^2. \end{aligned}$$

Therefore, for any $f, g \in \mathbb{R}^n$, we have

$$\begin{aligned} R'(f, g) &\geq R(f, g) - \lambda(\sqrt{B} + \sqrt{B'}) (B + B') \|f\|^2 \|g\|^2 \\ &\geq \left[C_o - \lambda(\sqrt{B} + \sqrt{B'}) (B + B') \right] \|f\|^2 \|g\|^2 \\ &\geq \left[C_o - \lambda \frac{A^2 + 6B^2}{2B} \left(\sqrt{B} + \sqrt{\frac{A^2 + 4B^2}{2B}} \right) \right] \|f\|^2 \|g\|^2 \end{aligned}$$

and $\{f'_k\}_{k=1}^m$ is a phase retrievable frame for $\mathbb{R}^n$ by [3, Theorem 2.4]. $\square$

The next theorem states that the norm retrievable property is not necessarily stable under small perturbations of the frame set.

THEOREM 2.6. *Suppose that $\{f_k\}_{k=1}^m$ is a norm retrievable frame for $\mathbb{R}^n$. If $\{f_k\}_{k=1}^m$ is not a phase retrievable frame for $\mathbb{R}^n$, then $\{f_k\}_{k=1}^m$ does not preserve the norm retrievable property under perturbations.*

Proof. If $\{f_k\}_{k=1}^m$ is not a phase retrievable frame for $\mathbb{R}^n$, then there exists a $I \subset [m]$ such that $\text{Span}\{f_k\}_{k \in I} \neq \mathbb{R}^n$ and $\text{Span}\{f_k\}_{k \in I^c} \neq \mathbb{R}^n$ by [4, Theorem 2.8]. Thus, there exist $0 \neq f \in \text{Span}\{f_k\}_{k \in I}^\perp$ and $0 \neq g \in \text{Span}\{f_k\}_{k \in I^c}^\perp$. Using the norm retrievability of $\{f_k\}_{k=1}^m$ we have $\langle f, g \rangle = 0$ by Theorem 2.3. Now let $\lambda > 0$ be given. We define the sequence $\{f'_k\}_{k=1}^m$ by

$$f'_k = \begin{cases} f_k - \frac{\lambda \langle f_k, g \rangle f}{2\sqrt{B}\|f\|\|g\|} & \text{if } k \in I \\ f_k & \text{if } k \in I^c \end{cases},$$

where B is the upper frame bound of $\{f_k\}_{k=1}^m$. Now $\max_{1 \leq k \leq m} \|f_k - f'_k\| < \lambda$.

If $\{f'_k\}_{k=1}^m$ is not a frame for $\mathbb{R}^n$, then $\{f'_k\}_{k=1}^m$ is not a norm retrievable frame for $\mathbb{R}^n$.

If $\{f'_k\}_{k=1}^m$ is a frame for $\mathbb{R}^n$, then we show that $\{f'_k\}_{k=1}^m$ is not a norm retrievable frame for $\mathbb{R}^n$. Indeed, It is easy to show that $\left\langle \frac{2\sqrt{B}\|g\|f}{\|f\|} + \lambda g, f'_k \right\rangle = 0$ for all $k \in I$ and $\langle g, f'_k \rangle = 0$ for all $k \in I^c$. Also we have

$$\left\langle \frac{2\sqrt{B}\|g\|f}{\|f\|} + \lambda g, g \right\rangle = \lambda \|g\|^2 > 0.$$

Thus, $\text{Span}\{f'_k\}_{k \in I}^\perp$ is not orthogonal to $\text{Span}\{f'_k\}_{k \in I^c}^\perp$, and hence, $\{f'_k\}_{k=1}^m$ is not a norm retrievable frame for $\mathbb{R}^n$ by Theorem 2.3. $\square$

Since every phase retrievable frame is also a norm retrievable frame, the following remark is immediate from Theorem 2.5 and Theorem 2.6.

REMARK 2.7. The norm retrievable property is stable under small perturbations of the frame set only for phase retrievable frames.

REFERENCES

- [1] S. Bahmanpour, J. Cahill, P.G. Casazza, J. Jasper, and L.M. Woodland. Phase retrieval and norm retrieval. Submitted.
- [2] R. Balan. Stability of phase retrievable frames. Preprint, arXiv:1308.5465, 2013.
- [3] R. Balan. Reconstruction of signals from magnitudes of redundant representations. Preprint, arXiv:1207.1134v1, 2012.
- [4] R. Balan, P.G. Casazza, and D. Edidin. On signal reconstruction without phase. *Applied and Computational Harmonic Analysis*, 20:345-356, 2006.
- [5] R.H. Bates and D. Mnyama. The status of practical Fourier phase retrieval. In: W.H. Hawkes (editor), *Advances in Electronics and Electron Physics*, 67:1-64, 1986.

- [6] C. Becchetti and L.P. Ricotti. *Speech Recognition Theory and C++ Implementation*. John Wiley & Sons Ltd, Chichester, England, 1999.
- [7] J.J. Benedetto. Frame decompositions, sampling, and uncertainty principle inequalities. In: J. Benedetto and M. Frazier (editors), *Wavelets: Mathematics and Applications*, CRC Press, Boca Raton, FL, 247–304, 1994.
- [8] P.G. Casazza and N. Leonhard. Classes of finite equal norm Parseval frames. *Contemporary Mathematics*, 451:11–32, 2008.
- [9] P.G. Casazza and L.M. Woodland. Phase retrieval by vectors and projections. *Contemporary Mathematics*, to appear.
- [10] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, Boston, 2002.
- [11] R.J. Duffin and A.C. Schaeffer. A class of nonharmonic Fourier series. *Transactions of the American Mathematical Society*, 72:341–366, 1952.
- [12] J.R. Fienup. Reconstruction of an object from the modulus of its Fourier transform. *Optics Letters*, 3:27–29, 1978.
- [13] C. Heil and D. Walnut. Continuous and discrete wavelet transform. *SIAM Review*, 31:628–666, 1969.
- [14] J.M. Renes, R. Blume-Kohout, A.J. Scott, and C.M. Caves. Symmetric informationally complete quantum measurements. *Journal of Mathematical Physics*, 45:2171–2180, 2004.
- [15] R. Young. *An Introduction to Nonharmonic Fourier Series*. Academic Press, New York, 1992.