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CHARACTERIZATIONS OF LINEAR MAPPINGS THROUGH ZERO PRODUCTS OR ZERO JORDAN PRODUCTS*

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Abstract. Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. A characterization of generalized derivations and generalized Jordan derivations from $\mathcal{A}$ into $\mathcal{M}$, through zero products or zero Jordan products, is given. Suppose that $\mathcal{M}$ is a unital left $\mathcal{A}$-module. It is investigated when a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ is a Jordan left derivation under certain conditions. It is also studied whether an algebra with a nontrivial idempotent is zero Jordan product determined, and Jordan homomorphisms, Lie homomorphisms and Lie derivations on zero Jordan product determined algebras are characterized.

Key words. Generalized derivation, Generalized Jordan derivation, Jordan left derivation, Zero Jordan product determined algebra.

AMS subject classifications. 15A86, 47A07, 47B47, 47B49.

1. Introduction. Throughout this paper, let $\mathcal{A}$ be a unital associative algebra over $\mathbb{F}$, where $\mathbb{F}$ is a field of characteristic not 2.

For each a, b in $\mathcal{A}$, we define the *Jordan product* by $a \circ b = ab + ba$ and the *Lie product* by $[a, b] = ab - ba$.

Suppose that $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *generalized derivation* if $\delta(ab) = \delta(a)b + a\delta(b) - a\delta(1)b$ for each a, b in $\mathcal{A}$, and δ is called a *generalized Jordan derivation* if $\delta(a\circ b) = \delta(a)\circ b + a\circ\delta(b) - a\delta(1)b - b\delta(1)a$ for each a, b in $\mathcal{A}$.

In [1, 2, 7, 8, 10, 11, 14, 15, 17], several authors consider the following conditions on a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$:

$(\mathbb{D}_1)$ $a, b \in \mathcal{A},\ ab = 0 \Rightarrow a\delta(b) + \delta(a)b = 0;$

$(\mathbb{D}_2)$ $a, b, c \in \mathcal{A},\ ab = bc = 0 \Rightarrow a\delta(b)c = 0;$

$(\mathbb{D}_3)$ $a, b \in \mathcal{A},\ ab = ba = 0 \Rightarrow a \circ \delta(b) + \delta(a) \circ b = 0;$

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and investigate whether these conditions characterize generalized derivations or generalized Jordan derivations.

We denote by $\mathfrak{L}(\mathcal{A})$ the linear span of all idempotents in $\mathcal{A}$, and by $\mathfrak{J}(\mathcal{A})$ the subalgebra of $\mathcal{A}$ generated algebraically by all idempotents in $\mathcal{A}$. $\mathcal{M}$ is said to have the *property* $\mathbb{P}_1$, if there is an ideal $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ of $\mathcal{A}$ such that

$$\{m \in \mathcal{M} : xm = mx = 0 \text{ for every } x \in \mathcal{J}\} = \{0\};$$

$\mathcal{M}$ is said to have the *property* $\mathbb{P}_2$, if there is an ideal $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ of $\mathcal{A}$ such that

$$\{m \in \mathcal{M} : xmy = 0 \text{ for each } x,\ y \in \mathcal{J}\} = \{0\};$$

$\mathcal{M}$ is said to have the *property* $\mathbb{P}_3$, if there is an ideal $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ of $\mathcal{A}$ such that

$$\{m \in \mathcal{M} : xmx = 0 \text{ for every } x \in \mathcal{J}\} = \{0\}.$$

It is clear that $\mathbb{P}_3 \Rightarrow \mathbb{P}_2 \Rightarrow \mathbb{P}_1$ and if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{M}$ has the properties $\mathbb{P}_1$-$\mathbb{P}_3$.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. By a *subspace lattice* on $\mathcal{H}$, we mean a collection $\mathcal{L}$ of closed subspaces of $\mathcal{H}$ with (0) and $\mathcal{H}$ in $\mathcal{L}$ such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. We disregard the distinction between a closed subspace and the orthogonal projection onto it.

Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$, define $\mathcal{P}_{\mathcal{L}} = \{E \in \mathcal{L} : E_- \not\supseteq E\}$, where $E_- = \vee\{F \in \mathcal{L} : F \not\supseteq E\}$ and let $E_+ = \cap\{F \in \mathcal{L} : F \not\subseteq E\}$. A subspace $\mathcal{L}$ is called a *completely distributive* if $L = \vee\{E \in \mathcal{L} : E_- \not\supseteq L\}$ for every $L \in \mathcal{L}$; $\mathcal{L}$ is called a $\mathcal{P}$*-subspace lattice* if $\vee\{E : E \in \mathcal{P}_{\mathcal{L}}\} = \mathcal{H}$ or $\cap\{E_- : E \in \mathcal{P}_{\mathcal{L}}\} = (0)$. For some properties of completely distributive subspace lattices and $\mathcal{P}$-subspace lattices, see [7, 13]. $\mathcal{L}$ is said to be a *commutative subspace lattice* if it consists of mutually commuting projections. A totally ordered subspace lattice $\mathcal{N}$ is called a *nest*.

We use Alg$\mathcal{L}$ to denote the algebra of all operators in $B(\mathcal{H})$ that leave members of $\mathcal{L}$ invariant.

Let $\mathcal{J}$ be an ideal of $\mathcal{A}$, we say that $\mathcal{J}$ is a *right separating set* (resp., *left separating set*) of $\mathcal{M}$ if for every m in $\mathcal{M}$, $\mathcal{J}m = \{0\}$ implies $m = 0$ (resp., $m\mathcal{J} = \{0\}$ implies $m = 0$). When $\mathcal{J}$ is a right separating set and a left separating set of $\mathcal{M}$, we call $\mathcal{J}$ a *separating set* of $\mathcal{M}$. By [7, 11], we know that if $\mathcal{A}$ and $\mathcal{M}$ satisfy one of the following conditions:

(1) $\mathcal{A} = \mathcal{B} \cap \text{Alg}\mathcal{N}$ and $\mathcal{M} = \mathcal{B}$, where $\mathcal{N}$ is a nest in a factor von Neumann algebra $\mathcal{B}$;

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(2) $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ with $(0)_+ \neq (0)$ and $\mathcal{H}_- \neq \mathcal{H}$, $\mathcal{M} = B(\mathcal{H})$;

(3) $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ with $\vee\{E : E \in \mathcal{P}_{\mathcal{L}}\} = \mathcal{H}$ and $\cap\{E_- : E \in \mathcal{P}_{\mathcal{L}}\} = (0)$, $\mathcal{M} = B(\mathcal{H})$,

then $\mathcal{M}$ has a separating set $\mathcal{J}$ with $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$, it is easy to show that $\mathcal{M}$ has the property $\mathbb{P}_2$. If $\mathcal{A}$ is a completely distributive commutative subspace lattice algebra and $\mathcal{M} = B(\mathcal{H})$, then $\mathcal{M}$ has the property $\mathbb{P}_3$.

This paper is organized as follows. In Section 2, we suppose that $\mathcal{M}$ has the property $\mathbb{P}_1$, $\mathbb{P}_2$ or $\mathbb{P}_3$ and characterize linear mappings that satisfy the condition $\mathbb{D}_1$, $\mathbb{D}_2$ or $\mathbb{D}_3$ through their action on zero products or zero Jordan products.

Let $\mathcal{M}$ be a left $\mathcal{A}$-module. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a *Jordan left derivation* if $\delta(a \circ b) = 2a\delta(b) + 2b\delta(a)$ for each a, b in $\mathcal{A}$. In [16], J. Li and J. Zhou show that if $\mathcal{M}$ has a right separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$, then every Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$ is zero.

The natural way to translate the conditions $\mathbb{D}_1$-$\mathbb{D}_3$ to the context of Jordan left derivations is to consider the following conditions on a linear mapping $\delta : \mathcal{A} \to \mathcal{M}$:

($\mathbb{J}_1$) $a, b \in \mathcal{A},\ ab = 0 \Rightarrow a\delta(b) + b\delta(a) = 0$;

($\mathbb{J}_2$) $a, b, c \in \mathcal{A},\ ab = bc = 0 \Rightarrow ac\delta(b) = 0$;

($\mathbb{J}_3$) $a, b \in \mathcal{A},\ ab = ba = 0 \Rightarrow a\delta(b) + b\delta(a) = 0$.

It is clear that $\mathbb{J}_1$ implies $\mathbb{J}_3$. In [16], J. Li and J. Zhou show that if $\mathcal{M}$ has a right separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$, then each linear mapping δ such that the condition $\mathbb{J}_1$ is zero under certain conditions.

In Section 3, we investigate whether the condition $\mathbb{J}_2$ or $\mathbb{J}_3$ characterizes Jordan left derivation and, as applications of Theorem 3.1, we generalize some results in [16].

In [6], M. Brešar et al. introduce the concepts of zero product determined algebras and zero Jordan product determined algebras, which can be used to study the linear mappings preserving zero product or zero Jordan product.

Let $\mathcal{X}$ be a linear space over $\mathbb{F}$. An algebra $\mathcal{A}$ is said to be *zero product determined* if every bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into any linear space $\mathcal{X}$ satisfying

$$\phi(a, b) = 0, \quad \text{whenever } ab = 0$$

can be written as $\phi(a, b) = T(ab)$, for some linear mapping T from $\mathcal{A}$ into $\mathcal{X}$.

Similarly, $\mathcal{A}$ is said to be *zero Jordan product determined* if every bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into any linear space $\mathcal{X}$ satisfying

$$a, b \in \mathcal{A},\ a \circ b = 0 \Rightarrow \phi(a, b) = 0$$

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can be written as $\phi(a,b) = T(a \circ b)$, for some linear mapping T from $\mathcal{A}$ into $\mathcal{X}$.

In Section 4, we investigate whether an algebra with a nontrivial idempotent is zero Jordan product determined. For a unital zero Jordan product determined algebra $\mathcal{A}$, we also characterize the linear mappings η on $\mathcal{A}$ satisfying that $\eta(a) \circ \eta(b) = 0$ whenever $ab = ba = 0$ or $\eta([a,b]) - [\eta(a), \eta(b)] = 0$ whenever $a \circ b = 0$. As applications of Theorems 4.7, 4.8 and 4.9, we generalize the main results in [17].

2. Generalized derivations and generalized Jordan derivations. In [5], Brešar shows that if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is zero product determined. In [8], Ghahramani proves that if $\mathcal{A} = \mathfrak{L}(\mathcal{A})$, then $\mathcal{A}$ is zero Jordan product determined. In [3], we show that if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is zero Jordan product determined. For more information on bilinear mappings from $\mathcal{A} \times \mathcal{A}$ into a vector space $\mathcal{X}$, we refer to [9, 12, 17].

The following two lemmas will be used repeatedly.

Lemma 2.1. [5, Theorem 4.1] *If ϕ is a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into a vector space $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ ab = 0 \Rightarrow \phi(a,b) = 0,$$

then we have that

$$\phi(a,x) = \phi(ax,1) \quad \text{and} \quad \phi(x,a) = \phi(1,xa)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$.

Lemma 2.2. *If ϕ is a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into a vector space $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ ab = ba = 0 \Rightarrow \phi(a,b) = 0,$$

then we have that

$$\phi(a,x) + \phi(x,a) = \phi(ax,1) + \phi(1,xa)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$.

Proof. By the definition of $\mathfrak{J}(\mathcal{A})$, we know that every x in $\mathfrak{J}(\mathcal{A})$ can be written as a linear combination of some elements $x_1, x_2, \ldots, x_k$ in $\mathfrak{J}(\mathcal{A})$ such that $x_k = p_{k_1} p_{k_2} \cdots p_{k_i}$, where $p_{k_1}, p_{k_2}, \ldots, p_{k_i}$ are idempotents in $\mathcal{A}$. Since ϕ is bilinear, to show the theorem, it is sufficient to prove that

$$\phi(a, p_1 p_2 \cdots p_n) + \phi(p_1 p_2 \cdots p_n, a) = \phi(a p_1 p_2 \cdots p_n, 1) + \phi(1, p_1 p_2 \cdots p_n a) \qquad (2.1)$$

for every a and idempotents p_i in $\mathcal{A}$.

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By [8, Theorem 3.5], we know that if $n = 1$, then (2.1) is true. For $n = k$, suppose that (2.1) is true.

Let $n = k + 1$. By $a = p_{k+1}ap_1 + p_{k+1}^{\perp}ap_1 + p_{k+1}ap_1^{\perp} + p_{k+1}^{\perp}ap_1^{\perp}$, it follows that

$$\begin{aligned}\phi(a, p_1p_2\cdots p_kp_{k+1}) =&\phi(p_{k+1}ap_1, p_2\cdots p_kp_{k+1}) - \phi(p_{k+1}ap_1, p_1^{\perp}p_2\cdots p_k)\\ &+ \phi(p_{k+1}^{\perp}ap_1, p_2\cdots p_kp_{k+1}) + \phi(p_{k+1}ap_1^{\perp}, p_1p_2\cdots p_k),\end{aligned}$$

and

$$\begin{aligned}\phi(p_1p_2\cdots p_kp_{k+1}, a) =&\phi(p_2\cdots p_kp_{k+1}, p_{k+1}ap_1) - \phi(p_1^{\perp}p_2\cdots p_k, p_{k+1}ap_1)\\ &+ \phi(p_2\cdots p_kp_{k+1}, p_{k+1}^{\perp}ap_1) + \phi(p_1p_2\cdots p_k, p_{k+1}ap_1^{\perp}).\end{aligned}$$

We have that

$$\begin{aligned}\phi(a, p_1p_2\cdots p_{k+1})+&\phi(p_1p_2\cdots p_{k+1}, a)\\ =&(\phi(p_{k+1}ap_1, p_2\cdots p_kp_{k+1}) + \phi(p_2\cdots p_kp_{k+1}, p_{k+1}ap_1))\\ &- (\phi(p_{k+1}ap_1, p_1^{\perp}p_2\cdots p_k) + \phi(p_1^{\perp}p_2\cdots p_k, p_{k+1}ap_1))\\ &+ (\phi(p_{k+1}^{\perp}ap_1, p_2\cdots p_kp_{k+1}) + \phi(p_2\cdots p_kp_{k+1}, p_{k+1}^{\perp}ap_1))\\ &+ (\phi(p_{k+1}ap_1^{\perp}, p_1p_2\cdots p_k) + \phi(p_1p_2\cdots p_k, p_{k+1}ap_1^{\perp})).\end{aligned}$$

By the inductive assumption, we obtain the following identity:

$$\begin{aligned}\phi(a, p_1p_2\cdots p_kp_{k+1})+&\phi(p_1p_2\cdots p_kp_{k+1}, a)\\ =&\phi(p_{k+1}ap_1p_2\cdots p_kp_{k+1}, 1) + \phi(1, p_2\cdots p_kp_{k+1}ap_1)\\ &- \phi(1, p_1^{\perp}p_2\cdots p_kp_{k+1}ap_1) + \phi(p_{k+1}^{\perp}ap_1p_2\cdots p_kp_{k+1}, 1)\\ &+ \phi(1, p_1p_2\cdots p_kp_{k+1}ap_1^{\perp}).\end{aligned}$$

It implies that

$$\phi(a, p_1p_2\cdots p_{k+1}) + \phi(p_1p_2\cdots p_{k+1}, a) = \phi(ap_1p_2\cdots p_{k+1}, 1) + \phi(1, p_1p_2\cdots p_{k+1}a).$$

Thus, (2.1) is true when $n = k + 1$. ☐

THEOREM 2.3. *Suppose that δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that the condition $\mathbb{D}_1$ holds. If $\mathcal{M}$ has the property $\mathbb{P}_1$, then δ is a generalized derivation and $a\delta(1) = \delta(1)a$ for every a in $\mathcal{A}$.*

Proof. Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi(a, b) = \delta(ab) - a\delta(b) - \delta(a)b$ for each a, b in $\mathcal{A}$. Then $ab = 0$ implies $\phi(a, b) = 0$.

Let $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ be an ideal of $\mathcal{A}$, a and b be in $\mathcal{A}$, x be in $\mathcal{J}$. Applying Lemma 2.1, we obtain $\phi(a, x) = \phi(ax, 1)$ and $\phi(x, 1) = \phi(1, x)$. Hence,

$$\delta(ax) - a\delta(x) - \delta(a)x = \delta(ax) - ax\delta(1) - \delta(ax) \tag{2.2}$$

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and

$$\delta(x) - \delta(x) - \delta(1)x = \delta(x) - \delta(x) - x\delta(1). \tag{2.3}$$

By (2.2) and (2.3), it follows that

$$\delta(ax) = a\delta(x) + \delta(a)x - ax\delta(1) \tag{2.4}$$

and

$$\delta(1)x = x\delta(1). \tag{2.5}$$

By (2.4), it follows that

$$\delta(abx) = \delta((ab)x) = ab\delta(x) + \delta(ab)x - abx\delta(1), \tag{2.6}$$

and on the other hand we have that

$$\delta(abx) = a\delta(bx) + \delta(a)bx - abx\delta(1) = ab\delta(x) + a\delta(b)x + \delta(a)bx - 2abx\delta(1). \tag{2.7}$$

Using (2.5), (2.6) and (2.7), we obtain

$$(\delta(ab) - a\delta(b) - \delta(a)b + ab\delta(1))x = 0.$$

Similarly, we have that

$$x(\delta(ab) - a\delta(b) - \delta(a)b + ab\delta(1)) = 0.$$

Since $\mathcal{M}$ has the property $\mathbb{P}_1$, it follows that

$$\delta(ab) = a\delta(b) + \delta(a)b - ab\delta(1). \tag{2.8}$$

Taking $a = 1$ in (2.8), we have that $\delta(1)b = b\delta(1)$ for every b in $\mathcal{A}$. Thus, δ is a generalized derivation. □

A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local derivation* if for every a in $\mathcal{A}$ there exists a derivation δ_a (depending on a) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(a) = \delta_a(a)$. It is clear that every local derivation satisfies the condition $\mathbb{D}_2$. Hence, the following result is extremely useful in studying the structure of local derivations.

THEOREM 2.4. *Suppose that δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that the condition $\mathbb{D}_2$ holds. If $\mathcal{M}$ has the property $\mathbb{P}_2$, then δ is a generalized derivation.*

Proof. Choosing a_0, b_0 in $\mathcal{A}$ such that $a_0b_0 = 0$. Define a bilinear mapping ϕ_1 from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi_1(a, b) = a\delta(ba_0)b_0$ for each a, b in $\mathcal{A}$. Then $ab = 0$ implies $\phi_1(a, b) = 0$.

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Let $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ be an ideal of $\mathcal{A}$, a be in $\mathcal{A}$ and x be in $\mathcal{J}$. Applying Lemma 2.1, we obtain

$$\phi_1(x,a) = \phi_1(1,xa) \text{ and } \phi_1(x,1) = \phi_1(1,x). \tag{2.9}$$

Since $\mathcal{J}$ is an ideal of $\mathcal{A}$, we have that

$$\phi_1(1,xa) = \phi_1(xa,1). \tag{2.10}$$

Hence, by (2.9) and (2.10), it follows that

$$x\delta(aa_0)b_0 = xa\delta(a_0)b_0. \tag{2.11}$$

Now fix $a_1 \in \mathcal{A}$ and $x \in \mathcal{J}$. By (2.11), it follows that $ab = 0$ implies $x\delta(a_1a)b - xa_1\delta(a)b = 0$ for each a, b in $\mathcal{A}$.

Define a bilinear mapping ϕ_2 from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{M}$ by $\phi_2(a,b) = x\delta(a_1a)b - xa_1\delta(a)b$ for each a, b in $\mathcal{A}$. Then $ab = 0$ implies $\phi_2(a,b) = 0$.

Let a_2 be in $\mathcal{A}$ and y be in $\mathcal{J}$. Similarly, we obtain $\phi_2(a_2,y) = \phi_2(a_2y,1) = \phi_2(1,a_2y)$. Hence,

$$x\delta(a_1a_2)y - xa_1\delta(a_2)y = x\delta(a_1)a_2y - xa_1\delta(1)a_2y. \tag{2.12}$$

By (2.12), we have that

$$x(\delta(a_1a_2) - a_1\delta(a_2) - \delta(a_1)a_2 + a_1\delta(1)a_2)y = 0 \tag{2.13}$$

for each a_1, a_2 in $\mathcal{A}$ and x, y in $\mathcal{J}$.

Since $\mathcal{M}$ has the property $\mathbb{P}_2$, by (2.13) it is easy to show that δ is a generalized derivation. $\square$

In [1], the authors show that in the case when $\mathcal{A}$ is a C^*-algebra and $\mathcal{M}$ is an essential Banach $\mathcal{A}$-bimodule, the condition $\mathbb{D}_3$ implies that δ is of the form $\delta = \Delta + \varphi$, where $\Delta : \mathcal{A} \to \mathcal{M}$ is a derivation and $\varphi : \mathcal{A} \to \mathcal{M}$ is a bimodule homomorphism. Applying the techniques in [8], we have the following result.

Theorem 2.5. *Suppose that δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that the condition $\mathbb{D}_3$ holds. If $\mathcal{M}$ has the property $\mathbb{P}_3$, then δ is a generalized Jordan derivation and $a\delta(1) = \delta(1)a$ for every a in $\mathcal{A}$.*

Proof. Let p be an idempotent in $\mathcal{A}$. By $p(1-p) = (1-p)p = 0$, it follows that

$$2\delta(p) + p\delta(1) + \delta(1)p = 2p\delta(p) + 2\delta(p)p. \tag{2.14}$$

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Multiplying p from the left and the right of (2.14), respectively, we obtain the following identities:

$$p\delta(1)p + \delta(1)p = 2p\delta(p)p \tag{2.15}$$

and

$$p\delta(1) + p\delta(1)p = 2p\delta(p)p. \tag{2.16}$$

Comparing (2.15) and (2.16), we have that $p\delta(1) = \delta(1)p$.

Let $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ be an ideal of $\mathcal{A}$, then by the definition of $\mathfrak{J}(\mathcal{A})$ we obtain $x\delta(1) = \delta(1)x$ for every x in $\mathcal{J}$. Hence, we have the following identities:

$$a\delta(1)x = ax\delta(1) = \delta(1)ax \tag{2.17}$$

and

$$x\delta(1)a = \delta(1)xa = xa\delta(1) \tag{2.18}$$

for every a in $\mathcal{A}$ and x in $\mathcal{J}$. By (2.17) and (2.18), we obtain

$$x(a\delta(1) - \delta(1)a)x = 0.$$

Since $\mathcal{M}$ has the property $\mathbb{P}_3$, it follows that $a\delta(1) = \delta(1)a$ for every a in $\mathcal{A}$.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by $\Delta(a) = \delta(a) - a\delta(1)$ for every a in $\mathcal{A}$. It is clear that $a \circ \Delta(b) + \Delta(a) \circ b = 0$ for each a, b in $\mathcal{A}$ with $ab = ba = 0$ and $\Delta(1) = 0$.

Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi(a, b) = a \circ \Delta(b) + \Delta(a) \circ b$. Then $ab = ba = 0$ implies $\phi(a, b) = 0$. Applying Lemma 2.2, we obtain

$$\phi(a, x) + \phi(x, a) = \phi(ax, 1) + \phi(1, xa)$$

for every a in $\mathcal{A}$ and x in $\mathcal{J}$. Hence, $\Delta(a \circ x) = a \circ \Delta(x) + \Delta(x) \circ a$ for every a in $\mathcal{A}$ and x in $\mathcal{J}$.

Next we prove that Δ is a Jordan derivation. Define $\{a, m, b\} = amb + bma$ and $\{a, b, m\} = \{m, b, a\} = abm + mba$ for each a, b in $\mathcal{A}$ and every m in $\mathcal{M}$.

Let a be in $\mathcal{A}$, x and y be in $\mathcal{M}$. By the proof of [8, Theorem 4.3], we have the following two identities:

$$\Delta\{x, a, y\} = \{\Delta(x), a, y\} + \{x, \Delta(a), y\} + \{x, a, \Delta(y)\}, \tag{2.19}$$

and

$$\Delta\{x, a^2, y\} = \{\Delta(x), a^2, y\} + \{x, a \circ \Delta(a), y\} + \{x, a^2, \Delta(y)\}. \tag{2.20}$$

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On the other hand, by (2.19) we have that

$$\Delta\{x, a^2, x\} = \{\Delta(x), a^2, x\} + \{x, \Delta(a^2), x\} + \{x, a^2, \Delta(x)\}. \tag{2.21}$$

Comparing (2.20) and (2.21), it follows that $\{x, \Delta(a^2), x\} = \{x, a \circ \Delta(a), x\}$. That is $x(\Delta(a^2) - a \circ \Delta(a))x = 0$.

Since $\mathcal{M}$ has the property $\mathbb{P}_3$, it follows that $\Delta(a^2) - a \circ \Delta(a) = 0$ for every a in $\mathcal{A}$. It means that δ is a generalized Jordan derivation. ▯

Remark 2.6. Suppose that $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, an example of such an algebra is the matrix algebra $M_n(\mathcal{B})$, where $n \geqslant 2$ and $\mathcal{B}$ is a unital algebra. It is easy to show that if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then Theorems 2.3 and 2.5 improve [17, Theorem 2.3]

3. Characterizations of Jordan left derivations. Let $\mathcal{M}$ be a unital left $\mathcal{A}$-module. In this section, we characterize the linear mappings satisfying the condition $\mathbb{J}_2$ or $\mathbb{J}_3$ holds. The main result of this section is the following theorem.

Theorem 3.1. *Suppose that δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that the condition $\mathbb{J}_3$ holds. If $\mathcal{M}$ has a right separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$, then $\delta(a) = a\delta(1)$ for every a in $\mathcal{A}$. Moreover, if $\delta(1) = 0$, then $\delta \equiv 0$.*

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let δ be as in* Theorem 3.1*. Then for every a, every idempotent p in $\mathcal{A}$, the following three statements hold:*

(1) $\delta(p) = p\delta(p) = p\delta(1)$;

(2) $\delta(pa) = p\delta(a)$;

(3) $\delta(ap) = p\delta(a) + (ap - pa)\delta(1)$.

Proof. Since $p(1-p) = (1-p)p = 0$, it follows that $p\delta(1-p) + (1-p)\delta(p) = 0$. Through a simple calculation, we have that $\delta(p) = p\delta(p) = p\delta(1)$.

To prove (2) and (3), define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by $\Delta(a) = \delta(a) - a\delta(1)$ for every a in $\mathcal{A}$. It is clear that $a\Delta(b) + b\Delta(a) = 0$ for each a, b in $\mathcal{A}$ with $ab = ba = 0$ and $\Delta(1) = 0$.

Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi(a, b) = a\Delta(b) + b\Delta(a)$ for each a, b in $\mathcal{A}$. Then $ab = ba = 0$ implies $\phi(a, b) = 0$.

Let a be in $\mathcal{A}$ and p be an idempotent in $\mathcal{A}$. Applying Lemma 2.2, we obtain

$$\phi(a, p) + \phi(p, a) = \phi(ap, 1) + \phi(1, pa),$$

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and hence,

$$2p\Delta(a) = \Delta(ap) + \Delta(pa). \tag{3.1}$$

Replacing a by ap and pa in (3.1), respectively, we have the following two identities:

$$2p\Delta(ap) = \Delta(ap) + \Delta(pap) \tag{3.2}$$

and

$$2p\Delta(pa) = \Delta(pa) + \Delta(pap). \tag{3.3}$$

Multiplying p from the left of (3.2), we obtain $2p\Delta(ap) = p\Delta(ap) + p\Delta(pap)$. Thus,

$$p\Delta(ap) = p\Delta(pap). \tag{3.4}$$

Similarly, we have that

$$p\Delta(pa) = p\Delta(pap). \tag{3.5}$$

Replacing a by $a(1-p)$ in (3.1), we obtain $2p\Delta(a - ap) = \Delta(pa - pap)$. Then by (3.5) we have that

$$p\Delta(a) = p\Delta(ap) \tag{3.6}$$

and $\Delta(pa) = \Delta(pap)$. Then by (3.6) we obtain

$$p\Delta(pa) = \Delta(pa) = \Delta(pap). \tag{3.7}$$

Multiplying p from the left of (3.1), we have that $2p\Delta(a) = p\Delta(ap) + p\Delta(pa)$. By (3.1), (3.3) and (3.7), it follows that $p\Delta(a) = p\Delta(pa) = \Delta(pa) = \Delta(ap)$. By the definition of Δ, it follows that

$$p(\delta(a) - a\delta(1)) = \delta(ap) - ap\delta(1) = \delta(pa) - pa\delta(1),$$

that is,

$$\delta(pa) = p\delta(a) = \delta(ap) - (ap - pa)\delta(1)$$

for every a and every idempotent p in $\mathcal{A}$. □

By the definition of $\mathfrak{J}(\mathcal{A})$, it is easy to show the following result.

Corollary 3.3. *Let δ be as in* Theorem 3.1. *If $\mathcal{B} \subseteq \mathfrak{J}(\mathcal{A})$, then for every s in $\mathcal{B}$ and every a in $\mathcal{A}$, we have that $\delta(sa) = s\delta(a)$.*

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In the following, we prove Theorem 3.1.

Proof of Theorem 3.1. By Corollary 3.3, we have that $\delta(sab) = s\delta(ab)$ and $\delta(sab) = \delta((sa)b) = sa\delta(b)$ for each a, b in $\mathcal{A}$ and every s in $\mathcal{J}$.

Hence, $s(\delta(ab) - a\delta(b)) = 0$ for each a, b in $\mathcal{A}$ and every s in $\mathcal{J}$. Since $\mathcal{J}$ is a right separating set of $\mathcal{M}$, it follows that $\delta(ab) = a\delta(b)$ for each a, b in $\mathcal{A}$. Taking $b = 1$, we obtain $\delta(a) = a\delta(1)$ for every a in $\mathcal{A}$. ▯

In [16], J. Li and J. Zhou show that if $\mathcal{M}$ has a right separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$, then each linear mapping δ satisfying $\mathbb{J}_1$ is zero under certain conditions. Since $\mathbb{J}_1$ implies $\mathbb{J}_3$, by using Theorem 3.1, we can improve [16, Theorem 3.3].

THEOREM 3.4. *Suppose that δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that the condition $\mathbb{J}_2$ holds. If $\mathcal{A} = \mathfrak{J}(\mathcal{A})$ and $\delta(1) = 0$, then $\delta \equiv 0$.*

Proof. Choose a_0, b_0 in $\mathcal{A}$ such that $a_0 b_0 = 0$. Define a bilinear mapping ϕ_1 from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi_1(a, b) = ab_0\delta(ba_0)$ for each a, b in $\mathcal{A}$. Then $ab = 0$ implies $\phi_1(a, b) = 0$. Let a be in $\mathcal{A}$, applying Lemma 2.1, we obtain

$$\phi_1(a, 1) = \phi_1(1, a),$$

and hence,

$$ab_0\delta(a_0) = b_0\delta(aa_0). \tag{3.8}$$

By (3.8), it follows that for each a, a_0, b_0 in $\mathcal{A}$, $a_0b_0 = 0$ implies $ab_0\delta(a_0) - b_0\delta(aa_0) = 0$.

Now fix $a_1 \in \mathcal{A}$, define a bilinear mapping ϕ_2 from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi_2(a, b) = a_1 b\delta(a) - b\delta(a_1 a)$ for each a, b in $\mathcal{A}$. Then $ab = 0$ implies $\phi_2(a, b) = 0$. Applying Lemma 2.1, we obtain

$$\phi_2(1, a_2) = \phi_2(a_2, 1)$$

for every a_2 in $\mathcal{A}$. Hence,

$$a_1 a_2 \delta(1) - a_2\delta(a_1) = a_1\delta(a_2) - \delta(a_1 a_2)$$

for each a_1, a_2 in $\mathcal{A}$. This means that δ is a left derivation if $\delta(1) = 0$. By Theorem 3.1, we have that $\delta \equiv 0$. ▯

4. Characterizing linear mappings through zero Jordan products. A linear mapping δ from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a *Lie derivation* if $\delta[a, b] = [\delta(a), b] + [a, \delta(b)]$ for each a, b in $\mathcal{A}$. A linear mapping η from $\mathcal{A}$ into an algebra $\mathcal{B}$ is said to be a *Jordan homomorphism* if $\eta(a \circ b) = \eta(a) \circ \eta(b)$ for each a, b in $\mathcal{A}$ and η is said to be a *Lie homomorphism* if $\eta[a, b] = [\eta(a), \eta(b)]$ for each a, b in $\mathcal{A}$.

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In this section, we investigate whether a unital algebra with nontrivial idempotents is zero Jordan product determined, and we characterize Jordan homomorphisms, Lie homomorphisms and Lie derivations on a unital zero Jordan product determined algebra.

Lemma 4.1. [3, Theorem 2.1] *If ϕ is a bilinear mapping from $\mathcal{A}\times\mathcal{A}$ into a vector space $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ \ a \circ b = 0 \Rightarrow \phi(a, b) = 0,$$

then we have that

$$\phi(a, x) = \frac{1}{2}\phi(ax, 1) + \frac{1}{2}\phi(xa, 1)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$. Thus, $\mathcal{A}$ is zero Jordan product determined if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$.

Similar to the proof of Lemma 4.1, we can obtain the following result.

Lemma 4.2. *If ϕ is a bilinear mapping from $\mathcal{A}\times\mathcal{A}$ into a vector space $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ \ a \circ b = 0 \Rightarrow \phi(a, b) = 0,$$

then we have that

$$\phi(x, a) = \frac{1}{2}\phi(1, ax) + \frac{1}{2}\phi(1, xa)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$. Thus, $\mathcal{A}$ is zero Jordan product determined if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$.

Let p be a nontrivial idempotent in $\mathcal{A}$ and $q = 1 - p$. We have the Peirce decomposition of $\mathcal{A}$ as follows:

$$\mathcal{A} = p\mathcal{A}p + p\mathcal{A}q + q\mathcal{A}p + q\mathcal{A}q.$$

It is easy to show that $p + paq$ is an idempotent for every a in $\mathcal{A}$, and hence, $p\mathcal{A}q = (p + p\mathcal{A}q) - p$ is contained in $\mathfrak{J}(\mathcal{A})$.

Recall an algebra $\mathcal{A}$ is *simple* if $\{0\}$ and $\mathcal{A}$ are the only two ideals of $\mathcal{A}$. By [4, p.11], we know that every simple algebra with a nontrivial idempotent is generated algebraically by its idempotents. By Lemma 4.1, we can obtain the following result immediately.

Corollary 4.3. *If $\mathcal{A}$ is a simple algebra with a nontrivial idempotent p, then $\mathcal{A}$ is zero Jordan product determined.*

Theorem 4.4. *If $p\mathcal{A}p$ and $q\mathcal{A}q$ are zero Jordan product determined, then $\mathcal{A}$ is zero Jordan product determined.*

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Proof. Suppose that $\mathcal{X}$ is a linear space and ϕ is a bilinear mapping from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{X}$ such that

$$a,b\in\mathcal{A},\ \ a\circ b=0\Rightarrow\phi(a,b)=0.$$

Let a and b be in $\mathcal{A}$. Since $p\mathcal{A}p$ is zero Jordan product determined and p is the unit in $p\mathcal{A}p$, we have a linear mapping T from $p\mathcal{A}p$ into $\mathcal{X}$ such that

$$\phi(pap,pbp)=T(papbp)+T(pbpap)=\frac{1}{2}\phi(papbp,p)+\frac{1}{2}\phi(pbpap,p).$$

By Lemma 4.1, we obtain

$$\phi(papbp,p)=\phi(papbp,1)$$

and

$$\phi(pbpap,p)=\phi(pbpap,1).$$

It follows that

$$\phi(pap,pbp)=\frac{1}{2}\phi(papbp,1)+\frac{1}{2}\phi(pbpap,1). \tag{4.1}$$

Similarly, we have that

$$\phi(qaq,qbq)=\frac{1}{2}\phi(qaqbq,1)+\frac{1}{2}\phi(qbqaq,1).$$

By Peirce decomposition, we obtain

$$\phi(a,b)=\phi(pap+paq+qap+qaq,pbp+pbq+qbp+qbq).$$

By Lemma 4.1 and (4.1), it follows that

$$\begin{aligned}\phi(pap,pbp+pbq+qbp+qbq)=&\phi(pap,pbp)+\phi(pap,pbq)\\&+\phi(pap,qbp)+\phi(pap,qbq)\\=&\frac{1}{2}\phi(papbp,1)+\frac{1}{2}\phi(pbpap,1)\\&+\frac{1}{2}\phi(papbq,1)+\frac{1}{2}\phi(qbpap,1)\\=&\frac{1}{2}\phi(papb,1)+\frac{1}{2}\phi(bpap,1).\end{aligned} \tag{4.2}$$

Similarly, we have the following identity:

$$\phi(qaq,pbp+pbq+qbp+qbq)=\frac{1}{2}\phi(bqaq,1)+\frac{1}{2}\phi(qaqb,1). \tag{4.3}$$

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By Lemma 4.2 we know that $\phi(1, x) = \phi(x, 1)$ for every $x \in \mathfrak{J}(\mathcal{A})$, and

$$\begin{aligned}\phi(paq, pbp + pbq + qbp + qbq) =& \frac{1}{2}\phi(1, pbpaq) + \frac{1}{2}\phi(1, paqbp)\\ &+ \frac{1}{2}\phi(1, qbpaq) + \frac{1}{2}\phi(1, paqbq)\\ =& \frac{1}{2}\phi(pbpaq, 1) + \frac{1}{2}\phi(paqbp, 1)\\ &+ \frac{1}{2}\phi(qbpaq, 1) + \frac{1}{2}\phi(paqbq, 1)\\ =& \frac{1}{2}\phi(bpaq, 1) + \frac{1}{2}\phi(paqb, 1).\end{aligned} \tag{4.4}$$

Similarly, we can obtain

$$\phi(qap, pbp + pbq + qbp + qbq) = \frac{1}{2}\phi(qapb, 1) + \frac{1}{2}\phi(bqap, 1). \tag{4.5}$$

By (4.2), (4.3), (4.4) and (4.5), it follows that

$$\phi(a, b) = \frac{1}{2}\phi(ab, 1) + \frac{1}{2}\phi(ba, 1). \tag{4.6}$$

Define a linear mapping T_0 from $\mathcal{A}$ into $\mathcal{X}$ by $T_0(a) = \frac{1}{2}\phi(a, 1)$ for every a in $\mathcal{A}$. By (4.6) we have that $\phi(a, b) = T_0(a \circ b)$ for each a, b in $\mathcal{A}$. □

Theorem 4.5. *If $\mathcal{A}$ is zero Jordan product determined and $p\mathcal{A}q\mathcal{A}p = \{0\}$, then $p\mathcal{A}p$ is zero Jordan product determined.*

Proof. Suppose that $\mathcal{X}$ is a linear space and ϕ is a bilinear mapping from $p\mathcal{A}p \times p\mathcal{A}p$ into $\mathcal{X}$ such that

$$a, b \in \mathcal{A},\ pap \circ pbp = 0 \Rightarrow \phi(pap, pbp) = 0.$$

Let a and b be in $\mathcal{A}$. Define a bilinear mapping φ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$ by $\varphi(a, b) = \phi(pap, pbp)$.

Since $p\mathcal{A}q\mathcal{A}p = \{0\}$, we obtain $pabp = papbp + paqbp = papbp$. Similarly, we have $pbap = pbpap$.

If $ab+ba = 0$, then $papbp+pbpap = pabp+pbap = 0$ and $\varphi(a, b) = \phi(pap, pbp) = 0$.

Since $\mathcal{A}$ is zero Jordan product determined, we have a linear mapping T from $\mathcal{A}$ into $\mathcal{X}$ such that $T(a \circ b) = \varphi(a, b)$.

Define a linear mapping T_0 from $p\mathcal{A}p$ into $\mathcal{X}$ by $T_0(pap) = T(a)$ for every a in $\mathcal{A}$.

Next we show that T_0 is well defined. Let a and b be in $\mathcal{A}$ such that $p(a-b)p = 0$. By the definition of φ, it follows that

$$2T(a - b) = T((a - b) \circ 1) = \varphi(a - b, 1) = \phi(p(a - b)p, p) = 0,$$

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it means that $T(a) = T(b)$.

Since $pabp = paqbp$ and $pbap = pbqbp$, we have that

$$T_0(pap \circ pbp) = T_0(p(ab + ba)p) = T(ab + ba) = \varphi(a, b) = \phi(pap, pbp)$$

for each a, b in $\mathcal{A}$. Thus, $p\mathcal{A}p$ is zero Jordan product determined. ∎

By Theorems 4.4 and 4.5, we have the following result.

Corollary 4.6. *Suppose that $p\mathcal{A}q\mathcal{A}p = \{0\}$ and $q\mathcal{A}p\mathcal{A}q = \{0\}$. Then $\mathcal{A}$ is zero Jordan product determined if and only if $p\mathcal{A}p$ and $q\mathcal{A}q$ are zero Jordan product determined.*

Remark 4.7. In [9], Ghahramani shows that the triangular algebra

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} : A \in \mathcal{A}, B \in \mathcal{B}, M \in \mathcal{M} \right\}$$

is zero Jordan product determined if and only if $\mathcal{A}$ and $\mathcal{B}$ are zero Jordan product determined. By Corollary 4.6, we can obtain this result immediately.

Theorem 4.8. *Let $\mathcal{A}$ be a unital zero Jordan product determined algebra and $\mathcal{B}$ be a unital algebra. Suppose that η is a linear mapping from $\mathcal{A}$ into $\mathcal{B}$ with $\eta(1) = 1$. If η satisfies*

$$a, b \in \mathcal{A}, \ ab = ba = 0 \Rightarrow \eta(a) \circ \eta(b) = 0,$$

then η is a Jordan homomorphism.

Proof. Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{B}$ by

$$\phi(a, b) = \eta(a \circ b) - \eta(a) \circ \eta(b) \tag{4.7}$$

for each a and b in $\mathcal{A}$. Then $ab = ba = 0$ implies $\phi(a, b) = 0$. By Lemma 2.2, we have that

$$\phi(a, b) + \phi(b, a) = \phi(ab, 1) + \phi(1, ba). \tag{4.8}$$

By (4.7) and (4.8), it is easy to show that

$$\phi(a, b) = \phi(b, a) \text{ and } \phi(a, b) = \frac{1}{2}\phi(ab + ba, 1).$$

It follows that $ab + ba = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is zero Jordan product determined, we have a linear mapping T from $\mathcal{A}$ into $\mathcal{B}$ such that $\phi(a, b) = T(a \circ b)$. It follows that

$$T(a \circ b) = \eta(a \circ b) - \eta(a) \circ \eta(b). \tag{4.9}$$

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Taking $b = 1$ in (4.9) and by $\eta(1) = 1$, we obtain $T(a) = 0$. Thus, by (4.9), we have that $\eta(a \circ b) = \eta(a) \circ \eta(b)$ for each a, b in $\mathcal{A}$. ∎

THEOREM 4.9. *Let $\mathcal{A}$ be a unital zero Jordan product determined algebra and $\mathcal{B}$ be a unital algebra. Suppose that η is a linear mapping from $\mathcal{A}$ into $\mathcal{B}$ with $\eta(1) = 1$. If η satisfies*

$$a, b \in \mathcal{A},\ a \circ b = 0 \Rightarrow \eta([a, b]) - [\eta(a), \eta(b)] = 0,$$

then η is a Lie homomorphism.

Proof. Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{B}$ by $\phi(a, b) = \eta([a, b]) - [\eta(a), \eta(b)]$ for each a and b in $\mathcal{A}$.

Then $a \circ b = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is zero Jordan product determined, we have a linear mapping T from $\mathcal{A}$ into $\mathcal{B}$ such that $\phi(a, b) = T(a \circ b)$. It follows that

$$T(a \circ b) = \eta([a, b]) - [\eta(a), \eta(b)]. \tag{4.10}$$

Taking $b = 1$ in (4.10), we obtain that:

$$2T(a) = -[\eta(a), \eta(1)] = 0.$$

It follows that $T(a) = 0$ for all $a \in \mathcal{A}$. Thus, by (4.10), we have that $\eta([a, b]) = [\eta(a), \eta(b)]$ for each a, b in $\mathcal{A}$. ∎

THEOREM 4.10. *Let $\mathcal{A}$ be a unital zero Jordan product determined algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ a \circ b = 0 \Rightarrow \delta([a, b]) - [a, \delta(b)] - [\delta(a), b] = 0,$$

then η is a Lie derivation.

Proof. Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by $\phi(a, b) = \delta([a, b]) - [a, \delta(b)] - [\delta(a), b]$ for each a and b in $\mathcal{A}$.

Then $a \circ b = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is zero Jordan product determined, we have a linear mapping T from $\mathcal{A}$ into $\mathcal{B}$ such that $\phi(a, b) = T(a \circ b)$. It follows that

$$T(a \circ b) = \delta([a, b]) - [a, \delta(b)] - [\delta(a), b]. \tag{4.11}$$

Taking $b = 1$ and $a = 1$ in (4.11), respectively, we obtain the following two identities:

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$$T(2a) = -[a, \delta(1)] \text{ and } T(2b) = -[\delta(1), b].$$

It implies that $2T(a) = -[\delta(1), a] = 0$. Thus, by (4.11), we have that $\delta([a, b]) = [a, \delta(b)] + [\delta(a), b]$ for each a, b in $\mathcal{A}$. □

REMARK 4.11. If $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then Theorems 4.8, 4.9 and 4.10 generalize [17, Theorems 2.1, 2.2 and 2.4].

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