Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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AN EXPANSION PROPERTY OF BOOLEAN LINEAR MAPS*

YAOKUN WU†, ZEYING XU‡, AND YINFENG ZHU‡

Abstract. Given a finite set K, a Boolean linear map on K is a map f from the set 2^K of all subsets of K into itself with $f(\emptyset)=\emptyset$ such that $f(A\cup B)=f(A)\cup f(B)$ holds for all $A,B\in 2^K$. For fixed subsets X,Y of K, to predict if Y is reachable from X in the dynamical system driven by f, one can assume the existence of nonnegative integers h with $f^h(X)=Y$, find an upper bound α for the minimum of all such assumed integers h, and test if Y really appears in $f^0(X),\ldots,f^\alpha(X)$. In order to get such an upper bound estimate, this paper establishes an expansion property for the Boolean linear map f. Namely, the authors find a lower bound on the size of $f^h(X)$ for any nonnegative integer h. Besides presenting several direct applications of the derived expansion property, this paper collects some related problems on Boolean linear dynamical systems, including problems on primitive multilinear maps and inhomogeneous topological Markov chains.

Key words. Diameter, Digraph, Girth, Phase space, Primitive exponent, Reachability, Wielandt's inequality.

AMS subject classifications. 05C20, 05C50, 68Q45.

1. Boolean linear maps. Let K be a finite nonempty set. We write $\binom{K}{m}$ for the set of all subsets of K of size m and we use 2^K for the set of all subsets of K, that is, $2^K=\cup_{m=0}^{|K|}\binom{K}{m}$. A *Boolean linear map on* K is a map f from 2^K to 2^K that satisfies the following two conditions:

$$\begin{cases} f(\emptyset)=\emptyset; \\ f(A)\cup f(B)=f(A\cup B) \text{ for all } A,B\in 2^K. \end{cases}$$

The set of all Boolean linear maps on K is denoted $\mathcal{B}_K$. Let $f\in\mathcal{B}_K$. Whenever we can do iterations, we encounter a dynamical system and so we can consider the dynamics driven by f. The *phase space* of f is the digraph with vertex set 2^K and arc set $\{A\to f(A):\ A\in 2^K\}$, for which we use the notation $\mathcal{PS}_f$. Given any nonnegative integer h, applying f iteratively h times yields the Boolean linear map f^h on K. For any given $X\in 2^K$, the evolutionary process of X under the force of f^h

*Received by the editors on August 16, 2015. Accepted for publication on April 12, 2016. Handling Editor: Bryan L. Shader.

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can be recorded as the unique directed walk of length h starting from X in $\mathcal{PS}_f$:

$$X = f^0(X) \to f^1(X) \to \cdots \to f^h(X).$$

Every weakly connected component of $\mathcal{PS}_f$ must be a cycle with some in-trees planted on its vertices – the cycle corresponds to the periodic points of the dynamics and each in-tree, namely a rooted tree with a unique path from every vertex to the root, removing its root attached on the cycle, corresponds to the transient part of the dynamics. But this does not mean that there is little to say/ask about $\mathcal{PS}_f$. Indeed, the reachability properties of $\mathcal{PS}_f$ encode all dynamical behaviors of f and have attracted much attention in the literature.

We often identify a Boolean linear map f on K with a digraph Γ whose vertex set is K and whose arc set is $\{x \to y : \ y \in f(x), x \in K\}$. One can think of the dynamical behavior of f as the topological skeleton of all random walks on Γ which surely reveals features of the geometry of Γ. Note that $f(A)$ is the union of the out-neighbors in Γ of all $x \in A$. We will freely switch between the language of digraphs and the language of linear maps and will later, instead of mentioning Γ, directly refer to the digraph f. It looks to be a fundamental problem to understand the relationship between the two digraphs, f and $\mathcal{PS}_f$, namely the property of f as a digraph and the property of f as a Boolean linear map. The digraph f stands for the local connection mechanism and the digraph $\mathcal{PS}_f$ displays the global evolving picture. As the science of things that keep moving, the theory of dynamical systems aims to relate a system's global behaviour to its local behaviour and the forces that shape it.

The main concern of this paper is the expansion property of a Boolean linear map f. More precisely, we will try to establish lower bounds of the size of the set reached in certain number of steps by applying f, say h steps, from a given set, say X, in terms of parameters of the digraph f. We can rephrase this problem in a dual way as follows. After knowing the size of a set Y which is reachable from X in $\mathcal{PS}_f$, we want to estimate the shortest time needed to go there from X, namely an upper bound of the smallest h such that $Y = f^h(X)$. Unlike the case of nonnegative real linear maps, where spectral techniques play an important role, we will use a pure combinatorial approach to study the expansion property in the Boolean linear case. Due to the fundamental importance of Boolean linear maps, these kind of problems arise in various guises in many fields [1, 2, 3, 5, 7, 16, 17, 23, 27, 31, 33, 39, 41, 42, 44, 46]. It is noteworthy that our phase space approach naturally suggests many interesting problems and conjectures, which will be reported in Section 3.

Let $\mathbb{N}$ stand for the set of nonnegative integers. For every $k \in \mathbb{N}$, we write $[k]$ for the set of the first k positive integers and we use $\mathcal{B}_k$ to represent the set of all Boolean linear maps on $[k]$. It is clear that the study of Boolean linear maps on a k-element set K is essentially a study of $\mathcal{B}_k$.

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In the course of investigating a control theoretic question, Coxson and Shapiro discovered a result on the reachability of monomial patterns [17, Theorem 3] and they conjectured a stronger result. This conjecture was later proved by Coxson, Larson and Schneider [16, Theorem 1], which we present below.

Theorem 1.1 (Coxson-Larson-Schneider). *Let k be a positive integer and let $f \in \mathcal{B}_k$ be a digraph. Take $X \subseteq [k]$ and $h \in \mathbb{N}$ such that $Y = f^h(X)$ is a singleton set. Then there exists $h' \in \mathbb{N}$ such that $h' \leq k-1$ and $Y = f^{h'}(X)$.*

The importance of a result like Theorem 1.1 can be seen as follows. Suppose that we have a reasonably good upper bound estimate of how long it takes to go from X to Y under the assumption that Y is reachable from X. Then, if we have no way to predict in advance whether or not X can evolve into Y, we can simply observe $f^h(X)$ for h below our estimate and check if $f^h(X) = Y$ has occurred. We will extend Theorem 1.1 in two ways in this paper; see Theorems 1.2 and 1.11.

Before presenting our main results, we start with some necessary definitions. Let K be a finite nonempty set and choose $f \in \mathcal{B}_K$. We use g_f to denote the *girth* of f, namely the length of a shortest cycle in f, and use L_f to denote the length of a longest path in f. As usual, we set $\mathrm{g}_f = \infty$ when there is no cycle in f. Let $\mathcal{R}_f$ be the map from $K \times \mathbb{N}$ to 2^K, called the *range map*, such that $\mathcal{R}_f(x,i) = f^i(x)$ for $(x,i) \in K \times \mathbb{N}$. For $x \in K$, let $\mathcal{R}^*_f(x)$ denote the set $\cup_{i\in\mathbb{N}} \mathcal{R}_f(x,i)$. If $\mathcal{R}^*_f(x) = K$ for all $x \in K$, then f is *strongly connected* or *irreducible*. The map $f \in \mathcal{B}_K$ is said to be *nontrivial* if $f(K) = K$. Note that the only trivial irreducible Boolean linear map is the digraph on one vertex without loops. If $y \in \mathcal{R}^*_f(x)$, we write $\mathrm{Dist}_f(x,y)$ for the minimum i such that $y \in \mathcal{R}_f(x,i)$. In general, for any $C \subseteq \mathcal{R}^*_f(x)$, we adopt the notation $\mathrm{Dist}_f(x,C)$ for $\min_{y\in C} \mathrm{Dist}_f(x,y)$. Define the *longest distance of f* to be

$$\max_{\substack{x,y\in K\\ y\in\mathcal{R}^*_f(x)}} \mathrm{Dist}_f(x,y)$$

and denote it by D_f. Note that

$$\mathrm{D}_f \leq \mathrm{L}_f \leq |K| - 1.$$

When f is irreducible, D_f is usually named as the *diameter* of f; when $\mathrm{g}_f < \infty$, say when f is both irreducible and nontrivial, it holds

$$\mathrm{g}_f - 1 \leq \mathrm{D}_f\,.$$

We say that $A \subseteq K$ is *recurrent for f* if A is on a cycle in $\mathcal{PS}_f$ and we assert that A is *transient for f* otherwise. The Boolean linear map f is *primitive* provided it is nontrivial and every walk in $\mathcal{PS}_f$ staring from a vertex $X \neq \emptyset$ will reach the vertex K. For a primitive map f, its *primitive exponent* is the minimum nonnegative

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A publication of the International Linear Algebra Society
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integer i such that $f^i(x) = K$ for all $x \in K$, which coincides with $\text{Ł}_{\mathcal{PS}_f}$ [8, §3.5]. We mention that coding theorists [37, Problem 14.4] define the minimum back-length of an irreducible digraph f and this parameter coincides with the primitive exponent when f is primitive.

To proceed, we further recall some concepts/results from combinatorial matrix theory [8]. If f is a nontrivial strongly connected digraph, its *cyclicity* is the greatest common divisor of the lengths of all cycles in f. Note that f is primitive if and only if it is an irreducible map with cyclicity 1. For every positive integer t and every strongly connected digraph f with cyclicity p, the digraph f^t consists of $\gcd(p,t)$ strongly connected components, every component of them inducing a digraph with cyclicity $p/\gcd(p,t)$. In particular, for a nontrivial strongly connected digraph f with cyclicity p, the p strongly connected components of f^p are called the *cyclicity classes* of f. Fix a nontrivial strongly connected digraph f on K. Bear in mind that a set A is recurrent for f if and only if it is the union of one or more cyclicity classes of f. An important consequence, which we often use implicitly, is that, if $\mathcal{R}_f(x,i)$ is transient for f, then so is any set $A \subseteq \mathcal{R}_f(x,h)$ for any $h \in [i]$.

At the very beginning of the study of primitive exponent, Dulmage and Mendelsohn [21, Theorem 1] already found that $\text{Ł}_{\mathcal{PS}_f} \leq (k-2)\,\mathrm{g}_f + k$ for every primitive map $f \in \mathcal{B}_k$; see the proof of Corollary 1.4. The next result is a slight generalization of both this girth bound and Theorem 1.1 for primitive digraphs.

THEOREM 1.2. *Let k be a positive integer and let f be a nontrivial irreducible map from $\mathcal{B}_k$. Given $h \in \mathbb{N}$ and $z \in [k]$ such that $\mathcal{R}_f(z,h)$ is transient for f, it holds*

$$(|\,\mathcal{R}_f(z,h)| - 1)\,\mathrm{g}_f + k - 1 \geq h. \tag{1.1}$$

If Eq. (1.1) *occurs as an equality, then for any shortest cycle C of f and any $y \in [k] \setminus \{z\}$, $\mathrm{Dist}_f(z,C) > \mathrm{Dist}_f(y,C)$.*

For any integer k greater than 1, we define the k-vertex *Wielandt digraph* W_k [47] to be the one consisting of a Hamilton cycle $1 \to 2 \to \cdots \to k \to 1$ plus an extra arc $k \to 2$; and we define the k-vertex *near-Wielandt digraph* $\widetilde{W_k}$ to be the one obtained from W_k by adding the arc $1 \to 3 \pmod k$. As an illustration, we draw $W_2, W_6, \widetilde{W_2}$ and $\widetilde{W_6}$ in Fig. 1.1.

EXAMPLE 1.3. In the Wielandt digraph W_k, $C = (2 \to 3 \to \cdots \to k \to 2)$ is the unique shortest cycle and 1 is the unique vertex whose shortest distance to C takes maximum value. According to Theorem 1.2, to get the equality in Eq. (1.1) for $f = W_k$, it is impossible to have $z \neq 1$. In the near-Wielandt digraph $\widetilde{W_k}$, $C = (2 \to 3 \to \cdots \to k \to 2)$ and $C' = (1 \to 3 \to 4 \to \cdots \to k \to 1)$ are the only two shortest cycles and the union of them cover all elements of $[k]$. Therefore, by Theorem 1.2, Eq. (1.1) is always a strict inequality for $f = \widetilde{W_k}$.

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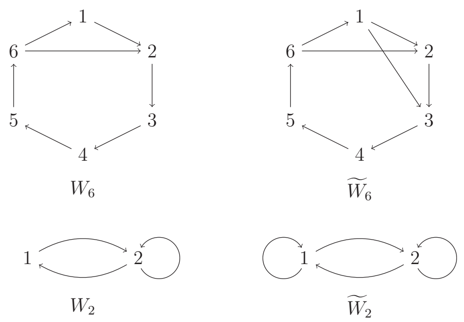

FIG. 1.1. *The Wielandt digraphs W_2, W_6 and the near-Wielandt digraphs $\widetilde{W}_2, \widetilde{W}_6$.*

As an application of Theorem 1.2, we can deduce the classic Wielandt's inequality for primitive exponent [21, 40, 47].

COROLLARY 1.4 (Wielandt's inequality). *Take an integer $k > 1$ and let $f \in \mathcal{B}_k$ be primitive. Then $L_{\mathcal{PS}_f} \leq (k-1)^2 + 1$, with equality if and only if f is isomorphic with the Wielandt digraph W_k.*

Let K be a nonempty finite set and take a primitive digraph $f \in \mathcal{B}_K$. For any $m, n \in [|K|]$, let $\alpha_{m,n}(f)$ be the minimum integer t such that

$$|\cap_{x\in X} \mathcal{R}_f(x, q)| \geq n$$

holds for all $q \geq t$ and all $X \in \binom{K}{m}$. Various special cases of this parameter $\alpha_{m,n}$ for primitive digraphs have been studied in the literature and we give a very quick overview in the next example.

EXAMPLE 1.5. Let K be a k-element set and let $f \in \mathcal{B}_K$ be a primitive map.

- A map $g \in \mathcal{B}_K$ is *scrambling* [24] if $g(i) \cap g(j) \neq \emptyset$ for every $i, j \in K$. The *scrambling index* [1, 34, 41] of f is the smallest positive integer s such that f^s is scrambling. It is clear that the scrambling index of f is $\alpha_{2,1}(f)$.
- Let $m \in [k]$. Following Huang and Liu [25, Proposition 2.2], we define the *mth upper scrambling index of f* to be the least positive integer t such that

  $$\cap_{x\in X} \mathcal{R}_f(x, q) \neq \emptyset$$

  holds for all $X \in \binom{K}{m}$ and $q \geq t$. This means that the mth upper scrambling index of f is just $\alpha_{m,1}(f)$. This parameter was studied much earlier by Schwartz, Liu and Bo [6, 29, 41, 42] in the context of common consequents.
- Pick $n \in [k]$. The *n-competition index* of f [14, 26] is the smallest integer t such that

  $$|\mathcal{R}_f(x, q) \cap \mathcal{R}_f(y, q)| \geq n$$

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for all $\{x, y\} \in \binom{K}{2}$ and $q \geq t$. This n-competition index of f equals $\alpha_{2,n}(f)$.

When $m = 2$, the next result was derived earlier by Kim [26, Theorem 9].

COROLLARY 1.6. *Let k be a positive integer and let $f \in \mathcal{B}_k$ be primitive. Then, it holds for every $m, n \in [k]$ that*

$$\alpha_{m,n}(f) \leq \begin{cases} k + \left(\left\lfloor \frac{km-k+n}{m} \right\rfloor - 2\right) \mathrm{g}_f, & \text{if } m \mid km - k + n; \\ k - 1 + \left(\left\lfloor \frac{km-k+n}{m} \right\rfloor - 1\right) \mathrm{g}_f, & \text{if } m \nmid km - k + n. \end{cases}$$

The nice upper bound of the scrambling index of primitive digraphs given in Corollary 1.7 was obtained by Schwarz [41, Theorem 2.3] and by Akelbek and Kirkland [2, Theorem 3.18] independently. Schwarz [41, Theorem 4.1] extended the result to more general digraphs. Two groups of researchers, Bo and Liu [6, Theorem 3.1], and Akelbek and Kirkland [2, Theorem 3.18], further showed that the only extremal digraph for which the equality happens in Eq. (1.2) is the Wielandt digraph W_k. We shall make use of Theorem 1.2 to provide a new proof of Eq. (1.2) which is shorter than the original proofs from [2, 41].

COROLLARY 1.7 (Schwarz-Akelbek-Kirkland). *Let k be an integer greater than 1 and let f be a primitive map from $\mathcal{B}_k$. Then*

$$\alpha_{2,1}(f) \leq \left\lceil \frac{(k-1)^2 + 1}{2} \right\rceil. \tag{1.2}$$

We present below two diameter bounds of the expansion property for irreducible Boolean linear maps (Theorem 1.8 and Theorem 1.10). Roughly speaking, they tell us that for a strongly connected digraph f, the vertices that appear in the unique walk starting from a singleton set in $\mathcal{PS}_f$ will increase size by at least $\frac{1}{\mathrm{D}_f}$ per step in average before falling into the final periodic part of its trajectory in $\mathcal{PS}_f$. Our deduction of the diameter bounds in Theorem 1.8 will be much more nontrivial than the proof of the girth bound in Theorem 1.2.

THEOREM 1.8. *Let k be a positive integer and let f be an irreducible map from $\mathcal{B}_k$. Take $z \in [k]$ and a positive integer h such that $\mathcal{R}_f(z, h)$ is transient for f. Then the following hold.*

(A) $h \leq |\mathcal{R}_f(z, h)| \, \mathrm{D}_f$.
(B) If $|\mathcal{R}_f(z, h)| > |\mathcal{R}_f(u, h-1)$ *holds for all* $u \in f(z)$*, then*

$$h \leq (|\mathcal{R}_f(z, h)| - 1) \, \mathrm{D}_f \, .$$

Fix a positive integer k and a primitive map $f \in \mathcal{B}_k$. For every $m \in [k]$, Liu [29,

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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Theorem 3.8] discovered that

$$\alpha_{m,1}(f) \leq \left\lfloor \frac{m-1}{m} k \right\rfloor (k-1) + 1. \tag{1.3}$$

The following corollary of Theorem 1.8 is a strengthening of Eq. (1.3). As with the estimate of $\alpha_{k,1}(f)$, we should mention that Schwartz [42, Theorem 1] provided a very short proof of a better bound of $\alpha_{k,1}(f) \leq k^2 - 3k + 3$.

Corollary 1.9. *Let k be a positive integer and let $f \in \mathcal{B}_k$ be a primitive map. For any $m, n \in [k]$, it holds*

$$\alpha_{m,n}(f) \leq \left\lfloor k - \frac{k-n+1}{m} \right\rfloor \mathrm{D}_f + 1.$$

In Theorem 1.8, we deal with transient reachable sets; the forthcoming Theorem 1.10 will be about recurrent reachable sets. Wielandt's inequality in Corollary 1.4 also follows directly from Theorem 1.10.

Theorem 1.10. *Let k be a positive integer and let f be a nontrivial irreducible map from $\mathcal{B}_k$ with cyclicity p. Let $C_1, \ldots, C_p$ be the cyclicity classes of f. Take $z \in [k]$ and $h \in \mathbb{N}$. If $\mathcal{R}_f(z, h-p) \subsetneq \mathcal{R}_f(z, h) = C_s$ for some $s \in [p]$, then*

$$|\mathcal{R}_f(z, h)| = |C_s| \geq \left\lceil \frac{h-p}{\mathrm{D}_f} \right\rceil + 1.$$

Finally, we can get to the main theorem of our paper, which again generalizes the afore-mentioned result of Coxson, Larson and Schneider (Theorem 1.1). The proof of Theorem 1.11 will be based on Theorems 1.8 and 1.10.

Theorem 1.11. *Let k be a positive integer and let $f \in \mathcal{B}_k$ be a digraph. Let $Y = f^h(X)$ where $X \subseteq [k]$ and $h \in \mathbb{N}$.*

- *(A) If Y falls into a strongly connected component of f, then there exists $h' \in \mathbb{N}$ such that $h' \leq |Y| L_f$ and $Y = f^{h'}(X)$.*
- *(B) If f itself is irreducible, then there exists $h' \in \mathbb{N}$ such that $h' \leq |Y| \mathrm{D}_f$ and $Y = f^{h'}(X)$.*

Consider the phase space of an irreducible map $f \in \mathcal{B}_K$. For any $Y \subseteq K$, we learn from Theorem 1.11(B) that we will either reach Y in $|Y| \mathrm{D}_f$ steps, or will never run into it at all throughout our endless life in $\mathcal{PS}_f$. If f is assumed to be primitive, Neufeld [33] and Shen [44] independently proved that we will always arrive at $Y = K$ in $\mathrm{D}_f^2 + 1$ steps, which is better than our previous bound of $|Y| \mathrm{D}_f = |K| \mathrm{D}_f$. This result is among the very few estimates of $\mathrm{Ł}_{\mathcal{PS}_f}$ in terms of D_f and is regarded as

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one of the best work in the area of exponents for nonnegative matrices [30, p. 3537]. It may be interesting to see if there is a bound for the number of steps to reach a reachable set Y which, when specified to $Y = K$, can match the diameter bound given by Neufeld and Shen.

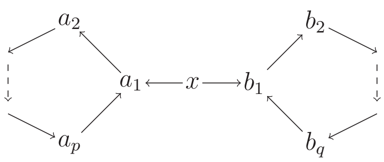

FIG. 1.2. *A digraph f with $p+q+1$ vertices and two cycles.*

EXAMPLE 1.12. Let f be the digraph in Fig. 1.2. Assume that p and q are coprime and $q > p$. The distance from the vertex $X = \{x\}$ to the vertex $Y = \{a_p, b_q\}$ in $\mathcal{PS}_f$ is pq, which can be much bigger than $|Y|\L_f = 2q$ when p is much bigger than 2. Note that Y is not contained in any strongly connected component of f and so this fact does not violate Theorem 1.11(A).

EXAMPLE 1.13. Let f be the digraph shown in Fig. 1.3. Observe that $(4, 3) = (\L_f, \mathrm{D}_f)$ and

$$\{a\} \to \{b, d\} \to \{c, e\} \to \{d\} \to \{e\}$$

is a length 4 path in $\mathcal{PS}_f$. This illustrates the necessity of the irreducibility assumption in Theorem 1.11(B) and that we cannot replace $\L_f$ by D_f in Theorem 1.11(A).

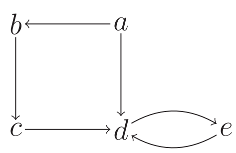

FIG. 1.3. *A digraph f for which $\mathcal{PS}_f$ contains a path of length $\L_f > \mathrm{D}_f$ ending at a singleton set.*

We have established bounds for the expansion property of a Boolean linear map which involve both girth and diameter. If we compare Theorem 1.2 with Theorem 1.8, say, we see that the diameter bound will be better for those irreducible f satisfying $\mathrm{g}_f \in \{\mathrm{D}_f, \mathrm{D}_f + 1\}$ unless f is a cycle. Similarly, Corollary 1.9 improves Corollary 1.6 for digraphs f satisfying $\mathrm{g}_f = \mathrm{D}_f + 1$. It seems that no characterization of those digraphs f with $\mathrm{g}_f = \mathrm{D}_f + 1$ is known. But one such primitive digraph is the lexicographic product of two n-cycles. In general, we have the following construction.

EXAMPLE 1.14. Let $G_1, \ldots, G_n$ be n nontrivial strongly connected digraphs on disjoint vertex sets with girth n and diameter $n-1$. Let G be the digraph whose vertex

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set is $\cup_{i\in[n]}V(G_i)$ and whose arc set is the the union of those arcs of $G_i, i \in [n]$, and those pairs $u \to v$ where $u \in V(G_i)$ and $v \in V(G_{i+1})$, $i \in [n]$, using the convention that $G_{n+1} = G_1$. It is clear that G is primitive and its diameter and girth are $n-1$ and n, respectively. Note that G may not be obtained from a lexicographic product construction.

Example 1.15. Recall the definition of the Wielandt digraphs given before Example 1.3. Here are some well-known basic properties of $f = W_k$.

1) $\mathrm{g}_f = \mathrm{D}_f = k-1$ and the cyclicity of f is $p = 1$.
2) For any positive integer h and any $z \in [k]$,

$$\mathcal{R}_f(z,h) = \left\{\underline{z+h}, \underline{z+h+1}, \ldots, \underline{z+h+\left\lfloor \frac{h+z-2}{k-1} \right\rfloor}\right\},$$

where $\underline{q}$ refers to the number $q - \left\lfloor \frac{q-1}{k} \right\rfloor k$.
3) For any positive integer h and any $z \in [k]$, $|\,\mathcal{R}_f(z,h)| = \min\{k, \left\lfloor \frac{h+z-2}{k-1} \right\rfloor + 1\}$.

They allow us to show the sharpness of some previous results in this section.

- Take $s \in [k-1]$. For $z = 1$ and $h = s(k-1)$, it holds $|\,\mathcal{R}_f(z,h)| = \left\lfloor \frac{s(k-1)-1}{k-1} \right\rfloor + 1 = s$, namely $h = |\,\mathcal{R}_f(z,h)|\,\mathrm{D}_f = (|\,\mathcal{R}_f(z,h)| - 1)\,\mathrm{g}_f + k - 1$. This indicates the sharpness of Theorem 1.2 and Theorem 1.8(A).
- Take $s \in [k-1]$. For $z = k$ and $h = (s-1)(k-1)$, it holds $|\,\mathcal{R}_f(z,h)| = \left\lfloor \frac{(s-1)(k-1)+k-2}{k-1} \right\rfloor + 1 = s$, namely $h = (|\,\mathcal{R}_f(z,h)| - 1)\,\mathrm{D}_f$. Moreover, $f(z) = \{1,2\}$, $|\,\mathcal{R}_f(1,h-1)| = \left\lfloor \frac{(s-1)(k-1)-1+1-2}{k-1} \right\rfloor + 1 = s-1$ and $|\,\mathcal{R}_f(2,h-1)| = \left\lfloor \frac{(s-1)(k-1)-1+2-2}{k-1} \right\rfloor + 1 = s-1$. So, $|\,\mathcal{R}_f(z,h)| > |\,\mathcal{R}_f(u,h-1)$ does hold for all $u \in f(z)$. This shows the sharpness of Theorem 1.8(B).
- Let $z = 1$ and $h = (k-1)^2 + 1$. Observe that $|\,\mathcal{R}_f(z,h)| = \left\lfloor \frac{(k-1)^2+1+1-2}{k-1} \right\rfloor + 1 = k$ and $|\,\mathcal{R}_f(z,h-p)| = \left\lfloor \frac{(k-1)^2+1-2}{k-1} \right\rfloor + 1 = k-1$. Consequently, $\mathcal{R}_f(z,h-1) \subsetneq \mathcal{R}_f(z,h) = [k]$. It is readily checked that $\frac{h-p}{\mathrm{D}_f} + 1 = \frac{(k-1)^2}{k-1} + 1 = k$, demonstrating the sharpness of Theorem 1.10.

In the next section, we provide the proofs of our results mentioned above. In Section 3, we discuss several interesting variants and generalizations of our work here. The discussions in Section 3 also provide more background for our work.

2. Proofs.

We call $f \in \mathcal{B}_k$ *non-shrinking* provided, for every $A \in 2^{[k]}$ and every $i \in \mathbb{N}$, $f^i(A)$ is never a proper subset of A. It is easy to see that if $f \in \mathcal{B}_k$ is strongly connected then it is non-shrinking. If f is non-shrinking and A is transient for f, then

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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$f^t(A) \setminus A$ is always nonempty for all positive integers t.

LEMMA 2.1. *Let k and ℓ be two positive integers and let $f \in \mathcal{B}_k$ be non-shrinking. Take $z \in [k]$ and $j \in \mathbb{N}$ such that $\mathcal{R}_f(z, \ell + j)$ is transient for f. If $z \in \mathcal{R}_f(z, \ell)$, then $\mathcal{R}_f(z, j) \subsetneq \mathcal{R}_f(z, \ell + j)$.*

Proof. As $\mathcal{R}_f(z, \ell + j)$ is transient for the non-shrinking map f, so is $\mathcal{R}_f(z, j)$. This tells us

$$\mathcal{R}_f(z, j) \neq f^{\ell}(\mathcal{R}_f(z, j)) = \mathcal{R}_f(z, \ell + j).$$

In addition, from $z \in \mathcal{R}_f(z, \ell)$ we derive that

$$\mathcal{R}_f(z, j) \subseteq f^{j}(\mathcal{R}_f(z, \ell)) = \mathcal{R}_f(z, \ell + j),$$

completing the proof. ◻

Proof of Theorem 1.2. If $h < k - 1$, Eq. (1.1) surely holds as a strict inequality. We proceed under the assumption that

$$h \geq k - 1. \tag{2.1}$$

Since $\mathcal{R}_f(z, h)$ is transient, we know that f cannot be a cycle, implying that

$$\mathrm{g}_f \in [k - 1]. \tag{2.2}$$

Let C be a cycle of f of length g_f. Write ℓ_z for $\mathrm{Dist}_f(z, C)$ and let a be a vertex on C such that $\ell_z = \mathrm{Dist}_f(z, a)$. We let P_{za} be one shortest path from z to a in f. Walking from z to a along P_{za} and then going around the cycle C for $\mathrm{g}_f - 1$ steps gives rise to a path of f, which we denote by P. Let ℓ_P be the length of P. It surely holds

$$k - 1 \geq \mathrm{L}_f \geq \ell_P = \ell_z + \mathrm{g}_f - 1. \tag{2.3}$$

Since $a \in \mathcal{R}_f(z, \ell_z)$, we find that

$$\mathcal{R}_f(a, h - \ell_z) \subseteq f^{h-\ell_z}(\mathcal{R}_f(z, \ell_z)) \subseteq \mathcal{R}_f(z, h), \tag{2.4}$$

and thus, as $\mathcal{R}_f(z, h)$ is transient for f, so is $\mathcal{R}_f(a, h - \ell_z)$. Let

$$m = \left\lceil \frac{h - \ell_z + 1}{\mathrm{g}_f} \right\rceil,$$

which, according to Eq. (2.1), is a positive integer. By virtue of $a \in \mathcal{R}_f(a, \mathrm{g}_f)$, Lemma 2.1 applies now to give

$$\mathcal{R}_f(a, h - \ell_z) \supsetneq \mathcal{R}_f(a, h - \ell_z - \mathrm{g}_f) \supsetneq \cdots \supsetneq \mathcal{R}_f(a, h - \ell_z - (m-1)\,\mathrm{g}_f) \neq \emptyset.$$

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A publication of the International Linear Algebra Society
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This together with Eq. (2.4) forces $m-1 \leq |\mathcal{R}_f(a, h-\ell_z)| - |\mathcal{R}_f(a, h-\ell_z-(m-1)\,\mathrm{g}_f)| \leq |\mathcal{R}_f(z,h)| - 1$, whence

$$m \leq |\mathcal{R}_f(z,h)|. \tag{2.5}$$

Substituting $\left\lceil \frac{h-\ell_z+1}{\mathrm{g}_f} \right\rceil$ for m in Eq. (2.5) yields

$$h \leq \ell_z + |\mathcal{R}_f(z,h)|\,\mathrm{g}_f - 1 = (|\mathcal{R}_f(z,h)| - 1)\,\mathrm{g}_f + (\ell_z + \mathrm{g}_f - 1).$$

This along with Eq. (2.3) establishes Eq. (1.1).

Now assume that Eq. (1.1) holds as an equality. Then checking the above proof yields that $h \geq k-1$ and that equality holds throughout Eq. (2.3). In view of Eq. (2.2), this tells us that z must be outside of C and the path P contains all vertices of f. Since P_{za} is a shortest path from z to the cycle C in f, we find that $\mathrm{Dist}_f(z,C) > \mathrm{Dist}_f(y,C)$ for every $y \in [k] \setminus \{z\}$. Since C is chosen to be any cycle of f of length g_f, the proof is complete. ☐

Proof of Corollary 1.4. For any $z \in [k]$ and $h \in \mathbb{N}$ with $|\mathcal{R}_f(z,h)| \leq k-1$, Theorem 1.2 tells us

$$h \leq (|\mathcal{R}_f(z,h)| - 1)\,\mathrm{g}_f + k - 1 \leq (k-2)\,\mathrm{g}_f + k - 1.$$

So z will reach the vertex $[k]$ in $\mathcal{PS}_f$ in at most $(k-2)\,\mathrm{g}_f + k$ steps. This means that

$$\mathrm{L}_{\mathcal{PS}_f} \leq (k-2)\,\mathrm{g}_f + k, \tag{2.6}$$

which is exactly the Dulmage-Mendelsohn bound [21, Theorem 1] mentioned before Theorem 1.2. Because f is primitive, we can deduce that $\mathrm{g}_f \leq k-1$, and so Eq. (2.6) gives $\mathrm{L}_{\mathcal{PS}_f} \leq (k-1)^2 + 1$.

To have $\mathrm{L}_{\mathcal{PS}_f} = (k-1)^2 + 1$, the girth of f must be $k-1$. But it is not hard to check that, up to isomorphism, W_k and $\widetilde{W}_k$ are the only two primitive digraphs from $\mathcal{B}_k$ with girth $k-1$ [21, Theorem 6] [30, Lemma 2.3]. From Example 1.3 we see that $\mathrm{L}_{\mathcal{PS}_f} < (k-1)^2 + 1$ for $f = \widetilde{W}_k$. On the other hand, it is straightforward to check that $\mathrm{L}_{\mathcal{PS}_f} = (k-1)^2 + 1$ for $f = W_k$ (See Example 1.15). ☐

Lemma 2.2. *If $S_1, \ldots, S_m$ are m subsets of a k-element set K such that $|S_i| \geq s$ and $ms - (m-1)k \geq n$, then $|\cap_{i\in[m]} S_i| \geq n$.*

Proof. This follows from the pigeon hole principle. ☐

Proof of Corollary 1.6. Let $s = \left\lfloor \frac{km-k+n}{m} \right\rfloor$ and let $t = km - k + n - ms$. We verify the claim in two cases separately.

Case 1. $m \mid km - k + n$, namely $t = 0$.

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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Let $h \geq (s-2)\,\mathrm{g}_f + k$. Pick any $z \in [k]$. If $\mathcal{R}_f(z,h) = [k]$, then it is immediate that $|\,\mathcal{R}_f(z,h)| = k \geq s$. If $\mathcal{R}_f(z,h) \subsetneq [k]$, then $\mathcal{R}_f(z,h)$ is transient and thus it follows from Theorem 1.2 that $|\,\mathcal{R}_f(z,h)| \geq s$. It is clear that $ms-(m-1)k = n$. So, the claim is readily verified using Lemma 2.2.

Case 2. $m \nmid km-k+n$, namely $t > 0$ and $s \leq k-1$.

Let $h \geq (s-1)\,\mathrm{g}_f + k - 1$ and let $z_1, \ldots, z_m$ be m elements from $[k]$. We claim that

$$|\,\mathcal{R}_f(z_i,h)| \geq s \tag{2.7}$$

for every $i \in [m]$. Indeed, if $\mathcal{R}_f(z_i,h)$ is transient for f, Eq. (2.7) is a consequence of Theorem 1.2; if $\mathcal{R}_f(z_i,h)$ is not transient for f, it is direct that $|\,\mathcal{R}_f(z_i,h)| = k > s$ and so Eq. (2.7) still follows. Applying Theorem 1.2 again, we know that there exists at most one $i \in [m]$ such that $|\,\mathcal{R}_f(z_i,h)| = s$. This gives

$$\sum_{i=1}^{m} |\,\mathcal{R}_f(z_i,h)| \geq ms + m - 1 \geq ms + t = (m-1)k + n.$$

By the pigeon hole principle, we derive

$$\left| \bigcap_{i\in[m]} \mathcal{R}_f(z_i,h) \right| \geq n,$$

which shows that $\alpha_{m,n}(f) \leq k - 1 + \left(\left\lfloor \frac{km-k+n}{m} \right\rfloor - 1 \right) \mathrm{g}_f$, as wanted. $\square$

Proof of Corollary 1.7. As f is primitive, we see that $\mathrm{g}_f \leq k-1$. Now we distinguish two cases.

Case 1. $\mathrm{g}_f \leq k-2$.

By Corollary 1.6,

$$\alpha_{2,1}(f) \leq \begin{cases} k + (\frac{k+1}{2} - 2)\,\mathrm{g}_f \leq (k^2-3k+6)/2, & \text{if } 2 \mid k+1; \\ k - 1 + (\frac{k}{2} - 1)\,\mathrm{g}_f \leq (k^2-2k+2)/2, & \text{if } 2 \mid k. \end{cases}$$

Since $\mathrm{g}_f \leq k-2$, we know that $k \geq 3$ and so $\lceil (k^2-3k+6)/2 \rceil \leq \lceil (k^2-2k+2)/2 \rceil$. This then leads to

$$\alpha_{2,1}(f) \leq \left\lceil \frac{k^2-2k+2}{2} \right\rceil = \left\lceil \frac{(k-1)^2+1}{2} \right\rceil.$$

Case 2. $\mathrm{g}_f = k-1$.

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A publication of the International Linear Algebra Society
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In this case, f is isomorphic to either W_k or $\widetilde{W}_k$ [21, Theorem 6] [30, Lemma 2.3]. Since W_k is a subgraph of $\widetilde{W}_k$, it holds $\alpha_{2,1}(\widetilde{W}_k) \le \alpha_{2,1}(W_k)$. Let $h = \left\lceil \frac{(k-1)^2+1}{2} \right\rceil$, $f = W_k$ and take a, b so that $1 \le a \le b \le k$. It remains to show $\mathcal{R}_f(a,h) \cap \mathcal{R}_f(b,h) \ne \emptyset$. By property 3) in Example 1.15,

$$|\mathcal{R}_f(z,h)| = \left\lfloor \frac{h+z-2}{k-1} \right\rfloor + 1 = \left\lfloor \frac{\left\lceil \frac{k^2}{2} \right\rceil + z - 2}{k-1} \right\rfloor \ge \left\lceil \frac{k}{2} \right\rceil \tag{2.8}$$

holds for $z \in [k]$ and

$$|\mathcal{R}_f(z,h)| = \left\lfloor \frac{h+z-2}{k-1} \right\rfloor + 1 = \left\lfloor \frac{\left\lceil \frac{k^2}{2} \right\rceil + z - 2}{k-1} \right\rfloor \ge \left\lceil \frac{k}{2} \right\rceil + 1 \tag{2.9}$$

holds when k is even and $z \in [k] \setminus [k/2]$. If $b \le \lceil k/2 \rceil$, property 2) in Example 1.15 along with Eq. (2.8) shows that $\mathcal{R}_f(a,h) \cap \mathcal{R}_f(b,h) \ne \emptyset$. If $b \ge \lceil k/2 \rceil + 1$, combining Eq. (2.8) and Eq. (2.9) yields $|\mathcal{R}_f(a,h)| + |\mathcal{R}_f(b,h)| > k$, finishing the proof. ◻

LEMMA 2.3. *Let k be a positive integer, f be a non-shrinking map from $\mathcal{B}_k$ and $T \in 2^{[k]}$ be a transient set for f. Suppose that there exist $x, z \in [k]$ and $i, j \in \mathbb{N}$ such that $\mathcal{R}_f(z, i+j) = \mathcal{R}_f(x,i) = T$ and $x \in \mathcal{R}_f^*(z)$. Then $j \le \mathrm{Dist}_f(z,x) \le \mathrm{D}_f$.*

Proof. Let $t = \mathrm{Dist}_f(z,x)$. Assume, for the sake of contradiction, that $j > t$. Noting that $x \in \mathcal{R}_f(z,t)$, we obtain

$$f^{j-t}\big(\mathcal{R}_f(z,t+i)\big) = \mathcal{R}_f(z,i+j) = T = \mathcal{R}_f(x,i) \subseteq f^i\big(\mathcal{R}_f(z,t)\big) = \mathcal{R}_f(z,t+i).$$

Since f is non-shrinking, equalities must hold throughout the above formula and so $f^{j-t}(T) = T$. But this then contradicts the fact that T is transient for f. ◻

LEMMA 2.4. *Let k be a positive integer and f be a nontrivial irreducible map from $\mathcal{B}_k$. Take $z \in [k]$ and $h \in \mathbb{N}$ such that $\mathcal{R}_f(z,h)$ is transient for f. Let ℓ be the length of a shortest cycle of f containing z. If there exists one vertex $a \in f(z)$ such that $\mathcal{R}_f(z,h) \supsetneq \mathcal{R}_f(a,h-1)$ holds, then $h \le (|\mathcal{R}_f(z,h)| - 1)\ell$.*

Proof. We write m for the positive integer $\lceil (h+1)/\ell \rceil$. By assumption, we have $z \in \mathcal{R}_f(z,\ell)$. Applying Lemma 2.1 yields

$$\mathcal{R}_f(z,h) \supsetneq \mathcal{R}_f(z,h-\ell) \supsetneq \cdots \supsetneq \mathcal{R}_f(z,h-(m-1)\ell) \ne \emptyset,$$

which implies that

$$m - 1 \le |\mathcal{R}_f(z,h)| - |\mathcal{R}_f(z,h-(m-1)\ell)|. \tag{2.10}$$

CASE 1. $|\mathcal{R}_f(z,h-(m-1)\ell)| \ge 2$.

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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It follows from Eq. (2.10) that $m-1 \le |\mathcal{R}_f(z,h)|-2$ and so $h < m\ell \le (|\mathcal{R}_f(z,h)|-1)\ell$, as desired.

Case 2. $|\mathcal{R}_f(z, h-(m-1)\ell)| = 1$.

By Eq. (2.10), $m \le |\mathcal{R}_f(z,h)|$. Therefore, it suffices to verify $h=(m-1)\ell$.

If $h=(m-1)\ell$ were not true, then $h>(m-1)\ell$ and so from $a \in f(z)$ we deduce that $\mathcal{R}_f(a, h-(m-1)\ell-1)$ is a subset of the singleton set $\mathcal{R}_f(z, h-(m-1)\ell)$. This implies that $\mathcal{R}_f(a, h-(m-1)\ell-1) = \mathcal{R}_f(z, h-(m-1)\ell)$. Consequently,

$$\begin{aligned}\mathcal{R}_f(a,h-1) &= f^{(m-1)\ell}\big(\mathcal{R}_f(a,h-(m-1)\ell-1)\big)\\ &= f^{(m-1)\ell}\big(\mathcal{R}_f(z,h-(m-1)\ell)\big)\\ &= \mathcal{R}_f(z,h),\end{aligned}$$

a contradiction with the assumption that $\mathcal{R}_f(z,h) \supsetneq \mathcal{R}_f(a,h-1)$. □

Proof of Theorem 1.8. If $\mathrm{D}_f = 1$, a routine check confirms both (A) and (B). In the following, we always assume that $\mathrm{D}_f \ge 2$.

Let j be the maximum nonnegative integer t such that

- $t \le h$, and
- $\mathcal{R}_f(z,t)$ contains an element x for which $\mathcal{R}_f(x,h-t) = \mathcal{R}_f(z,h)$.

Let $i = h-j$. Pick $x \in \mathcal{R}_f(z,j)$ such that $\mathcal{R}_f(x,i) = \mathcal{R}_f(z,h)$. By setting $T = \mathcal{R}_f(z,h)$, we can deduce from Lemma 2.3 that

$$j \le \mathrm{D}_f\,. \tag{2.11}$$

Let us use (A_α) and (B_α) to stand for the statements in claim (A) and claim (B), respectively, under the additional assumption of $|\mathcal{R}_f(z,h)| = \alpha$. Our goal is to verify (A_α) for all positive integers α and (B_α) for all integers α greater than 1. We proceed by induction on $|\mathcal{R}_f(z,h)| = \alpha$ and divide the proof into the following three steps.

Step 1. Verify (A_1).

If $|\mathcal{R}_f(z,h)| = 1$, we have $i=0$ and so

$$\begin{aligned}h &= i+j\\ &= j\\ &\le \mathrm{D}_f && \text{(By Eq. (2.11))}\\ &= |\mathcal{R}_f(z,h)|\,\mathrm{D}_f\,.\end{aligned}$$

This proves (A_1).

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A publication of the International Linear Algebra Society
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Step 2. For $\alpha \geq 2$, verify (B_α) under the assumption that (B_β) holds for all possible $\beta \in \{2, \ldots, \alpha-1\}$ and (A_β) holds for all positive integers $\beta \leq \alpha - 1$.

We assume $|\mathcal{R}_f(z,h)| = \alpha \geq 2$ and

$$|\mathcal{R}_f(z,h)| > |\mathcal{R}_f(u,h-1)| \tag{2.12}$$

for all $u \in f(z)$. If $h \leq \mathrm{D}_f$, there is nothing to prove. So, assume

$$h \geq \mathrm{D}_f + 1 \tag{2.13}$$

hereafter.

Case 1. The vertex z is on a cycle of f whose length is no more than D_f.

Thanks to Lemma 2.4, $h \leq (\alpha - 1)\,\mathrm{D}_f$.

Case 2. All cycles of f passing through z have lengths at least $\mathrm{D}_f + 1$.

Take two vertices a and b from $f(z)$ such that it holds

$$\begin{cases} \mathcal{R}_f(a,h-1) \subsetneq \mathcal{R}_f(z,h); \\ \mathcal{R}_f(b,h-1) \subsetneq \mathcal{R}_f(z,h); \\ \mathcal{R}_f(a,h-1) \neq \mathcal{R}_f(b,h-1). \end{cases} \tag{2.14}$$

Let

$$\beta = \min\{|\mathcal{R}_f(a,h-1)|, |\mathcal{R}_f(b,h-1)|\}.$$

Eq. (2.14) says that $\beta \leq \alpha - 1$ and so our task is to address both the case of $\beta \leq \alpha - 2$ and the case of $\beta = \alpha - 1$.

Case 2.1. $\beta \leq \alpha - 2$.

Without loss of generality, assume that $\beta = |\mathcal{R}_f(a,h-1)|$. It follows from (A_β) that $h - 1 \leq |\mathcal{R}_f(a,h-1)|\,\mathrm{D}_f = \beta\,\mathrm{D}_f \leq (\alpha-2)\,\mathrm{D}_f$. Clearly, this gives $h \leq (\alpha-2)\,\mathrm{D}_f + 1 < (\alpha-1)\,\mathrm{D}_f$.

Case 2.2. $\beta = |\mathcal{R}_f(a,h-1)| = |\mathcal{R}_f(b,h-1)| = \alpha - 1$.

Since $\max\{\mathrm{Dist}_f(a,z), \mathrm{Dist}_f(b,z)\} \leq \mathrm{D}_f$, we conclude from our standing assumption for Case 2 that

$$z \in \mathcal{R}_f(a,\mathrm{D}_f) \cap \mathcal{R}_f(b,\mathrm{D}_f). \tag{2.15}$$

In view of Eqs. (2.13) and (2.15), we have

$$\begin{cases} \mathcal{R}_f(a,h-1) = f^{h-\mathrm{D}_f-1}(\mathcal{R}_f(a,\mathrm{D}_f)) \supseteq \mathcal{R}_f(z,h-\mathrm{D}_f-1); \\ \mathcal{R}_f(b,h-1) = f^{h-\mathrm{D}_f-1}(\mathcal{R}_f(b,\mathrm{D}_f)) \supseteq \mathcal{R}_f(z,h-\mathrm{D}_f-1). \end{cases} \tag{2.16}$$

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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Combining (2.12), (2.14) and (2.16) yields

$$|\,\mathcal{R}_f(z, h-\mathrm{D}_f-1)| \le \alpha - 2. \tag{2.17}$$

This means that we only need to treat the following two cases.

Case 2.2.1. $|\,\mathcal{R}_f(z, h-\mathrm{D}_f-1)| \le \alpha - 3$.

Applying (A_γ) for $\gamma = |\,\mathcal{R}_f(z, h-\mathrm{D}_f-1)| \le \alpha-3$, we have $h-\mathrm{D}_f-1 \le \gamma\,\mathrm{D}_f \le (\alpha-3)\,\mathrm{D}_f$ and so $h \le (\alpha-3)\,\mathrm{D}_f+\mathrm{D}_f+1 = (\alpha-2)\,\mathrm{D}_f+1 < (\alpha-1)\,\mathrm{D}_f$.

Case 2.2.2. $|\,\mathcal{R}_f(z, h-\mathrm{D}_f-1)| = \alpha - 2$.

It follows from Eq. (2.17) that $\alpha \ge 3$. So, (B_α) holds trivially when $h \le 2\,\mathrm{D}_f$. Let us assume below that

$$h \ge 2\,\mathrm{D}_f+1 \ge \mathrm{D}_f+2. \tag{2.18}$$

Eq. (2.18) allows us to derive from $a \in f(z)$ that

$$\mathcal{R}_f(a, h-\mathrm{D}_f-2) \subseteq \mathcal{R}_f(z, h-\mathrm{D}_f-1). \tag{2.19}$$

If $|\,\mathcal{R}_f(a, h-\mathrm{D}_f-2)| = \alpha-2$, then it holds $\mathcal{R}_f(a, h-\mathrm{D}_f-2) = \mathcal{R}_f(z, h-\mathrm{D}_f-1)$ and hence

$$\begin{aligned}\mathcal{R}_f(a, h-1) &= f^{\mathrm{D}_f+1}\big(\mathcal{R}_f(a, h-\mathrm{D}_f-2)\big)\\ &= f^{\mathrm{D}_f+1}\big(\mathcal{R}_f(z, h-\mathrm{D}_f-1)\big)\\ &= \mathcal{R}_f(z,h),\end{aligned}$$

a contradiction to Eq. (2.14). Recalling that $|\,\mathcal{R}_f(z, h-\mathrm{D}_f-1)| = \alpha-2$, we can now infer from Eq. (2.19) that $|\,\mathcal{R}_f(a, h-\mathrm{D}_f-2)| \le \alpha-3$.

Applying (A_γ) for $\gamma = |\,\mathcal{R}_f(a, h-\mathrm{D}_f-2)| \le \alpha-3$, we acquire $h-\mathrm{D}_f-2 \le \gamma\,\mathrm{D}_f \le (\alpha-3)\,\mathrm{D}_f$. This implies

$$h \le (\alpha-3)\,\mathrm{D}_f+\mathrm{D}_f+2 \le (\alpha-1)\,\mathrm{D}_f,$$

as was to be shown.

Step 3. For $\alpha \ge 2$, verify (A_α) under the assumption that (B_α) holds.

We assume $|\,\mathcal{R}_f(z,h)| = \alpha > 2$ and so $i \ge 1$. According to our choice of i and j, we see that $|\,\mathcal{R}_f(x,i)| = |\,\mathcal{R}_f(z,h)| = \alpha$ while $|\,\mathcal{R}_f(u,i-1)| < |\,\mathcal{R}_f(z,h)| = \alpha$ for every $u \in f(x)$. Utilizing (B_α) gives $i \le (|\,\mathcal{R}_f(x,i)|-1)\,\mathrm{D}_f = \alpha\,\mathrm{D}_f$. Together with Eq. (2.11), this implies

$$h = i + j \le (\alpha-1)\,\mathrm{D}_f+\mathrm{D}_f = \alpha\,\mathrm{D}_f,$$

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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proving (A_α). ◻

Proof of Corollary 1.9. Let $s = \left\lfloor k - \frac{k-n+1}{m} \right\rfloor + 1$. Take any $z \in [k]$ and any $h \geq (s-1)\,\mathrm{D}_f + 1$. By Lemma 2.2, our task boils down to showing that $|\,\mathcal{R}_f(z,h)| \geq s$ and $ms - (m-1)k \geq n$.

Firstly, if $\mathcal{R}_f(z,h)$ is transient for f, then Theorem 1.8(A) asserts

$$|\,\mathcal{R}_f(z,h)| \geq \left\lfloor k - \frac{k-n+1}{m} \right\rfloor + 1 = s;$$

while if $\mathcal{R}_f(z,h)$ is recurrent for f, then

$$|\,\mathcal{R}_f(z,h)| = k \geq s.$$

Secondly, letting $p = \left\lfloor \frac{k-n+1}{m} \right\rfloor$ and $q = k - n + 1 - pm$, we have

$$\begin{aligned} ms - (m-1)k &= m(k - \left\lceil \frac{k-n+1}{m} \right\rceil + 1) - (m-1)k \\ &= k + m - m \left\lceil \frac{k-n+1}{m} \right\rceil \\ &= k + m - mp - m \left\lceil \frac{q}{m} \right\rceil \\ &= n + q - 1 + m(1 - \left\lceil \frac{q}{m} \right\rceil) \\ &\geq n. \quad \square \end{aligned}$$

Proof of Theorem 1.10. We may assume that $z \in C_1$ and that $f(C_i) \subseteq C_j$ holds for all $i, j \in [p]$ satisfying $j - i \equiv 1 \pmod{p}$. From $\mathcal{R}_f(z,h) = C_s$ we derive that $s \equiv h + 1 \pmod{p}$.

Taking into account $\mathcal{R}_f(z,h-p) \subsetneq C_s$, we know that $\mathcal{R}_f(z,h-p)$ is transient for f. Therefore, Theorem 1.8 gives

$$|\,\mathcal{R}_f(z,h-p)| \geq \frac{h-p}{\mathrm{D}_f},$$

and so it holds $|\,\mathcal{R}_f(z,h)| \geq |\,\mathcal{R}_f(z,h-p)| + 1 \geq \frac{h-p}{\mathrm{D}_f} + 1$, as wanted. ◻

Lemma 2.5. *Let f be a strongly connected digraph with cyclicity p. If G has at least $p+1$ vertices, then $L_f \geq \mathrm{D}_f \geq p$.*

Proof. The digraph f has p cyclicity classes and so one of them must have size larger than 1. Take two different vertices x and y from this cyclicity class. The shortest path from x to y has length at least p and so we are done. ◻

Proof of Theorem 1.11. Claim (B) is immediate from Theorem 1.8, Theorem 1.10 and Lemma 2.5. We now focus on claim (A).

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Let h' be the minimum number such that $f^{h'}(X) = f^h(X)$. We aim to show that $h' \leq |Y| \text{Ł}_f$.

Let C be the strongly connected component of f which contains Y. Let $\overline{f}$ be the Boolean linear map on C such that $\overline{f}(y) = f(y) \cap C$ for every $y \in C$. Let p be the cyclicity of $\overline{f}$. Let

$$j = \min\{t \in \mathbb{N} : \ f^t(X) \subseteq C\}.$$

Case 1. There exists $y \in f^j(X)$ such that $f^{h'-j}(y) = \overline{f}^{h'-j}(y)$ is transient for $\overline{f}$.

Take the maximum nonnegative integer $i \leq h' - j$ such that we can find $z \in f^i(y)$ satisfying $f^{h'-j-i}(z) = f^{h'-j}(y)$. From Lemma 2.3 we get $i \leq \mathrm{Dist}_{\overline{f}}(y, z) = \mathrm{Dist}_f(y, z)$ and hence

$$i + j \leq \text{Ł}_f \tag{2.20}$$

follows. In light of Theorem 1.8 (B), we have $h' - j - i \leq (|f^{h'-j-i}(z)| - 1) \, \mathrm{D}_{\overline{f}} \leq (|Y| - 1) \, \mathrm{D}_{\overline{f}}$. This combined with Eq. (2.20) leads to

$$h' \leq (|Y| - 1) \, \mathrm{D}_{\overline{f}} + \text{Ł}_f \tag{2.21}$$

and so $h' \leq |Y| \text{Ł}_f$, as desired.

Case 2. For every $y \in f^j(X)$, $f^{h-j}(y) = \overline{f}^{h-j}(y)$ is recurrent for $\overline{f}$.

Case 2.1. $h' - j < p$.

It is clear that $h' < j + p - 1 \leq \text{Ł}_f$.

Case 2.2. $h' - j \geq p$.

By the minimality of h', there exists $y \in f^j(X)$ such that

$$f^{h'-j}(y) \supsetneq f^{h'-j-p}(y). \tag{2.22}$$

This means that $f^{h'-j-p}(y)$ is transient for f and so the argument deployed to obtain Eq. (2.21) demonstrates that

$$\begin{aligned} h' - p &\leq (|f^{h'-j-p}(y)| - 1) \, \mathrm{D}_{\overline{f}} + \text{Ł}_f \\ &\leq (|f^{h'-j}(y)| - 2) \, \mathrm{D}_{\overline{f}} + \text{Ł}_f \\ &\leq (|Y| - 2) \, \mathrm{D}_{\overline{f}} + \text{Ł}_f. \end{aligned} \tag{2.23}$$

As a consequence of Eq. (2.22), $|C| > p$ holds and so we could apply Lemma 2.5 to get $p \leq \text{Ł}_f$. It then follows from Eq. (2.23) that $h' \leq |Y| \text{Ł}_f$. $\square$

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3. Further research. The problem considered in this paper leaves vast possibilities of further research. We close this paper by addressing some related topics in the sequel.

3.1. Inhomogeneous. Given a finite set K, a Markov chain on the finite state space K is governed by a probability transition map, which naturally corresponds to a Boolean linear map f for which $y \in f(x)$ if and only if the Markov chain has positive probability to get to state $y \in K$ from state $x \in K$ in one step. For an inhomogeneous Markov chain, we will have several probability transition maps and so we need to consider Boolean linear dynamical system generated by several Boolean linear maps.

Let K be a nonempty finite set and let $\mathcal{F}$ be a subset of $\mathcal{B}_K$. The *phase space* of $\mathcal{F}$ has 2^K as its vertex set and has the union of the arc set of $\mathcal{PS}_f, f \in \mathcal{F}$, as its arc set. The Boolean linear dynamical system $(2^K, \mathcal{F})$ is *primitive* provided every walk of length $2^k - 2$ in $\mathcal{PS}_{\mathcal{F}}$ starting from a vertex inside $2^K \setminus \{\emptyset\}$ will reach K. If $\mathcal{F}$ is primitive, $\text{Ł}_{\mathcal{PS}_{\mathcal{F}}}$ is called the *primitive exponent* of $\mathcal{F}$. A Boolean linear map f on K is *essential* if $f(i) \neq \emptyset$ for every $i \in K$. The next conjecture is an inhomogeneous version of Theorem 1.11.

Conjecture 3.1. *Let $\mathcal{F}$ be a primitive set of essential Boolean linear maps on $[k]$. For any $A, B \in 2^{[k]}$, it holds $\text{Dist}_{\mathcal{PS}_{\mathcal{F}}}(A, B) \leq |B|k^{|\mathcal{F}|}$ as long as A can reach B in $\mathcal{PS}_{\mathcal{F}}$.*

If Conjecture 3.1 is correct, then for every integer $k \geq 2$ we can obtain

$$k \geq \gamma_k \geq \left\lceil \frac{\log_2(2^k - 2)}{\log_2 k} \right\rceil - 1,$$

where γ_k is the minimum size of a primitive set $\mathcal{F}$ of essential maps from $2^{[k]}$ to $2^{[k]}$ satisfying $\text{Ł}_{\mathcal{PS}_{\mathcal{F}}} = 2^k - 2$ [15, 49]. We suggest another conjecture which is a bit stronger than Conjecture 3.1 for the case of $|B| = 1$.

Conjecture 3.2. *Let k be a positive integer. Let $\mathcal{F}$ be a primitive set of essential Boolean linear maps on $[k]$ and take $x \in [k]$. Assume that*

$$X_1 \to X_2 \to \cdots \to X_{s+1}$$

is a walk of length s in $\mathcal{PS}_{\mathcal{F}}$ such that $x \notin X_i$ for $i \in [s+1]$. Then $s \leq k^{|\mathcal{F}|}$.

The ensuing example shows that we cannot improve the claim $s \leq k^{|\mathcal{F}|}$ in Conjecture 3.2 to $s \leq \prod_{f \in \mathcal{F}} \text{D}_f$.

Example 3.3. [Ziqing Xiang] Let m and n be two positive integers. Consider digraphs $f_{m,n}$ and $g_{m,n}$ on the same vertex set $V = \{x\} \cup \{w_{i,j} : i \in [m], j \in [n]\}$ with arc sets as specified below:

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

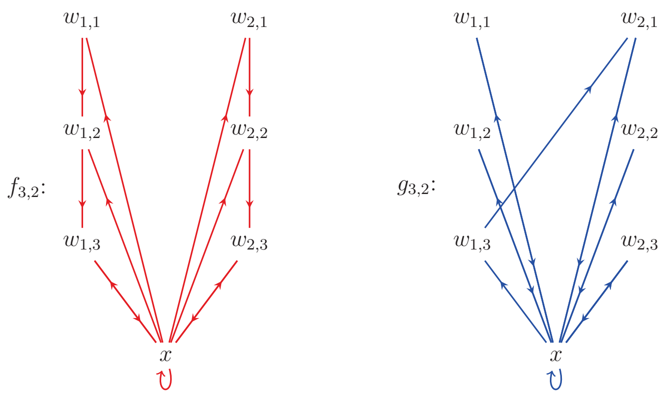

FIG. 3.1. *The two digraphs $f_{3,2}$ and $g_{3,2}$ as described in Example 3.3.*

- $f_{m,n}(w_{i,j}) = \{w_{i,j+1}\}$ for $i \in [m], j \in [n-1]$;
- $f_{m,n}(w_{i,n}) = \{x\}$ for $i \in [m]$;
- $f_{m,n}(x) = g_{m,n}(x) = V$;
- $g_{m,n}(w_{i,n}) = \{w_{i+1,1}\}$ for $i \in [m-1]$;
- $g_{m,n}(w_{i,j}) = \{x\}$ for $i \in [m], j \in [n-1]$;
- $g_{m,n}(w_{m,n}) = \{x\}$.

It is easy to check that $\mathrm{D}_{f_{m,n}} = m$, $\mathrm{D}_{g_{m,n}} = 2$ and $\{f_{m,n}, g_{m,n}\}$ is primitive. In $\mathcal{PS}_{f_{m,n},g_{m,n}}$,

$$X_1 \to X_2 \to \cdots \to X_{mn} \tag{3.1}$$

is a path of length mn, where

$$X_t = \{w_{\lceil \frac{t}{m} \rceil, t-(\lceil \frac{t}{m} \rceil -1)m}\}$$

for $t \in [mn]$. Note that $x \notin X_t$ for all $t \in [mn]$ and the length of the path displayed in Eq. (3.1) is larger than $\mathrm{D}_{f_{m,n}} \mathrm{D}_{g_{m,n}}$ provided $n > 2$.

3.2. Higher-order. In an order-t Markov chain [36, 45], the future state of the process depends on the past t states linearly. When $t = 1$, an order-t Markov chain becomes the usual Markov chain. To be more precise, an order-t Markov chain on a finite set K is specified by a probability transition hypermatrix

$$M = (M_{p_1,\ldots,p_{t+1}})_{p_1,\ldots,p_{t+1} \in K} \in \mathbb{R}^{\overbrace{K \times \cdots \times K}^{t+1}}, \tag{3.2}$$

where $M_{p_1,\ldots,p_{t+1}}$ are nonnegative numbers such that

$$\sum_{p_1 \in K} M_{p_1,\ldots,p_{t+1}} = 1$$

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A publication of the International Linear Algebra Society
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for all $p_2, \ldots, p_{t+1} \in K$. If the probability distribution of states at time $T+1, \ldots, T+t$ are $x^1, \ldots, x^t$ and if x_k represents the probability of taking state $k \in K$ in the distribution x, then in the order-t Markov chain governed by M, the probability distribution vector at time $T+t+1$ is given by $Mx^t \cdots x^1$, where

$$(Mx^t \cdots x^1)_k = \sum_{k_1, \ldots, k_t \in K} M_{kk_t \ldots k_1} x^t_{k_t} \cdots x^1_{k_1} \tag{3.3}$$

for all $k \in K$.

Replacing those vectors in Eq. (3.3) by their supports, we naturally yield the concept of Boolean multilinear maps as a models of a nonparametric version of higher-order Markov chains. A *Boolean t-linear map* on a finite set K is a map f from $(2^K)^t$ to 2^K such that

$$f(A_1, \ldots, A_t) = \emptyset$$

as long as one of $A_1, \ldots, A_t$ is empty, and that

$$\cup_{i_1 \cdots i_t \in [2]^t} f\big(A_{i_1}(1), \ldots, A_{i_t}(t)\big) = f\big(A_1(1) \cup A_2(1), \ldots, A_1(t) \cup A_2(t)\big)$$

for all $A_1(1), A_2(1), \ldots, A_1(t), A_2(t) \in 2^K$. We write $\mathcal{B}^t_K$ for the set of all Boolean t-linear maps on K.

Pick $f \in \mathcal{B}^t_K$. Let us extend several definitions for Boolean linear maps to the multilinear case so that we can consider possible generalizations of those results in Section 1. Let $\mathcal{R}_f$ be the map from $K^t \times \mathbb{N}$ to 2^K, called the *range map* of f, such that $\mathcal{R}_f(k_1, \ldots, k_t; i-1) = k_i$ for $i \in [t]$ and $\mathcal{R}_f(k_1, \ldots, k_t; i) = f\big(\mathcal{R}_f(k_1, \ldots, k_t; i-t), \ldots, \mathcal{R}_f(k_1, \ldots, k_t; i-1)\big)$ for $i \geq t$. The *digraph* of f, denoted by Γ_f, has vertex set K^t and arc set

$$\{(x_1, \ldots, x_t) \to (x_2, \ldots, x_t, x_{t+1}) : \; x_1, \ldots, x_t \in K, x_{t+1} \in f(x_1, \ldots, x_t)\}.$$

For $A = (A_1, \ldots, A_t) \in (2^K)^t$, let

$$\overrightarrow{f}(A) = \big(A_2, \ldots, A_t, f(A_1, \ldots, A_t)\big).$$

The *phase space* of f, denoted by $\mathcal{PS}_f$, has $(2^K)^t$ as its vertex set and has

$$\{X \to \overrightarrow{f}(X) : \; X \in (2^K)^t\}$$

as its arc set. The Boolean t-linear map f is called *primitive* if every vertex from $(2^K \setminus \{\emptyset\})^t$ will reach $(K, \ldots, K)$ in $\mathcal{PS}_f$. If f is primitive, the length of a longest path ending at $(K, \ldots, K)$ in $\mathcal{PS}_f$ is called the *primitive exponent* of f. Note that $(\emptyset, \ldots, \emptyset)$ and $(K, \ldots, K)$ are the only two vertices lying on a cycle of $\mathcal{PS}_f$ when f

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A publication of the International Linear Algebra Society
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is primitive. For primitive Boolean t-linear maps, Chen and Wu [13] have recently obtained generalizations of Theorems 1.2 and 1.8.

It is worth mentioning that $\{\Gamma_f : f \in \mathcal{B}_K^t\}$ coincides with the set of spanning subdigraphs of the $|K|$-nary t-dimensional De Bruijn digraph [9, 19]. In symbolic dynamics, the construction of Γ_f from a $f \in \mathcal{B}_K^t$ appears when recoding a t-step shift of finite type as a 1-step shift of finite type (edge shift) [28, Theorem 2.3.2] and is used as a symbolic trajectory of a higher-order Markov chain [28, §2.3]. Indeed, for this multilinear map f, letting

$$\mathscr{F} = \{x_1 \cdots x_{t+1} \in K^{t+1} : x_{t+1} \notin f(x_1, \ldots, x_t)\},$$

the shift of finite type $\mathsf{X}_{\mathscr{F}}$ [28, Chapter 2] and its dynamical properties should have close relation with the expansion property of f and this connection deserves further study. For given $f \in \mathcal{B}_K^t$, Γ_f itself is a Boolean linear map on K^t and, in essence, everything about the t-linear map f can be deduced from the knowledge of the linear map Γ_f. However, it seems nontrivial to tell the shape of $\mathcal{PS}_f$ even after knowing the shape of $\mathcal{PS}_{\Gamma_f}$.

Example 3.4. We specify a Boolean 2-linear map f on $K = [2]$ by depicting Γ_f in Fig. 3.2. Its phase space $\mathcal{PS}_f$ is given in Fig. 3.3. Viewing Γ_f as a Boolean 1-linear map on K^2, we draw its phase space $\mathcal{PS}_{\Gamma_f}$ in Fig. 3.4.

$$\begin{tikzcd}
11 \arrow[r] & 12 \arrow[dl] \arrow[d] \\
21 \arrow[u] & 22
\end{tikzcd}$$

Fig. 3.2. *The digraph Γ_f of a Boolean 2-linear map f on $K = [2]$.*

For any $f \in \mathcal{B}_K^t$, it is not hard to find that f is primitive whenever Γ_f is primitive. The next example says that the converse is not true. It also reflects the difficulty of really understanding an order-t Markov chain on K from its encoded order-1 Markov chain on K^t.

Example 3.5. Let $K = [2]$ and let $f \in \mathcal{B}_K^3$ be the one whose digraph Γ_f is as shown on the left hand side of Fig. 3.5. Note that Γ_f is even not strongly connected. On the right hand side of Fig. 3.5 we display part of the phase space $\mathcal{PS}_f$ of f, namely the subdigraph spanned by $(2^K \setminus \{\emptyset\})^3$. It can be seen there that the longest path ending at KKK in $\mathcal{PS}_f$ has length 7 and so we conclude that f is primitive with primitive exponent 7.

To comprehend those slowly expanding Boolean t-linear maps as well as the links

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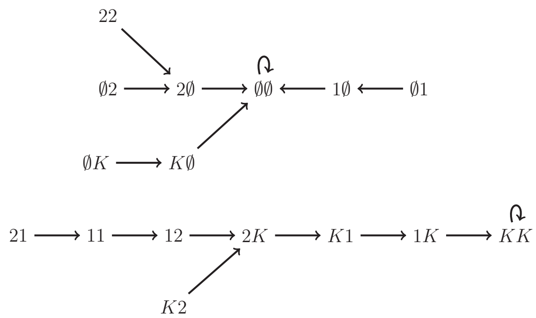

FIG. 3.3. *The phase space $\mathcal{PS}_f$ for the Boolean* 2*-linear map f on $K = [2]$ indicated in Fig. 3.2.*

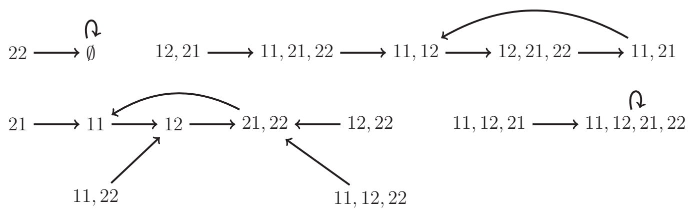

FIG. 3.4. *The phase space $\mathcal{PS}_{\Gamma_f}$ for the Boolean* 1*-linear map Γ_f on the set $K \times K = \{11, 12, 21, 22\}$ shown in Fig. 3.2.*

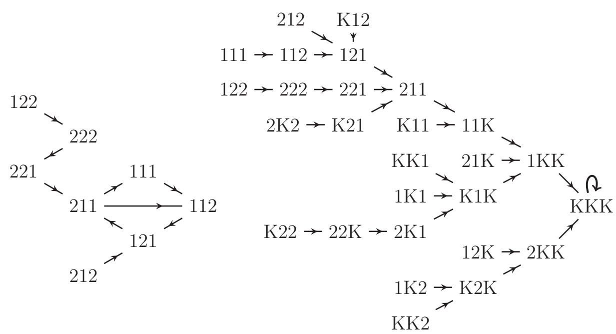

FIG. 3.5. *The digraph and part of the phase space of the* 3*-linear map in Example 3.5.*

between them and other objects [3, 5, 13, 39, 48], it may be important to search for a generalization of Wielandt's inequality.

CONJECTURE 3.6. *For any positive integer k, the maximum primitive exponent for which a primitive Boolean* 2*-linear map on $[k]$ can achieve is $O(k^2)$.*

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A publication of the International Linear Algebra Society
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The hypermatrix M in Eq. (3.2) is often defined in terms of tensors [11]. Both the spectral and combinatorial properties of tensors have been subjects of recent interest. Following the popular usage of notation, we may thus call the elements of $\mathcal{B}_K^t$ Boolean tensors. Chang, Pearson and Zhang initiated a study of primitive nonnegative tensors [10, Definition 2.6]. Since the primitiveness of a nonnegative tensor only relies on the zero pattern of the tensor, their definition naturally yields a definition of a class of Boolean tensors, which we call CPZ-primitive Boolean tensors. Our definition of primitive Boolean multilinear maps (Boolean tensors) is different from the definition of CPZ-primitive Boolean tensors. Indeed, the primitive Boolean tensor f given in Example 3.5 is not a CPZ-primitive Boolean tensor, which can be seen by appealing to [18, Theorem 4], not mentioning to be a strongly primitive Boolean tensor as defined in [43, Definition 4.3]. The primitivity concept introduced in this section will be systematically examined in the framework of multivariable graph theory in [48].

3.3. Increasing paths. Let K be a finite nonempty set and let $f \in \mathcal{B}_K$ be an irreducible map. A path in $\mathcal{PS}_f$ is *increasing* if the sizes of the vertices along the path never decreases. Motivated by Seymour's second neighborhood conjecture [12], we may want to estimate the length of a longest increasing path in $\mathcal{PS}_f$. We say that a transient vertex X is *lucky* if there exists an increasing path in $\mathcal{PS}_f$ which contains X and a recurrent vertex. A *Garden-of-Eden* in $\mathcal{PS}_f$ is a vertex X of $\mathcal{PS}_f$ for which there exists no Y such that $f(Y) = X$. Let $\mathcal{T}_f$ denote the set of transient vertices for f. It is not hard to see that there exists a set S of Garden-of-Edens in $\mathcal{PS}_f$ such that $\{f^{i+1}(X) : \ X \in S, i \in \mathbb{N}\} \cap \mathcal{T}_f$ are all the lucky vertices. What is the distribution of the lucky vertices in $\mathcal{PS}_f$? Is there any nontrivial lower bound for the number of lucky vertices in $\mathcal{PS}_f$?

3.4. An extremal problem. For any $f \in \mathcal{B}_k$ and any positive integer t, let us define

$$\sigma_t(f) = \sum_{i \in [k]} |\,\mathcal{R}_f(i,t)|.$$

To understand the expansion property of f, the prior subsection examines the increasing paths in $\mathcal{PS}_f$. We now look at the sequence $(\sigma_1(f), \sigma_2(f), \ldots)$. We say that f is *strictly expanding* if for every positive integer t, either $\sigma_t(f) < \sigma_{t+1}(f)$ or $\sigma_t(f) = k^2$; we say that f is *weakly expanding* if $\sigma_t(f) \le \sigma_{t+1}(f)$ for every positive integer t.

Conjecture 3.7. *Let k be an integer no smaller than 4 and let $f \in \mathcal{B}_k$ be a primitive map.*

- *If $\sigma_1(f) > k^2 - 4k + 7$, then f is strictly expanding.*
- *If $\sigma_1(f) > k^2 - 5k + 10$, then f is weakly expanding.*

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A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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For any $R, C \in 2^{[k-1]}$, define $f_{k;R,C}$ to be the element in $\mathcal{B}_k$ so that

$$f_{k;R,C}(i) = \begin{cases} [k-1], & \text{if } \ i = k; \\ C \cup \{k\}, & \text{if } \ i \in R; \\ \{k\}, & \text{if } \ i \in [k-1] \setminus R. \end{cases}$$

When $\{|R|, |C|\} = \{k-3, k-4\}$, we can check that $\sigma_1(f_{k;R,C}) = k^2 - 5k + 10$, $\sigma_2(f_{k;R,C}) = k^2 - 5$, $\sigma_3(f_{k;R,C}) = k^2 - 6$, $\sigma_4(f_{k;R,C}) = k^2$. When $|R| = |C| = k-3$, we can check that $\sigma_1(f_{k;R,C}) = k^2 - 4k + 7$, $\sigma_2(f_{k;R,C}) = k^2 - 4$, $\sigma_3(f_{k;R,C}) = k^2 - 4$, $\sigma_4(f_{k;R,C}) = k^2$. Computer experiments suggest that these constructions essentially give all those primitive maps f from $\mathcal{B}_k$ with $\sigma_1(f) = k^2 - 5k + 10$ which are not weakly expanding and all those primitive maps f from $\mathcal{B}_k$ with $\sigma_1(f) = k^2 - 4k + 7$ which are not strictly expanding.

3.5. Tropical. For a Boolean linear map f, an interesting parameter is the largest height of the in-trees attached to the cycles in $\mathcal{PS}_f$ – this turns out to be the primitive exponent when f is primitive. Generalizing many results about this parameter in the Boolean case, some results about it in the tropical case have been established [32]. Analogously, it should be an interesting project to extend the expansion property for Boolean linear maps to the tropical setting.

3.6. Beyond. Besides expansion property, some shrinking properties, say zero controllability and synchronizing property, also attract lots of attentions. Actually, by relaxing the linear condition $f(A \cup B) = f(A) \cup f(B)$ to the supermodular condition $f(A \cup B) \supseteq f(A) \cup f(B)$ for all set A and B, almost all results regarding expansion property in this paper are still valid. We emphasize here the Boolean linear maps as many interesting shrinking properties can be studied on them as well.

From the viewpoint of the phase space structure of a discrete dynamical system, a wide range of active research field, whose background vary from positive switched system in control theory to models of biological regulation networks and others, can be put in a unified setting [3, 4, 5, 7, 20, 22, 23, 27, 35, 38, 39, 46, 49]. It should be safe to conclude that the reachability problem of the phase spaces of various discrete dynamical systems is a world with many secrets to be explored.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 381-407, June 2016

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