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ORTHOGONAL BASES OF BRAUER SYMMETRY CLASSES OF TENSORS FOR GROUPS HAVING CYCLIC SUPPORT ON NON-LINEAR BRAUER CHARACTERS*

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Abstract. This paper provides some properties of Brauer symmetry classes of tensors. A dimension formula is derived for the orbital subspaces in the Brauer symmetry classes of tensors corresponding to the irreducible Brauer characters of the groups whose non-linear Brauer characters have support being a cyclic group. Using the derived formula, necessary and sufficient condition are investigated for the existence of an o-basis of dicyclic groups, semi-dihedral groups, and also those things are reinvestigated on dihedral groups. Some criteria for the non-vanishing elements in the Brauer symmetry classes of tensors associated to those groups are also included.

Key words. Brauer symmetry classes of tensors, Orthogonal basis, Semi-dihedral groups, Dicyclic groups.

AMS subject classifications. 20C30, 15A69.

1. Introduction.

During the past decades, there are many papers devoted to study symmetry classes of tensors, see, for example, [1]–[9]. One of the active research topics is the investigation of a special basis (called an o-basis) for the classes. This basis consists of decomposable symmetrized tensors that are images of the symmetrizer using an irreducible character of a given group. In [10], Randall R. Holmes and A. Kodithuwakku studied symmetry classes of tensors using an irreducible Brauer character of the dihedral group instead of an ordinary irreducible character and gave necessary and sufficient conditions for the existence of an o-basis. A classical method to provide the conditions applies the dimension of the orbital subspaces in order to find an o-basis for each orbit separately. A main tool for computing the dimension of symmetry classes using ordinary characters is the Freese's theorem [9]. Unfortunately, the symmetrizer using Brauer characters is not (in general) idempotent, so the Freese's theorem can not be applied directly. However, for the case of non-linear Brauer characters of dihedral groups, the authors in [10] decomposed them into a

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sum of ordinary characters and used the generalized Freese's theorem to bound the dimension.

One common property for all non-linear Brauer characters of dihedral groups is their vanishing outside some cyclic subgroups. Many finite groups, including dicyclic groups and semi-dihedral groups, satisfy this property. In this paper, we investigate the existence of an o-basis of Brauer symmetry classes of tensors associated with the groups having the stated property. Some properties of symmetry classes of tensors symetrized using a complex value function are stated. For the non-linear Brauer characters, we decompose the orbital subspaces of Brauer symmetry classes of tensors into an orthogonal direct sum of smaller factors and then provide a dimension formula for each of them. The necessary and sufficient condition for the existence of an o-basis for dicyclic groups, semi-dihedral groups and dihedral groups are investigated and reinvestigated as an application of the formula. Some criteria for the non-vanishing elements in the Brauer symmetry classes of tensors associated to these groups are also included.

2. Preliminaries. Let G be a subgroup of the full symmetric group S_m and p be a fixed prime number. An element of G is *p-regular* if its order is not divisible by p. Denote by $\hat{G}$ the set of all p-regular elements of G. Let $\mathrm{IBr}(G)$ denote the set of irreducible Brauer characters of G. A Brauer character is a certain function from $\hat{G}$ to $\mathbb{C}$ associated with an FG-module where F is a suitably chosen field of characteristic p. The Brauer character is irreducible if the associated module is simple. A conjugacy class of G consisting of p-regular elements is called a p-regular class. The number of irreducible Brauer characters of G equals the number of p-regular classes of G. Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. (Unless preceded by the word *Brauer*, the word *character* always refers to an ordinary character.) If the order of G is not divisible by p, then $\hat{G} = G$ and $\mathrm{IBr}(G) = \mathrm{Irr}(G)$. Let S be a subset of G containing the identity element e and let $\phi : S \to \mathbb{C}$ be a fixed function. Statements below involving ϕ hold if ϕ is a character of G (in which case $S = G$) and also if ϕ is a Brauer character of G (in which case $S = \hat{G}$). During the last few years, many very interesting results on the topic of Brauer characters have been found (see e.g. [13] and [15]–[22]).

Let V be a k-dimensional complex inner product space and $\{e_1, \ldots, e_k\}$ be an orthonormal basis of V. Let Γ_k^m be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$, with $1 \leq \alpha_i \leq k$. Define the action of G on Γ_k^m by

$$\alpha\sigma = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)}).$$

We denote by G_α the *stabilizer subgroup* of α, i.e., $G_\alpha = \{\sigma \in G | \alpha\sigma = \alpha\}$. The space $V^{\otimes m}$ is a left $\mathbb{C}G$-module with the action given $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ $(\sigma \in G, \gamma \in \Gamma_k^m)$ extended linearly. The inner product on V induces an inner product on $V^{\otimes m}$ which

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is G-invariant and, with respect to this inner product, the set $\{e_\alpha | \alpha \in \Gamma_k^m\}$ is an orthonormal basis for $V^{\otimes m}$, where $e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$.

The *symmetrizer* corresponding to ϕ and $S \subseteq G$ is the element s_ϕ of the group algebra $\mathbb{C}G$ given by

$$s_\phi = \frac{\phi(e)}{|S|} \sum_{\sigma \in S} \phi(\sigma)\sigma. \tag{2.1}$$

Corresponding to ϕ and $\alpha \in \Gamma_k^m$, the *standard (or decomposable) symmetrized tensor* is

$$e_\alpha^\phi = s_\phi e_\alpha = \frac{\phi(e)}{|S|} \sum_{\sigma \in S} \phi(\sigma) e_{\alpha\sigma^{-1}}. \tag{2.2}$$

The *symmetry class of tensors* associated with ϕ and $S \subseteq G$ is

$$V_\phi(G) = s_\phi V^{\otimes m} = \langle e_\alpha^\phi | \alpha \in \Gamma_k^m \rangle.$$

If ϕ is a Brauer character, we refer to $V_\phi(G)$ as a *Brauer symmetry class of tensors*. The orbital subspace of $V_\phi(G)$ corresponding to $\alpha \in \Gamma_k^m$ is

$$V_\alpha^\phi(G) = \langle e_{\alpha\sigma}^\phi | \sigma \in G \rangle.$$

An *o-basis* of a subspace W of $V_\phi(G)$ is an orthogonal basis of W of the form $\{e_{\alpha_1}^\phi, \ldots, e_{\alpha_t}^\phi\}$ for some $\alpha_i \in \Gamma_k^m$. By convention, the empty set is regarded as an o-basis of the zero subspace of $V_\phi(G)$. Let $\Delta = \Delta_G$ be a set of representatives of the orbits of Γ_k^m under the action of G.

The following critical theorem is used to reduce the task of investigation on the existence of an o-basis.

Theorem 2.1. *We have an orthogonal sum decomposition*

$$V_\phi(G) = \sum_{\alpha \in \Delta} V_\alpha^\phi(G).$$

Proof. See [10, Thm. 1.1]. ☐

The induced inner product on $V_\phi(G)$ can be calculated via the formula below, which is an adaptation from the Theorem 1.2 in [10].

Theorem 2.2. *For every $\alpha \in \Gamma_k^m$ and $\sigma_1, \sigma_2 \in G$, we have*

$$\langle e_{\alpha\sigma_1}^\phi, e_{\alpha\sigma_2}^\phi \rangle = \frac{|\phi(e)|^2}{|S|^2} \sum_{\mu \in S} \sum_{\tau \in \sigma_1 \mu^{-1} S \sigma_2^{-1} \cap G_\alpha} \phi(\mu) \overline{\phi(\mu \sigma_1^{-1} \tau \sigma_2)}. \tag{2.3}$$

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Proof. For $\alpha \in \Gamma_k^m$ and $\sigma_1, \sigma_2 \in G$, we have

$$\begin{aligned}
\langle e^{\phi}_{\alpha\sigma_1}, e^{\phi}_{\alpha\sigma_2}\rangle &= \frac{|\phi(e)|^2}{|S|^2}\sum_{\mu\in S}\sum_{\rho\in S}\phi(\mu)\overline{\phi(\rho)}\langle e_{\alpha\sigma_1\mu^{-1}}, e_{\alpha\sigma_2\rho^{-1}}\rangle \\
&= \frac{|\phi(e)|^2}{|S|^2}\sum_{\mu\in S}\sum_{\rho\in S}\phi(\mu)\overline{\phi(\rho)}\langle e_{\alpha\sigma_1\mu^{-1}\rho\sigma_2^{-1}}, e_{\alpha}\rangle \\
&= \frac{|\phi(e)|^2}{|S|^2}\sum_{\mu\in S}\sum_{\rho\in S, \sigma_1\mu^{-1}\rho\sigma_2^{-1}\in G_\alpha}\phi(\mu)\overline{\phi(\rho)} \\
&= \frac{|\phi(e)|^2}{|S|^2}\sum_{\mu\in S}\sum_{\tau\in \sigma_1\mu^{-1}S\sigma_2^{-1}\cap G_\alpha}\phi(\mu)\overline{\phi(\mu\sigma_1^{-1}\tau\sigma_2)},
\end{aligned}$$

where $\tau = \sigma_1\mu^{-1}\rho\sigma_2^{-1}$. □

The following is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $\sigma_1, \sigma_2 \in G, S \subseteq G$ and $\phi = \psi\mid_S$, where ψ is a linear character of G. If $G_\alpha = \{e\}$ and $A = \{\mu \in S \mid e \in \sigma_1\mu^{-1}S\sigma_2^{-1}\} \neq \emptyset$, then*

$$\langle e^{\phi}_{\alpha\sigma_1}, e^{\phi}_{\alpha\sigma_2}\rangle \neq 0.$$

In the following sections, we also need the lemma and propositions below.

Lemma 2.4. *Assume that S is closed under conjugation by elements of G and that ϕ is constant on the conjugacy classes of G. For each $\alpha \in \Gamma_k^m$ and $\sigma \in G$, we have $\sigma e^{\phi}_{\alpha} = e^{\phi}_{\alpha\sigma^{-1}}$.*

Proof. See [10, Lem. 1.3]. □

As an immediate consequence of this lemma, we have the following proposition.

Proposition 2.5. *Let $\phi : S \longrightarrow \mathbb{C}$ be a fixed function equipped the assumption of Lemma 2.4. If $B = \{e^{\phi}_{\alpha g_1}, e^{\phi}_{\alpha g_2}, \ldots, e^{\phi}_{\alpha g_k}\}$ is an o-basis of $V^{\phi}_{\alpha}(G)$, then, for each $g \in G$,*

$$g \cdot B = \{e^{\phi}_{\alpha g_1 g^{-1}}, e^{\phi}_{\alpha g_2 g^{-1}}, \ldots, e^{\phi}_{\alpha g_k g^{-1}}\}$$

is also an o-basis of $V^{\phi}_{\alpha}(G)$.

Proposition 2.6. *Let $\phi : S \longrightarrow \mathbb{C}$ be a fixed function. Also, let C contained in S be a subgroup of G. If $G_\gamma = \{e\}$ and $\phi(s) = 0$ for all $s \in S - C$, then*

$$V^{\phi}_{\gamma}(G) = \langle e^{\phi}_{\gamma g} \mid g \in C\rangle \oplus \langle e^{\phi}_{\gamma g} \mid g \in G - C\rangle.$$

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Proof. If we choose $\sigma_1 \in C$ and $\sigma_2 \in G \setminus C$ in (2.3), we get nonzero term only if $\mu \in C$ and $\mu\sigma_1^{-1}\sigma_2 \in C$, which is impossible, since C is a group. Thus, the two spaces are orthogonal. $\square$

PROPOSITION 2.7. *Let S be a subgroup of G and $\phi : S \longrightarrow \mathbb{C}$ be a nonzero constant function on S. Then, for each $\alpha \in \Gamma^m_{\dim V}$,*

$$V^\phi_\alpha(G) = \langle e^\phi_{\alpha\sigma} | \sigma \in G\rangle$$

has an o-basis and so does $V_\phi(G)$.

Proof. Suppose $\phi(s) = c \in \mathbb{C}$ for all $s \in S$. Since S is a group and by Theorem 2.2, we have that, for $\sigma, \tau \in G$,

$$\begin{aligned}\langle e^\phi_{\alpha\sigma}, e^\phi_{\alpha\tau}\rangle &= \frac{|c|^2}{|S|^2} \sum_{\mu\in S} \sum_{\delta\in\sigma\mu^{-1}S\tau^{-1}\cap G_\alpha} |c|^2 \\ &= \frac{|c|^4}{|S|^2} \sum_{\mu\in S} |\sigma S\tau^{-1} \cap G_\alpha| \\ &= \frac{|c|^4 |\sigma S\tau^{-1} \cap G_\alpha|}{|S|}.\end{aligned}$$

We have $G_\alpha \cap \sigma S\tau^{-1} = \emptyset$ or $G_\alpha \cap \sigma S\tau^{-1} \neq \emptyset$, for each $\sigma, \tau \in G$. For the latter case, we have $\sigma\mu\tau^{-1} \in G_\alpha$ for some $\mu \in S$. Thus, for each $b \in S$,

$$\alpha\sigma b = \alpha(\sigma\mu\tau^{-1})(\tau\mu^{-1}b) = \alpha\tau g, \text{ for some } g = \mu^{-1}b \in S.$$

Hence, $\{\alpha\sigma b | b \in S\} = \{\alpha\tau b | b \in S\}$. Since S is a group and $\phi(s) = c$ for all $s \in S$, we have

$$e^\phi_{\alpha\sigma} = \frac{c^2}{|S|} \sum_{s\in S} e_{\alpha s} = e^\phi_{\alpha\tau}.$$

This implies that, for $\sigma, \tau \in G$, $e^\phi_{\alpha\sigma} = e^\phi_{\alpha\tau}$ or $\langle e^\phi_{\alpha\sigma}, e^\phi_{\alpha\tau}\rangle = 0$, which yields that $V^\phi_\alpha(G)$ has an o-basis and by Theorem 2.1, we complete the proof. $\square$

3. Dimension formula. In this section, we let G be a finite group, $S \subseteq G$ and C be a subgroup of G contained in S. Let $\phi : G \longrightarrow \mathbb{C}$ be a function such that $\phi(\sigma) \neq 0$ for each $\sigma \in C$ but $\phi(S \setminus C) = 0$. Thus, under this assumption, the induced inner product (2.3) becomes

$$\langle e^\phi_{\alpha\sigma_1}, e^\phi_{\alpha\sigma_2}\rangle = \frac{|\phi(e)|^2}{|S|^2} \sum_{\mu\in C} \sum_{\tau\in\sigma_1 C\sigma_2^{-1}\cap G_\alpha} \phi(\mu)\overline{\phi(\mu\sigma_1^{-1}\tau\sigma_2)} \tag{3.1}$$

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for every $\alpha \in \Gamma_k^m$ and $\sigma_1, \sigma_2 \in G$. If $\sigma_1 C \sigma_2^{-1} \cap G_\alpha = \emptyset$, then $\langle e^\phi_{\alpha\sigma_1}, e^\phi_{\alpha\sigma_2}\rangle = 0$. This motivates us to define a relation on G: for each $\alpha \in \triangle$,

$$\sigma_1 \sim_\alpha \sigma_2 \iff \sigma_1 \in G_\alpha \sigma_2 C \tag{3.2}$$

for all $\sigma_1, \sigma_2 \in G$. It is not hard to check that $\sim_\alpha$, for each $\alpha \in \triangle$, is an equivalent relation.

Now, we set $[\sigma]$ as the equivalent class containing σ, R^G_α the set of representative of $G/\sim_\alpha$ and $V^\phi_\alpha([\sigma]) := \langle e^\phi_{\alpha g} | g \in [\sigma]\rangle$. It is clear that $V^\phi_\alpha([\sigma])$ is a subspace of $V^\phi_\alpha(G)$.

Lemma 3.1. *The space $V^\phi_\alpha(G)$ has an o-basis if and only if for each $\sigma \in R^G_\alpha$, $V^\phi_\alpha([\sigma])$ has an o-basis.*

Proof. Suppose that $V^\phi_\alpha([\sigma])$ has an o-basis, for each $\sigma \in R^G_\alpha$. To show that the space $V^\phi_\alpha(G)$ has an o-basis, it suffices to prove that $V^\phi_\alpha([\sigma])$'s are orthogonal. Now, let $\sigma_1, \sigma_2 \in G$ and $\alpha \in \Delta$. If $\sigma_1 C \sigma_2^{-1} \cap G_\alpha \neq \emptyset$, then $\sigma_1 \in G_\alpha \sigma_2 C$. Hence, if $\sigma_1 \nsim_\alpha \sigma_2$, then $\sigma_1 C \sigma_2^{-1} \cap G_\alpha = \emptyset$. In other words, if $[\sigma_1] \neq [\sigma_2]$, then $\langle e^\phi_{\alpha\sigma_1}, e^\phi_{\alpha\sigma_2}\rangle = 0$. The other implication is clear. ◻

For the following propositions, denote $\langle e^\phi_{\gamma g} | g \in C\rangle$ by $V^\phi_\gamma(C)$.

Proposition 3.2. *The space $V_\phi(G)$ has an o-basis if and only if for each $\gamma \in \Delta$, $V^\phi_\gamma(C)$ has an o-basis.*

Proof. For each $[\sigma] \in G/\sim_\alpha$, we have that

$$\begin{aligned} V^\phi_\alpha([\sigma]) &= \langle e^\phi_{\alpha g} | g \in [\sigma]\rangle \\ &= \langle e^\phi_{\alpha g} | g \in G_\alpha \sigma C\rangle \\ &= \langle e^\phi_{\alpha\sigma h} | h \in C\rangle \\ &= \langle e^\phi_{\gamma h} | h \in C\rangle; \quad \gamma = \alpha\sigma \\ &= V^\phi_\gamma(C). \end{aligned}$$

By Lemma 3.1 and (2.1), we finish the proof. ◻

To determine the dimension of $V^\phi_\gamma(C)$, for each $\gamma \in \Delta$, we introduce a relation $\sim^*_\gamma$ on C by: for each $\sigma_1, \sigma_2 \in C$,

$$\sigma_1 \sim^*_\gamma \sigma_2 \iff \sigma_1 \sigma_2^{-1} \in G_\gamma. \tag{3.3}$$

It is obvious that $\sim^*_\gamma$ is an equivalent relation. Now, we have:

Proposition 3.3. *If $C/\sim^*_\gamma = \{[\sigma_1], [\sigma_2], \ldots, [\sigma_{t_\gamma}]\}$, then $\dim(V^\phi_\gamma(C)) = \operatorname{rank}(M_\gamma)$, where $(M_\gamma)_{ij} := \sum_{h \in C \cap G_\gamma} \phi(h\sigma_i\sigma_j)$ and $1 \le i, j \le t_\gamma$.*

Proof. For each $j \in \{1, 2, \ldots, t_\gamma\}$ and $g_1, g_2 \in [\sigma_j]$, we have that $g_1 = cg_2$ for

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some $c \in G_\gamma$. Thus,

$$\begin{array}{rcl} e^{\phi}_{\gamma g_1} & = & \frac{\phi(e)}{|S|}\sum_{\sigma\in C}\phi(\sigma g_1)e_{\gamma\sigma^{-1}} \\ & = & \frac{\phi(e)}{|S|}\sum_{\sigma\in C}\phi(\sigma c g_2)e_{\gamma\sigma^{-1}} \\ & = & \frac{\phi(e)}{|S|}\sum_{\tau c^{-1}\in C}\phi(\tau g_2)e_{\gamma c\tau^{-1}};\tau := \sigma c \\ & = & \frac{\phi(e)}{|S|}\sum_{\tau\in C}\phi(\tau g_2)e_{\gamma\tau^{-1}} = e^{\phi}_{\gamma g_2}. \end{array}$$

Hence, $V^{\phi}_{\gamma}(C) = \langle e^{\phi}_{\gamma\sigma_j} | j = 1,2,\ldots,t_\gamma\rangle$. Moreover, note that $e_{\gamma g_1^{-1}} = e_{\gamma g_2^{-1}}$ if $g_1, g_2 \in [\sigma_i]$. This yields

$$e^{\phi}_{\gamma g} = \frac{\phi(e)}{|S|}\sum_{i=1}^{t_\gamma}\left(\sum_{\sigma\in[\sigma_i]}\phi(\sigma g)\right)e_{\gamma\sigma_i^{-1}},$$

for each $g \in C$. However, $\sum_{\sigma\in[\sigma_i]}\phi(\sigma g) = \sum_{h\in C\cap G_\gamma}\phi(h\sigma_i g)$. So, we have

$$e^{\phi}_{\gamma\sigma_j} = \frac{\phi(e)}{|S|}\sum_{i=1}^{t_\gamma}\left(\sum_{h\in C\cap G_\gamma}\phi(h\sigma_i\sigma_j)\right)e_{\gamma\sigma_i^{-1}}, \qquad 1 \le j \le t_\gamma.$$

The result follows by $(M_\gamma)_{ij} := \sum_{h\in C\cap G_\gamma}\phi(h\sigma_i\sigma_j)$ for $1 \le i, j \le t_\gamma$. $\square$

In particular, as a special case of Proposition 3.3, i.e., if C is a cyclic subgroup of G, we obtain a dimension formula for $V^{\phi}_{\gamma}(C)$.

THEOREM 3.4. *Let $C = \langle\tau\rangle \subseteq S$ be a cyclic subgroup of G such that $C/\sim^*_\gamma = \{[\tau],[\tau^2],\ldots,[\tau^{t_\gamma}]\}$. Denote $v_j = \sum_{h\in C\cap G_\gamma}\phi(h\tau^{t_\gamma-j})$ and*

$$d_\gamma = \left|\left\{s \in \{0,1,2,\ldots,t_\gamma-1\} \mid \sum_{j=0}^{t_\gamma-1} v_j e^{\frac{2\pi sji}{t_\gamma}} = 0\right\}\right|.$$

Then $t_\gamma = \frac{|C|}{|C\cap G_\gamma|}$ and $\dim(V^{\phi}_{\gamma}(C)) = t_\gamma - d_\gamma$.

Proof. Note that under the equivalent relation $\sim^*_\gamma$ with $C/\sim^*_\gamma = \{[\tau],[\tau^2],\ldots,[\tau^{t_\gamma}]\}$, we have that $[\tau^k] = \{\sigma \in C \mid \sigma\tau^{-k} \in G_\gamma\} = \{h\tau^k \mid h \in C\cap G_\gamma\}$. So, $|[\tau^k]| = |C\cap G_\gamma|$ for all $k = 1,2,\ldots,t_\gamma$, and hence,

$$t_\gamma = |C/\sim^*_\gamma| = \frac{|C|}{|C\cap G_\gamma|}.$$

By rank nullity theorem and Proposition 3.3,

$$\dim(V^{\phi}_{\gamma}(C)) = \mathrm{rank}(M_\gamma) = t_\gamma - \mathrm{nullity}(\mathrm{M}_\gamma) = t_\gamma - d_\gamma,$$

where $d_\gamma := \mathrm{nullity}(\mathrm{M}_\gamma)$.

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To determine d_γ, we observe that, if C is a cyclic subgroup of G, then M_γ can be reduced to a circulant matrix M_γ^{cir} by swamping some columns of M_γ. Precisely, $M_\gamma^{cir} = (v_0, v_1, \ldots, v_{t_\gamma - 1})$, where, for each $j = 0, 1, \ldots, t_\gamma - 1$,

$$v_j = \sum_{h \in C \cap G_\gamma} \phi(h\tau^{t_\gamma - j}).$$

It is well known that (see e.g. [12]).

$$\text{nullity}(M_\gamma^{cir}) = \deg[\gcd(P_v(x), x^{t_\gamma} - 1)],$$

where $P_v(x) = \sum_{j=0}^{t_\gamma - 1} v_j x^j$. Note that the set of all roots (over field $\mathbb{C}$) of $x^{t_\gamma} - 1$ is $U := \{e^{\frac{2\pi si}{t_\gamma}} \mid 0 \le s < t_\gamma\}$. Thus, common factors of $P_v(x)$ and $x^{t_\gamma} - 1$ must have roots in U and hence,

$$\begin{aligned} \deg[\gcd(P_v(x), x^{t_\gamma} - 1)] &= |\{s \in \mathbb{Z} \mid 0 \le s < t_\gamma \text{ and } P_v(e^{\frac{2\pi si}{t_\gamma}}) = 0\}| \\ &= |\{s \in \mathbb{Z} \mid 0 \le s < t_\gamma \text{ and } \textstyle\sum_{j=0}^{t_\gamma - 1} v_j e^{\frac{2\pi sji}{t_\gamma}} = 0\}|. \end{aligned}$$

Since the rank is invariant under column operations, $d_\gamma = \text{nullity}(M_\gamma^{cir})$ and thus the result follows. □

4. Dicyclic group T_{4n}. The dicyclic group T_{4n} is defined as follows:

$$T_{4n} = \langle r, s | r^{2n} = e, r^n = s^2, s^{-1}rs = r^{-1} \rangle.$$

Explicitly, all elements of the group T_{4n} may be given by $T_{4n} = \{r^i, sr^i | 0 \le i < 2n\}$. By the classical Cayley theorem, T_{4n} can be embedded in S_{4n}. Precisely,

$$\begin{aligned} r &= (\ 1 \quad 2 \quad 3 \quad \cdots \quad 2n\ )(\ 2n+1 \quad 2n+2 \quad 2n+3 \quad \cdots \quad 4n\ ) \\ s &= (\ 1 \quad 2n+1 \quad n+1 \quad 3n+1\ )(\ 2 \quad 4n \quad n+2 \quad 3n\ ) \\ &\quad (\ 3 \quad 4n-1 \quad n+3 \quad 3n-1\ ) \cdots (\ n-1 \quad 3n+3 \quad 2n-1 \quad 2n+3\ ) \\ &\quad (\ n \quad 3n+2 \quad 2n \quad 2n+2\ ). \end{aligned}$$

T_{4n} has $n+3$ conjugacy classes which are

$$\{e\},\ \{r^k, r^{2n-k}\},\ 1 \le k \le n,\ \{sr^{2k} \mid 0 \le k \le n-1\},\ \{sr^{2k+1} \mid 0 \le k \le n-1\}$$

and the ordinary irreducible character of T_{4n} are given by (see [4]):

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Table I: The character table for T_{4n}, when n is even.

Characters	$r^k(0 \leq k \leq n)$	s	rs
χ_0	1	1	1
χ_1	$(-1)^k$	1	-1
χ_2	1	-1	-1
χ_3	$(-1)^k$	-1	1
ψ_j, where $1 \leq j \leq n-1$	$2\cos\left(\frac{kj\pi}{n}\right)$	0	0

Table II: The character table for T_{4n}, when n is odd.

Characters	$r^k(0 \leq k \leq n)$	s	rs
χ'_0	1	1	1
χ'_1	$(-1)^k$	i	-i
χ'_2	1	-1	-1
χ'_3	$(-1)^k$	-i	i
ψ'_j, where $1 \leq j \leq n-1$	$2\cos\left(\frac{kj\pi}{n}\right)$	0	0

Write $2n = lp^t$ with l an integer not divisible by p (where p is our fixed prime number). We have

$$\hat{G} = \begin{cases} \{r^{jp^t}, sr^k \mid 0 \leq j < l, 1 \leq k \leq 2n\}, & \text{if } p \neq 2; \\ \{r^{jp^t} \mid 0 \leq j < l\}, & \text{if } p = 2. \end{cases}$$

Thus, the p-regular classes of G are

$$\begin{cases} \{r^{jp^t}, r^{(l-j)p^t}\};\ 0 \leq j \leq \frac{l}{2}, \{sr^{2k} \mid 1 \leq k \leq n\}, \{sr^{2k+1} \mid 0 \leq k \leq n-1\}, & \text{if } p \neq 2; \\ \{r^{jp^t}, r^{(l-j)p^t}\};\ 0 \leq j \leq \frac{l-1}{2} & \text{if } p = 2. \end{cases}$$

For each j and h, denote

$$\hat{\psi}_j = \psi_j|_{\hat{G}}, \quad \hat{\chi}_h = \chi_h|_{\hat{G}} \text{ and } \hat{\psi}'_j = \psi'_j|_{\hat{G}}, \quad \hat{\chi}'_h = \chi'_h|_{\hat{G}},$$

and define $\epsilon = \begin{cases} 4, & \text{if } p \neq 2; \\ 1, & \text{if } p = 2. \end{cases}$

PROPOSITION 4.1. *The complete list of irreducible Brauer characters of T_{4n} for even n is*

$$\hat{\chi}_h \quad (0 \leq h < \epsilon), \quad \hat{\psi}_j \quad \left(1 \leq j < \frac{l}{2}\right),$$

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and for odd n is

$$\hat{\chi'}_h \quad (0 \leq h < \epsilon), \quad \hat{\psi'}_j \quad \left(1 \leq j < \frac{l}{2}\right).$$

Proof. We first note that the restriction of a character of T_{4n} to $\widehat{T}_{4n}$ is a Brauer character and the number of all the irreducible Brauer characters is the number of p-regular classes of T_{4n}. Also, since T_{4n} is solvable, by the Fong-Swan theorem, any irreducible Brauer character of T_{4n} is the restriction of an ordinary irreducible character of T_{4n}.

The linear characters $\widehat{\chi}_h$'s and $\widehat{\chi'}_h$'s are obviously irreducible and distinct, by the character tables above. For characters, $\widehat{\psi}_j$ and $\widehat{\psi'}_j$, of dimension two we claim that they are all distinct and irreducible for all $1 \leq j < \frac{l}{2}$. By the character tables above, there is no need to separate the proof into the case of odd n, even n or $p = 2$, $p \neq 2$, since $\widehat{\psi}_j$ and $\widehat{\psi'}_j$ are agree on the columns r^k's and their values are zero outside these columns.

For the irreduciblity issue, we suppose for a contradiction that

$$\widehat{\psi}_j = \widehat{\chi}_h + \widehat{\chi}_k,$$

for some $0 \leq h, k < \epsilon$ and $1 \leq j < \frac{l}{2}$. Since $1 \leq j < \frac{l}{2}$, $l > 2$ and $r^{2p^t} \in \widehat{T}_{4n}$. So, we can evaluate both sides of the above equation at r^{2p^t} and obtain that

$$2\cos\left(\frac{2p^t j\pi}{n}\right) = 2,$$

which is impossible because $\cos\left(\frac{2p^t j\pi}{n}\right) < 1$ for all $1 \leq j < \frac{l}{2}$.

Analogously, for the issue of distinction, we suppose for a contradiction that $\widehat{\psi}_j = \widehat{\psi}_i$, for some $1 \leq i < j < \frac{l}{2}$. We now evaluate both sides by r^{p^t}, which yields that

$$\cos\left(\frac{p^t j\pi}{n}\right) = \cos\left(\frac{p^t i\pi}{n}\right).$$

It implies that, for $\frac{p^t j\pi}{n}$ and $\frac{p^t j\pi}{n}$, their difference or their sum must be a multiple of 2π. However, this is not the case because $1 \leq i < j < \frac{l}{2}$. $\square$

THEOREM 4.2. *Let $G = T_{4n}$, $0 \leq h < \epsilon$ where $\epsilon = 4$ if $p \neq 2$ and $\epsilon = 1$ if $p = 2$, and put $\phi = \hat{\chi}_h$ or $\hat{\chi'}_h$. The space $V_\phi(G)$ has an o-basis if and only if at least one of the following holds:*

(i) $\dim V = 1$,

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(ii) $p = 2$,

(iii) $2n$ *is not divisible by* p.

Proof. (i) If $\dim V = 1$, then $V_\phi(G) = \langle e^\phi_\alpha \mid \alpha \in \Gamma^{4n}_1\rangle$ has only at most one generator, namely, e^ϕ_α where $\alpha = (1, 1, 1 \ldots, 1)$. So, $\dim V_\phi(G) \le 1$, and thus, $V_\phi(G)$ has an o-basis.

(ii) If $p = 2$ then $\widehat{G} = \langle r^{p^t}\rangle$. Since $\widehat{G}$ is a subgroup of G and ϕ is constant on $\widehat{G}$, it follows by Proposition 2.7 that $V_\phi(G)$ has an o-basis.

(iii) Assume $p \neq 2$ and $2n$ is not divisible by p. Then $\hat{G} = G$ and consequently, these characters will be ordinary linear characters. Thus, $V_\phi(G)$ has an o-basis.

Conversely, we assume that $\dim V > 1$ and $p \neq 2$ and $2n$ is divisible by p. So, $r \notin \widehat{G}$ and $\widehat{G} = \{r^{jp^t}, sr^k \mid 0 \le j < l, 1 \le k \le 2n\} = \widehat{G}^{-1}$. We will show that $V_\phi(G)$ does not have an o-basis. For $\alpha = (1, 2, \ldots, 2, 2) \in \Gamma^{4n}_{\dim V}$, we have $G_\alpha = \{e\}$. Now, we concentrate on $\langle e^\phi_{\alpha\sigma}, e^\phi_\alpha\rangle$, for each $\sigma \in G$. We observe that $A = \{\mu \in \widehat{G} \mid e \in \sigma\mu^{-1}\widehat{G}\} = \{\mu \in \widehat{G} \mid \sigma \in \widehat{G}\mu\}$. Since $r^i = (sr^n)(sr^i) \in \widehat{G}^2$ for each $0 \le i < 2n$, $G \subseteq \widehat{G}^2$ and hence $A \neq \emptyset$. Thus, by Corollary 2.3, we have

$$\langle e^\phi_{\alpha\sigma}, e^\phi_\alpha\rangle \neq 0 \text{ for each } \sigma \in G. \tag{4.1}$$

Next, we claim that $\{e^\phi_{\alpha r}, e^\phi_\alpha\} \subseteq V^\phi_\alpha(G)$ is a linearly independent set. We can set $e^\phi_\alpha = \sum_\delta c_\delta e_\delta$ and $e^\phi_{\alpha r} = \sum_\delta d_\delta e_\delta$ as $\{e_\delta | \delta \in \Gamma^{4n}_{\dim V}\}$ forms a basis for $V^{\otimes 4n}$. Since $\widehat{G}^{-1} = \widehat{G}$,

$$e^\phi_\alpha = \frac{\phi(1)}{|\widehat{G}|}\sum_{\sigma\in\widehat{G}}\phi(\sigma^{-1})e_{\alpha\sigma}.$$

Since $G_\alpha = \{e\}$, the elements $\alpha\sigma$ with $\sigma \in G$ are distinct. Also, since $r \notin \widehat{G}$, $\alpha\sigma \neq \alpha r$ for all $\sigma \in \widehat{G}$, which yields that $c_r = 0$. On the other hand, $G_{\alpha r} = r^{-1}G_\alpha r = \{e\}$, so for $r \in \widehat{G}$, $(\alpha r)\sigma = \alpha r$ if and only if $\sigma = e$. This implies that $d_r = \frac{1}{|\widehat{G}|} \neq c_r = 0$, which implies that $\{e^\phi_{\alpha r}, e^\phi_\alpha\} \subseteq V^\phi_\alpha(G)$ is a linearly independent set. Hence, $\dim V^\phi_\alpha(G) \ge 2$.

By Proposition 2.5, if $V^\phi_\alpha(G)$ has an o-basis, then it has an o-basis containing e^ϕ_α, but, by (4.1), this is not the case. So, $V^\phi_\alpha(G)$ does not have an o-basis, and by Theorem 2.1, we complete the proof. ☐

For higher dimensional irreducible Brauer characters $\phi : \widehat{G} \longrightarrow \mathbb{C}$, we see that if $\dim V = 1$, then $V_\phi(G) = \langle e^\phi_\alpha \mid \alpha \in \Gamma^{4n}_1\rangle$ has only at most one generator, namely, e^ϕ_α where $\alpha = (1, 1, 1 \ldots, 1)$. So, $\dim V_\phi(G) \le 1$, and thus, $V_\phi(G)$ has an o-basis. If $\dim V > 1$, we investigate a necessary condition of the existence of an o-basis for

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dicyclic groups as follows.

Proposition 4.3. *For $G = T_{2(lp^t)}$ with $C \cap G_\gamma =< r^{t_\gamma p^t} >$ where $t_\gamma = \frac{l}{|C \cap G_\gamma|}$, $\gamma \in \Delta$ and $\phi = \hat{\psi}_b, \hat{\psi}'_b$, where $1 \le b < \frac{l}{2}$, we have that*

$$\dim(V_\gamma^\phi(C)) = \begin{cases} 2, & \text{if } \frac{bt_\gamma}{l} \in \mathbb{Z}; \\ 0, & \text{if } \frac{bt_\gamma}{l} \notin \mathbb{Z}. \end{cases}$$

Proof. Since G has cyclic support with $C =< r^{p^t} >$ and $C \cap G_\gamma =< r^{t_\gamma p^t} >$, for each ϕ, we can compute the dimension by using Proposition 3.4. By character tables and basic trigonometry identities, we compute that

$$\begin{array}{rcl} v_j = \sum_{h \in C \cap G_\gamma} \phi(h\tau^{t_\gamma - j}) & = & \sum_{m=1}^{l/t_\gamma} \phi(r^{(mt_\gamma - j)p^t}) \\ & = & \sum_{m=1}^{l/t_\gamma} 2\cos\left(m\left(\frac{2bt_\gamma}{l}\right)\pi - \left(\frac{2bj}{l}\right)\pi\right) \\ & = & \begin{cases} 2(\frac{l}{t_\gamma})\cos\left(\left(\frac{2bj}{l}\right)\pi\right), & \frac{bt_\gamma}{l} \in \mathbb{Z}; \\ 0, & \frac{bt_\gamma}{l} \notin \mathbb{Z}. \end{cases} \end{array}$$

So, if $\frac{bt_\gamma}{l} \notin \mathbb{Z}$, then $d_\gamma = t_\gamma$ and thus $\dim(V_\gamma^\phi(C)) = t_\gamma - t_\gamma = 0$.

For d_γ in which $\frac{bt_\gamma}{l} \in \mathbb{Z}$, we have that $\sum_{j=0}^{t_\gamma - 1} v_j e^{\frac{2\pi sji}{t_\gamma}} = 0$ if and only if

$$\sum_{j=0}^{t_\gamma - 1} 2\cos\left(\left(\frac{2bj}{l}\right)\pi\right)\cos\left(\left(\frac{2sj}{t_\gamma}\right)\pi\right) \text{ and } \sum_{j=0}^{t_\gamma - 1} 2\cos\left(\left(\frac{2bj}{l}\right)\pi\right)\sin\left(\left(\frac{2sj}{t_\gamma}\right)\pi\right),$$

are simultaneously zero. Since $\frac{bt_\gamma}{l} \in \mathbb{Z}$, the second sum is always zero and the first sum is zero for all $0 \le s < t_\gamma$ except for $\frac{b}{l} \pm \frac{s}{t_\gamma} \in \mathbb{Z}$; (i.e., except for $s = t_\gamma - \frac{bt_\gamma}{l}$ or $s = \frac{bt_\gamma}{l}$), because $0 < \frac{b}{l} + \frac{s}{t_\gamma} < 2$ and $-1 < \frac{b}{l} - \frac{s}{t_\gamma} < 1$. Hence, $d_\gamma = t_\gamma - 2$, and thus, the results follow. ▯

There is no surprise with the assertion that $\dim(V_\gamma^\phi(C)) = 0$ for which $\frac{bt_\gamma}{l} \notin \mathbb{Z}$ because:

Proposition 4.4. *For $G = T_{2(lp^t)}$ with $C =< r^{p^t} >$ and $C \cap G_\gamma =< r^{t_\gamma p^t} >$ where $t_\gamma = \frac{l}{|C \cap G_\gamma|}$, $\gamma \in \Delta$ and $\phi = \hat{\psi}_b, \hat{\psi}'_b$, where $1 \le b < \frac{l}{2}$, we have that, for each $\sigma \in C$,*

$$e_{\gamma\sigma}^\phi = 0 \quad \textit{if and only if} \quad \frac{bt_\gamma}{l} \notin \mathbb{Z}.$$

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Proof. Let $\sigma \in C$ and $\gamma \in \Delta$. By (3.1),

$$\begin{aligned}\langle e^{\phi}_{\gamma\sigma}, e^{\phi}_{\gamma\sigma}\rangle &= \frac{|\phi(e)|^2}{|\widehat{G}|^2}\sum_{\mu\in C}\sum_{\tau\in C\cap G_\gamma}\phi(\mu)\overline{\phi(\mu\tau)}\\ &= 4\frac{|\phi(e)|^2}{|\widehat{G}|^2}\sum_{j=1}^{l}\sum_{k=1}^{l/t_\gamma}\cos\left(\frac{2jb\pi}{l}\right)\cos\left(2k\left(\frac{bt_\gamma}{l}\right)\pi+\frac{2jb\pi}{l}\right)\\ &= \begin{cases}(\frac{4l}{t_\gamma})(\frac{|\phi(e)|^2}{|\widehat{G}|^2})\sum_{j=1}^{l}\cos^2\left(\frac{2jb\pi}{l}\right), & \text{if } \frac{bt_\gamma}{l}\in\mathbb{Z};\\ 0, & \text{if } \frac{bt_\gamma}{l}\notin\mathbb{Z},\end{cases}\end{aligned}$$

which completes the proof. ◻

Now, by the above propositions, we achieve the main conclusion.

THEOREM 4.5. *Let $G = T_{4n}$, where $2n = lp^t$ with l an integer not divisible by p and let $\phi = \hat{\psi}_b, \hat{\psi}'_b$, where $1 \le b < \frac{l}{2}$. Then, $V_\phi(G)$ has an o-basis if and only if $\nu_2(\frac{2b}{l}) < 0$.*

Proof. By Proposition 3.2, it is enough to focus on $V^{\phi}_{\gamma}(C)$. Also, in the proof of Proposition 3.3, we have $V^{\phi}_{\gamma}(C) = \langle e^{\phi}_{\gamma\sigma_j} | j = 1, 2, \ldots, t_\gamma\rangle$, where $\sigma_j = r^{jp^t}$ and $t_\gamma = \frac{|C|}{|C\cap G_\gamma|}$. Again, by (3.1) and the character tables, we compute that, for $1 \le i, j \le t_\gamma$,

$$\begin{aligned}\langle e^{\phi}_{\gamma\sigma_i}, e^{\phi}_{\gamma\sigma_j}\rangle = 0 &\iff \sum_{g\in C}\phi(g\sigma_i)\overline{\phi(g\sigma_j)} = 0\\ &\iff \sum_{k=0}^{l-1}\phi(r^{(k+i)p^t})\overline{\phi(r^{(k+j)p^t})} = 0\\ &\iff \sum_{k=0}^{l-1}2\cos\left((k+i)\frac{2b}{l}\pi\right)\cos\left((k+j)\frac{2b}{l}\pi\right) = 0\\ &\iff l\cos\left((i-j)\frac{2b}{l}\pi\right) = 0.\end{aligned}$$

By Proposition 4.3, $\dim(V^{\phi}_{\gamma}(C)) = 2$ for each γ such that $\frac{bt_\gamma}{l} \in \mathbb{Z}$. So, if $V_\phi(G)$ contains an o-basis, then there exist γ and distinct $1 \le i, j \le t_\gamma$ such that $\cos\left((i-j)\frac{2b}{l}\pi\right) = 0$, which clearly implies that $\nu_2(\frac{2b}{l}) < 0$. On the other hand, suppose $\nu_2(\frac{2b}{l}) = -k$, for some $k \in \mathbb{N}$. Then $\frac{b}{l} = \frac{m}{2^{k+1}}$ for some odd integer m. Since the existence of an o-basis depends on γ for which $\frac{bt_\gamma}{l} \in \mathbb{Z}$, 2^{k+1} is always a divisor of t_γ. Thus, we can choose $i_0 = 2^{k-1} + 1$ and $j_0 = 1$ so that $\cos\left((i_0 - j_0)\frac{2b}{l}\pi\right) = 0$. By Proposition 4.4, $e^{\phi}_{\gamma\sigma_{i_0}}$ and $e^{\phi}_{\gamma\sigma_{j_0}}$ are non zero and hence, by the above fact, $\{e^{\phi}_{\gamma r^{(2^{k-1}+1)p^t}}, e^{\phi}_{\gamma r^{p^t}}\}$ forms an o-basis for $V^{\phi}_{\gamma}(C)$. ◻

5. Dihedral group D_m. We first collect some facts about the Brauer characters of the dihedral groups D_{2n} from [10]. We follow the notions of [10] in this section. A presentation of the dihedral groups D_m having order $2m$, is given by $D_m =< r, s \mid r^m = s^2 = 1, srs = r^{-1} >$. The ordinary character table of D_m is:

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Table III: The character table of D_m.

Characters	r^k	sr^k
ψ_0	1	1
ψ_1	1	-1
ψ_2	$(-1)^k$	$(-1)^k$
ψ_2	$(-1)^k$	$(-1)^{k+1}$
χ_h $(1 \leq h < \frac{m}{2})$	$2\cos\frac{2\pi kh}{m}$	0

We write $m = lp^t$, where l is not divisible by prime number p, as before. The set of all p-regular elements of D_m are

$$\widehat{D_m} = \begin{cases} \{r^{jp^t}, sr^k \mid 0 \leq j < l, 0 \leq k < \frac{m}{2}\}, & p \neq 2; \\ \{r^{jp^t} \mid 0 \leq j < l\}, & p = 2. \end{cases}$$

The complete list of irreducible Brauer characters of D_m is, [10],

$$\hat{\psi}_j \ (0 \leq j < \epsilon), \ \hat{\chi}_h \ (1 \leq h < \tfrac{l}{2}),$$

where $\hat{\psi}_j = \psi_j \mid_{\widehat{D_m}}$, $\hat{\chi}_j = \chi_h \mid_{\widehat{D_m}}$ and

$$\epsilon = \begin{cases} 4, & l \text{ even}, p \neq 2; \\ 2, & l \text{ odd}, p \neq 2; \\ 1, & p = 2. \end{cases}$$

Necessary and sufficient condition for the existence of an o-basis for Brauer characters of dimension one is provided in [10]. Precisely, for $\phi = \hat{\psi}_j, \hat{\psi'}_j$, where $0 \leq j < \epsilon$, the space $V_\phi(D_m)$ has an o-basis if and only if $\dim V = 1$ or $p = 2$ or m is not divisible by p.

Necessary and sufficient condition for the existence of an o-basis for Brauer character of dimension two for D_m can be found in [10]. But it can also be obtained by very similar method applied on T_{4n} as we presented in §4. This is because ϕ has a cyclic support for each $\phi = \hat{\chi}_h$, where $0 \leq h < \frac{l}{2}$, and all values in the character tables of both groups are consistent on $C =< r^k >$. Thus, by changing m to $2n$ and h to b, each step of the computation for dimensions of $V_\gamma^\phi(D_m)$ and the condition for the existence becomes the same. This yields

THEOREM 5.1. *Let $G = D_m$, where $m = lp^t$ with l an integer not divisible by p and let $\phi = \hat{\chi}_h$, where $1 \leq h < \frac{l}{2}$. Then, $V_\phi(G)$ has an o-basis if and only if $\nu_2(\frac{2h}{l}) < 0$. Also, for each $\sigma \in C$, $e^\phi_{\gamma\sigma} \neq 0$ if and only if $\frac{ht_\gamma}{l} \in \mathbb{Z}$.*

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6. Irreducible Brauer character of SD_{8n}. A presentation for SD_{8n} for $n \geq 2$ is given by $SD_{8n} =< a, b \mid a^{4n} = b^2 = e, bab = a^{2n-1} >$. All $8n$ elements of SD_{8n} may be given by

$$SD_{8n} = \{e, a, a^2, \ldots, a^{4n-1}, b, ba, ba^2, \ldots, ba^{4n-1}\}.$$

The embedding of SD_{8n} into the symmetric group S_{4n} is given by $T(a)(t) := \overline{t+1}$ and $T(b)(t) := \overline{(2n-1)t}$, where $\overline{m}$ is the remainder of m divided by $4n$. We write $4n = lp^t$ with prime p and integer l not divisible by p and denote by $\widehat{SD_{8n}}$ the set of all p-regular elements of SD_{8n}. It is not hard to see that

$$\widehat{SD_{8n}} = \begin{cases} \{a^{jp^t}, ba^k \mid 0 \leq j < l; 0 \leq k < 4n\}, & \text{if } p \neq 2; \\ \{a^{jp^t} \mid 0 \leq j < l\}, & \text{if } p = 2. \end{cases}$$

By direct calculation, we have the following property.

Proposition 6.1. *The p-regular classes of SD_{8n}, $n \geq 2$ and $4n = lp^t$, are as follows:*

Case 1: p is odd prime.

- *If n is even (i.e., $\frac{l}{8} \in \mathbb{Z}$), then there are $\frac{l}{2} + 3$ p-regular classes. Precisely,*
  - *2 classes of size one being $\{e\}$ and $\{a^{\frac{l}{2}p^t}\}$,*
  - *$\frac{l}{4}-1$ classes of size two being $[a^{jp^t}] = \{a^{jp^t}, a^{(l-j)p^t}\}; j \in \{2, 4, 6, \ldots, \frac{l}{2}-2\}$,*
  - *$\frac{l}{8}$ classes of size two being $[a^{jp^t}] = \{a^{jp^t}, a^{(\frac{l}{2}-j)p^t}\}; j \in \{1, 3, 5, \ldots, \frac{l}{4}-1\}$,*
  - *$\frac{l}{8}$ classes of size two being $[a^{jp^t}] = \{a^{jp^t}, a^{(\frac{3l}{2}-j)p^t}\}; j \in \{\frac{l}{2}+1, \frac{l}{2}+3, \ldots, \frac{l}{2}+\frac{l}{4}-1\}$ and*
  - *2 classes of size $2n$ being $[b] = \{ba^{2i} \mid i = 0, 1, 2, \ldots, 2n-1\}$ and $[ba] = \{ba^{2i+1} \mid i = 0, 1, 2, \ldots, 2n-1\}$.*
- *If n is odd (i.e., $\frac{l}{4}$ is odd), then there are $\frac{l}{2} + 6$ p-regular classes. Precisely,*
  - *4 classes of size one being $\{e\}$, $\{a^{\frac{l}{4}p^t}\}$, $\{a^{\frac{l}{2}p^t}\}$ and $\{a^{\frac{3l}{4}p^t}\}$,*
  - *$\frac{l}{4}-1$ classes of size two being $[a^{jp^t}] = \{a^{jp^t}, a^{(l-j)p^t}\}; j \in \{2, 4, 6, \ldots, \frac{l}{2}-2\}$,*
  - *$\frac{l-4}{8}$ classes of size two being $[a^{jp^t}] = \{a^{jp^t}, a^{(\frac{l}{2}-j)p^t}\}; j \in \{1, 3, 5, \ldots, \frac{l}{4}-2\}$,*
  - *$\frac{l-4}{8}$ classes of size two being $[a^{jp^t}] = \{a^{jp^t}, a^{(\frac{3l}{2}-j)p^t}\}; j \in \{\frac{l}{2}+1, \frac{l}{2}+3, \ldots, \frac{l}{2}+\frac{l}{4}-2\}$ and*
  - *4 classes of size n being $[b] = \{ba^{4i} \mid i = 0, 1, 2, \ldots, n-1\}$, $[ba] = \{ba^{4i+1} \mid i = 0, 1, 2, \ldots, n-1\}$, $[ba^2] = \{ba^{4i+2} \mid i = 0, 1, 2, \ldots, n-1\}$ and $[ba^3] = \{ba^{4i+3} \mid i = 0, 1, 2, \ldots, n-1\}$.*

Case 2: $p = 2$. There are $\frac{l+1}{2}$ p-regular classes. Precisely, there is 1 class of size one, $\{e\}$, and there are $\frac{l-1}{2}$ classes of size two, $\{a^{jp^t}, a^{(l-j)p^t}\}$; $1 \leq j \leq \frac{l-1}{2}$.

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The ordinary irreducible character of SD_{8n} are given by (see [11]):

Table IV: The character table for SD_{8n}, when n is even.

Conjugacy classes, Characters	$[a^r]$; $r \in C_1$	$[a^r]$; $r \in C^{\dagger}_{odd}$	$[b]$	$[ba]$
χ_0	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1
ψ_h, where $h \in C^{\dagger}_{even}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2\cos\left(\frac{hr\pi}{2n}\right)$	0	0
ψ_h, where $h \in C^{\dagger}_{odd}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2i\sin\left(\frac{hr\pi}{2n}\right)$	0	0

Table V: The character table for SD_{8n}, when n is odd.

Conjugacy classes, Characters	$[a^r]$; $r \in C_1$	$[a^r]$; $r \in C^{odd}_{2,3}$	$[b]$	$[ba]$	$[ba^2]$	$[ba^3]$
χ'_0	1	1	1	1	1	1
χ'_1	1	1	-1	-1	-1	-1
χ'_2	1	-1	1	-1	1	-1
χ'_3	1	-1	-1	1	-1	1
χ'_4	$(-1)^{\frac{r}{2}}$	i^r	1	i	-1	$-i$
χ'_5	$(-1)^{\frac{r}{2}}$	i^r	-1	$-i$	1	i
χ'_6	$(-1)^{\frac{r}{2}}$	$(-i)^r$	1	$-i$	-1	i
χ'_7	$(-1)^{\frac{r}{2}}$	$(-i)^r$	-1	i	1	$-i$
ψ'_h, where $h \in C^{\dagger}_{even}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2\cos\left(\frac{hr\pi}{2n}\right)$	0	0	0	0
ψ'_h, where $h \in C^{\dagger}_{odd}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2i\sin\left(\frac{hr\pi}{2n}\right)$	0	0	0	0

where $C_1 = \{0, 2, 4, \ldots, 2n\}$, $C^{\dagger}_{even} := C_1 \setminus \{0, 2n\}$, $C^{odd}_{2,3} = \{1, 3, 5, \ldots, n, 2n+1, 2n+3, 2n+5, \ldots, 3n\}$, $C^{\dagger}_{odd} = \{1, 3, 5, \ldots, n-1, 2n+1, 2n+3, 2n+5, \ldots, 3n-1\}$.

For each k and h, put $\widehat{\chi}_k = \chi_k \mid_{\widehat{SD}_{8n}}$, $\widehat{\chi'}_k = \chi'_k \mid_{\widehat{SD}_{8n}}$ and $\widehat{\psi}_k = \psi_k \mid_{\widehat{SD}_{8n}}$, $\widehat{\psi'}_k = \psi'_k \mid_{\widehat{SD}_{8n}}$. Moreover, for odd prime p and $4n = lp^t$ such that l is not divisible

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by p, we define $E := \{2, 4, 6, \dots, \frac{l}{2} - 2\}$, $O_1^\epsilon := \{1, 3, 5, \dots, \frac{l}{4} - \epsilon\}$, $O_2^\epsilon := \{\frac{l}{2} + 1, \frac{l}{2} + 3, \dots, \frac{l}{2} + \frac{l}{4} - \epsilon\}$, where $\epsilon = 1$ if n is even and $\epsilon = 2$ if n is odd.

Proposition 6.2. *Let $IBr(SD_{8n})$ be the set of all distinct irreducible Brauer characters of SD_{8n}. Then,*

$$IBr(SD_{8n}) = \begin{cases} \{\widehat{\chi}_k, \widehat{\psi}_{jp^t} \mid 0 \le k \le 3, j \in E \cup O_1^1 \cup O_2^1\}, & \textit{if } p \neq 2 \textit{ and } n \textit{ is even;} \\ \{\widehat{\chi'}_k, \widehat{\psi'}_{jp^t} \mid 0 \le k \le 7, j \in E \cup O_1^2 \cup O_2^2\}, & \textit{if } p \neq 2 \textit{ and } n \textit{ is odd;} \\ \{\widehat{\chi}_0, \widehat{\psi}_{jp^t} \mid 0 \le j \le \frac{l-1}{2}\}, & \textit{if } p = 2 \textit{ and } n \textit{ is even.} \\ \{\widehat{\chi'}_0, \widehat{\psi'}_{jp^t} \mid 0 \le j \le \frac{l-1}{2}\}, & \textit{if } p = 2 \textit{ and } n \textit{ is odd.} \end{cases}$$

Proof. We first note that the restriction of a character of SD_{8n} to $\widehat{SD}_{8n}$ is a Brauer character and the order of the set $\mathrm{IBr}(SD_{8n})$ is the number of p-regular classes of SD_{8n}. Also, since SD_{8n} is solvable, by Fong-Swan theorem, any element in $\mathrm{IBr}(SD_{8n})$ is the restriction of an ordinary irreducible character of SD_{8n}.

Each $\widehat{\chi}_k$'s and $\widehat{\chi'}_k$'s are obviously irreducible and clearly distinct, by the character tables above. For characters of dimension two, $\widehat{\psi}_{jp^t}$ where p is an odd prime and n is even, we claim that those are irreducible. We suppose for a contradiction that $\widehat{\psi}_{jp^t} = \widehat{\chi}_i + \widehat{\chi}_k$ for some $j \in E \cup O_1^1 \cup O_2^2$ and $0 \le i, k \le 3$. Evaluating both sides at a^{2p^t} yields that

$$2 \cos \frac{jp^t \cdot 2p^t \pi}{2n} = 2.$$

That is $\cos \frac{4jp^t \pi}{l} = 1$, so $2j$ is a multiple of l. However, since $2j < l$ for $j \in E \cup O_1^1$ and $l < 2j < 2l$ for $j \in O_2^1$, this is a contradiction. We use similar arguments to show that all the remaining cases, $\widehat{\psi}_{jp^t}$'s are irreducible.

Next, we aim to show that all elements in $\mathrm{IBr}(SD_{8n})$ shown in the proposition are distinct. For the case odd prime p and even n, we suppose that $\widehat{\psi}_{jp^t} = \widehat{\psi}_{ip^t}$ for some $i, j \in E \cup O_1^1 \cup O_2^1$. It is clear (by the character table) that i, j either both are even or both are odd. If i, j are even, we evaluate both sides at a^{p^t} and then we get

$$\sin \frac{p^t(i+j)\pi}{l} \sin \frac{p^t(j-i)\pi}{l} = 0.$$

Since $gcd(l, p^t) = 1$ and $\frac{i+j}{l}$ and $\frac{j-i}{l}$ can not be positive integers for each $i, j \in E$, $i = j$. If i, j are odd, we evaluate both sides at a^{p^t}, and then we get

$$\sin \frac{p^t(j-i)\pi}{l} \cos \frac{p^t(j+i)\pi}{l} = 0.$$

Since $gcd(l, p^t) = 1$ and $\frac{i+j}{l} \neq \frac{l}{2}, \frac{3l}{2}$ for $i, j \in O_1^1 \cup O_2^1$ and $\frac{j-i}{l}$ can not be positive integer, $i = j$. Again, similar arguments work for all the remaining cases. ∎

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7. Existence of an o-basis for the class of tensors using a Brauer character of the SD_{8n}. In the following theorem, we denote

$$\epsilon = \begin{cases} 3 & \text{if} \quad p \neq 2, \quad n \ \text{ even} \\ 7 & \text{if} \quad p \neq 2, \quad n \ \text{ odd} \\ 1 & \text{if} \quad p = 2. \end{cases}$$

THEOREM 7.1. *Let* $\dim V > 2$, $G = SD_{8n}$, $0 \leq j \leq \epsilon$, *and put* $\phi = \hat{\chi}_j$ *or* $\hat{\chi'}_j$. *Then, space* $V_\phi(G)$ *has an o-basis if and only if* $p = 2$ *or* $4n$ *is not divisible by* p.

Proof. If $p = 2$ then $\widehat{G} = \langle a^{p^t} \rangle$. Since $\widehat{G}$ is a subgroup of G and ϕ is constant on $\widehat{G}$, by Proposition 2.7, $V_\phi(G)$ has an o-basis. Assume $p \neq 2$ and $4n$ is not divisible by p. Then $\hat{G} = G$ and consequently, these characters will be ordinary linear characters. Thus, $V_\phi(G)$ has an o-basis.

Conversely, suppose that $p \neq 2$ and $4n$ is divisible by p. We aim to show that $V_\phi(G)$ does not have an o-basis by showing that there exists $\alpha \in \Gamma_k^{4n}$ such that $V_\alpha^\phi(G)$ does not have an o-basis and then apply Theorem 2.1 to conclude the results.

Let $\alpha = (1, 2, 2, \ldots, 2, 3)$. Since $\dim V > 2$ and $4n \geq 4$, $\alpha \in \Gamma_{\dim V}^{4n}$. We also choose a representative Δ so that $\alpha \in \Delta$. We observe that to fix α, each $\sigma \in G$ must fix the first and the last position of α. It is clear that element of the form a^k satisfying the condition is only e. For elements of the form ba^k, they must satisfy $T(ba^k)(1) = 1$ and $T(ba^k)(4n) = 4n$. By using $T(ba^k)(t) = \overline{(2n-1)(k+t)}$, we conclude that $G_\alpha = \{e\}$. Since ϕ is a restriction of a linear character and $G_\alpha = \{e\}$, by Corollary 2.3, to show that $\langle e_{\alpha\sigma}^\phi, e_\alpha^\phi \rangle \neq 0$ for each $\sigma \in G$, it suffices to show that $A = \{\mu \in \widehat{G} | e \in \sigma\mu^{-1}\widehat{G}\} \neq \emptyset$. This is simple because $p \neq 2$ and $4n$ is divisible by p, so $\widehat{G} = \{ba^k | 0 \leq k < m\} = \widehat{G}^{-1}$ and then $A = \{\mu \in \widehat{G} | \sigma \in \widehat{G}\mu\}$. Since $e \in \sigma\mu^{-1}\widehat{G}$ if and only if $\sigma \in \widehat{G}^{-1}\mu = \widehat{G}\mu$ and thus for arbitrary $0 \leq k < 4n$, we have $a^k = ba^0ba^k \in \widehat{G}^2$ and $ab^k = a^0ba^k \in \widehat{G}^2$, so $G \subseteq \widehat{G}^2$. That is $A \neq \emptyset$. So,

$$\langle e_{\alpha\sigma}^\phi, e_\alpha^\phi \rangle \neq 0 \quad \text{for each } \sigma \in G. \tag{7.1}$$

Next, to show that $\{e_{\alpha a}^\phi, e_\alpha^\phi\} \subseteq V_\alpha^\phi(G)$ is a linearly independent set, we set $e_\alpha^\phi = \sum_\delta c_\delta e_\delta$ and $e_{\alpha a}^\phi = \sum_\delta d_\delta e_\delta$, as $\{e_\delta | \delta \in \Gamma_k^m\}$ forms a basis for $V^{\otimes m}$. Since $\widehat{G}^{-1} = \widehat{G}$,

$$e_\alpha^\phi = \frac{\phi(1)}{|\widehat{G}|} \sum_{\sigma \in \widehat{G}} \phi(\sigma^{-1}) e_{\alpha\sigma}.$$

Since $G_\alpha = \{e\}$, the elements $\alpha\sigma$ with $\sigma \in G$ are distinct. Also, since $a \notin \widehat{G}$, $\alpha\sigma \neq \alpha a$ for all $\sigma \in \widehat{G}$, which yields that $c_a = 0$. On the other hand, $G_{\alpha a} = a^{-1}G_\alpha a = \{e\}$,

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so for $a \in \widehat{G}$, $(\alpha a)\sigma = \alpha a$ if and only if $\sigma = e$. This implies that $d_a = \frac{1}{|\widehat{G}|} \neq c_{\alpha a} = 0$. Thus, $\{e^{\phi}_{\alpha a}, e^{\phi}_{\alpha}\} \subseteq V^{\phi}_{\alpha}(G)$ is a linearly independent set, and hence, $\dim V^{\phi}_{\alpha}(G) \geq 2$.

By Proposition 2.5, if $V^{\phi}_{\alpha}(G)$ were to have an o-basis, then it would have an o-basis containing e^{ϕ}_{α}, but, by (7.1), this is not the case. So, $V^{\phi}_{\alpha}(G)$ does not have an o-basis, which completes the proof. ◻

Remark 7.2. Theorem 7.1 shows that if $dimV > 2$, unlike the case for an irreducible character, it is possible that $V_{\phi}(G)$ has no o-basis when ϕ is a linear Brauer character. This holds when $dimV = 2$ as well. To observe this, we let $p \neq 2$ and $4n$ is divisible by p and $\phi = \hat{\chi}_0$ or $\hat{\chi'}_0$. Consider $\alpha = (1, 2, \ldots, 2, 2) \in \Gamma^{4n}_{\dim V}$. Thus, for such α, we have $G_{\alpha} = \{1, a^{2n+2}b\}$. Now by similar calculations done in Theorem 7.1 we have $\langle e^{\phi}_{\alpha\sigma_1}, e^{\phi}_{\alpha\sigma_2}\rangle \neq 0$.

For the remaining of this section, we denote

$$\Pi = \begin{cases} \{jp^t | j \in E \cup O^1_1 \cup O^1_2\}, & \text{if } p \neq 2 \text{ and } n \text{ even;} \\ \{jp^t | j \in E \cup O^2_1 \cup O^2_2\}, & \text{if } p \neq 2 \text{ and } n \text{ odd;} \\ \{jp^t | 0 \leq j \leq \frac{l-1}{2}\}, & \text{if } p = 2. \end{cases}$$

For $V_{\phi}(SD_{8n})$, where $\phi = \hat{\psi}_h, \hat{\psi'}_h$ such that $h \in \Pi$ is even, the condition for the existence can be obtained in the same manner as T_{4n} and D_m. This is because ϕ has a cyclic support and all values in the character tables of those groups are consistence on $C =< a^r >$ if h is even. Then, we have:

Theorem 7.3. *Let $G = SD_{8n}$, where $4n = lp^t$ with l an integer not divisible by p and let $\phi = \hat{\psi}_h, \hat{\psi'}_h$ such that $h \in \Pi$ be even. Then, $V_{\phi}(G)$ has an o-basis if and only if $\nu_2(\frac{2h}{l}) < 0$. Also, for each $\sigma \in C$, $e^{\phi}_{\gamma\sigma} \neq 0$ if and only if $\frac{ht_{\gamma}}{l} \in \mathbb{Z}$.*

For the case where $h \in \Pi$ is odd, we first compute the dimension of $V^{\phi}_{\gamma}(SD_{8n})$.

Proposition 7.4. *Let $G = SD_{8n}$, where $4n = lp^t$ with $p \nmid l$ and let $\phi = \hat{\psi}_h, \hat{\psi'}_h$ such that $h \in \Pi$ be odd. For $\gamma \in \Delta$ such $C \cap G_{\gamma} =< a^{t_{\gamma}p^t} >$, where $t_{\gamma} = \frac{l}{|C \cap G_{\gamma}|}$, then*

$$\dim(V^{\phi}_{\gamma}(C)) = \begin{cases} 4, & \text{if } \frac{ht_{\gamma}}{l} \in \mathbb{Z}; \\ 0, & \text{if } \frac{ht_{\gamma}}{l} \notin \mathbb{Z}. \end{cases}$$

Proof. Since G has cyclic support with $C =< a^{p^t} >$ and $C \cap G_{\gamma} =< a^{t_{\gamma}p^t} >$, for each ϕ, we can compute the dimension by using Proposition 3.4. By character tables, we compute $v_j = \sum_{h \in C \cap G_{\gamma}} \phi(h\tau^{t_{\gamma}-j})$, for the different case of j and t_{γ}. If t_{γ} is odd,

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then, for even j,

$$\begin{array}{rcl} v_j & = & \sum_{m=1}^{l/t_\gamma} \phi(r^{(mt_\gamma - j)p^t}) \\ & = & i\sum_{k=1}^{l/2t_\gamma} 2\sin\left((2k-1)(\frac{2ht_\gamma}{l})\pi - (\frac{2hj}{l})\pi\right) \\ & & + \sum_{k=1}^{l/2t_\gamma} 2\cos\left((2k)(\frac{2ht_\gamma}{l})\pi - (\frac{2hj}{l})\pi\right) \\ & = & 0, \end{array}$$

and for odd j,

$$\begin{array}{rcl} v_j & = & \sum_{m=1}^{l/t_\gamma} \phi(r^{(mt_\gamma - j)p^t}) \\ & = & \sum_{k=1}^{l/2t_\gamma} 2\cos\left((2k-1)(\frac{2ht_\gamma}{l})\pi - (\frac{2hj}{l})\pi\right) \\ & & + i\sum_{k=1}^{l/2t_\gamma} 2\sin\left((2k)(\frac{2ht_\gamma}{l})\pi - (\frac{2hj}{l})\pi\right) \\ & = & 0, \end{array}$$

since $\frac{2ht_\gamma}{l} \notin \mathbb{Z}$ (because h, t_γ are odd and then $4 \mid l$). So, if t_γ is odd (i.e., $\frac{ht_\gamma}{l} \notin \mathbb{Z}$), then $d_\gamma = t_\gamma$ and thus $\dim(V_\gamma^\phi(C)) = t_\gamma - t_\gamma = 0$.

Similarly, if t_γ is even, we compute that

$$v_j = \begin{cases} 0, & \frac{ht_\gamma}{l} \notin \mathbb{Z}; \\ -\frac{2il}{t_\gamma}\sin\left(\frac{2hj}{l}\pi\right), & \frac{ht_\gamma}{l} \in \mathbb{Z} \text{ and } j \text{ odd} \\ \frac{2l}{t_\gamma}\cos\left(\frac{2hj}{l}\pi\right), & \frac{ht_\gamma}{l} \in \mathbb{Z} \text{ and } j \text{ even.} \end{cases}$$

So, if t_γ is even and $\frac{ht_\gamma}{l} \notin \mathbb{Z}$, then $d_\gamma = t_\gamma$ and thus $\dim(V_\gamma^\phi(C)) = t_\gamma - t_\gamma = 0$. For d_γ in which t_γ is even and $\frac{ht_\gamma}{l} \in \mathbb{Z}$, we have that $\sum_{j=0}^{t_\gamma - 1} v_j e^{\frac{2\pi sji}{t_\gamma}} = 0$ if and only if

$$\frac{l}{t_\gamma}\left[\sum_{k=0}^{\frac{t_\gamma}{2}-1} 2\cos\left(\left(\frac{4hk}{l}\right)\pi\right)\cos\left(\left(\frac{4sk}{t_\gamma}\right)\pi\right) + \sum_{k=0}^{\frac{t_\gamma}{2}-1} 2\sin\left(\left(\frac{2h(2k+1)}{l}\right)\pi\right)\sin\left(\left(\frac{2s(2k+1)}{t_\gamma}\right)\pi\right)\right],$$

and

$$\frac{il}{t_\gamma}\left[\sum_{k=0}^{\frac{t_\gamma}{2}-1} 2\cos\left(\left(\frac{4hk}{l}\right)\pi\right)\sin\left(\left(\frac{4sk}{t_\gamma}\right)\pi\right) - \sum_{k=0}^{\frac{t_\gamma}{2}-1} 2\sin\left(\left(\frac{2h(2k+1)}{l}\right)\pi\right)\cos\left(\left(\frac{2s(2k+1)}{t_\gamma}\right)\pi\right)\right],$$

are simultaneously zero. Since $\frac{ht_\gamma}{l} \in \mathbb{Z}$, the second sum is always zero and the first sum is zero for each $0 \le s < t_\gamma$ except for $2\left(\frac{h}{l} + \frac{s}{t_\gamma}\right) \in \mathbb{Z}$ or $2\left(\frac{h}{l} - \frac{s}{t_\gamma}\right) \in \mathbb{Z}$. Since $0 \le \frac{s}{t_\gamma} < 1$,

$$\frac{2h}{l} \le 2\left(\frac{h}{l} + \frac{s}{t_\gamma}\right) < \frac{2h}{l} + 2 \text{ and } \frac{2h}{l} - 2 < 2\left(\frac{h}{l} - \frac{s}{t_\gamma}\right) \le \frac{2h}{l}.$$

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Precisely, if s belongs to

$$\left\{s_1 := \frac{t_\gamma}{2}\lceil\frac{2h}{l}\rceil - \frac{ht_\gamma}{l}, s_2 := \frac{t_\gamma}{2}\lceil\frac{2h}{l}\rceil - \frac{ht_\gamma}{l} + \frac{t_\gamma}{2}, s_3 := \frac{ht_\gamma}{l} - \frac{t_\gamma}{2}\lfloor\frac{2h}{l}\rfloor, s_4 := \frac{ht_\gamma}{l} - \frac{t_\gamma}{2}\lfloor\frac{2h}{l}\rfloor + \frac{t_\gamma}{2}\right\},$$

then the first sum will be not zero. Here, $\lceil r\rceil$ and $\lfloor r\rfloor$ are the ceiling function and floor function of the real number r, respectively. Since $t_\gamma > 0$ and $\frac{4h}{l} \notin \mathbb{Z}$ for each odd $h \in \Pi$, s_1, s_2, s_3, s_4 are all distinct. Hence, $d_\gamma = t_\gamma - 4$, and thus, the results follow. ∎

Now, we have:

THEOREM 7.5. *Let $G = SD_{8n}$, where $4n = lp^t$ with $p \nmid l$ and let $\phi = \hat{\psi}_h, \hat{\psi}'_h$, where $h \in \Pi$ be odd. If $\dim(V) > 1$, then, $V_\phi(G)$ does not have an o-basis. Also, for each $\sigma \in C$, $e^\phi_{\gamma\sigma} \neq 0$ if and only if $\frac{ht_\gamma}{l} \in \mathbb{Z}$.*

Proof. By Proposition 3.2, it is enough to focus on $V^\phi_\gamma(C)$. Also, in the proof of Proposition 3.3, we have $V^\phi_\gamma(C) = \langle e^\phi_{\gamma\sigma_j} | j = 1, 2, \ldots, t_\gamma\rangle$, where $\sigma_j = a^{jp^t}$ and $t_\gamma = \frac{|C|}{|C\cap G_\gamma|}$. By (3.1) and the character tables, we compute that, for even i, j such $1 \le i, j \le t_\gamma$,

$$\begin{aligned}
\langle e^\phi_{\gamma\sigma_i}, e^\phi_{\gamma\sigma_j}\rangle = 0 \iff & \textstyle\sum_{g\in C} \phi(g\sigma_i)\overline{\phi(g\sigma_j)} = 0\\
\iff & \textstyle\sum_{k=0}^{l-1} \phi(a^{(k+i)p^t})\overline{\phi(a^{(k+j)p^t})} = 0\\
\iff & 2\left[\textstyle\sum_{k=0}^{\frac{l}{2}-1} 2\cos\left((2k+i)\frac{2h}{l}\pi\right)\cos\left((2k+j)\frac{2h}{l}\pi\right)\right]\\
& +2\left[\textstyle\sum_{k=0}^{\frac{l}{2}-1} 2\sin\left((2k+1+i)\frac{2h}{l}\pi\right)\cos\left((2k+1+j)\frac{2h}{l}\pi\right)\right] = 0\\
\iff & 2\left[\textstyle\sum_{k=0}^{\frac{l}{2}-1} \cos\left((4k+i+j)\frac{2h}{l}\pi\right) + \sum_{k=0}^{\frac{l}{2}-1}\cos\left((i-j)\frac{2h}{l}\pi\right)\right]\\
& +2\left[\textstyle\sum_{k=0}^{\frac{l}{2}-1} \cos\left((i-j)\frac{2h}{l}\pi\right) - \sum_{k=0}^{\frac{l}{2}-1}\cos\left((4k+i+j)\frac{2h}{l}\pi\right)\right] = 0\\
\iff & 2l\cos\left((i-j)\tfrac{2h}{l}\pi\right) = 0 \quad (\text{since } \tfrac{4h}{l} \notin \mathbb{Z}).
\end{aligned}$$

Similar arguments work well for the remaining cases. Thus, we can conclude that

$$\langle e^\phi_{\gamma\sigma_i}, e^\phi_{\gamma\sigma_j}\rangle = 0 \iff \begin{cases} \cos\left((i-j)\frac{2h}{l}\pi\right) = 0, & \text{if } i, j \text{ are both even or both odd;}\\ \sin\left((i-j)\frac{2h}{l}\pi\right) = 0, & \text{if ortherwise.}\end{cases} \tag{7.2}$$

We consider $\gamma = (1, 2, 2, \ldots, 2) \in \Gamma^{4n}_{\dim(V)}$. Since $\dim(V) > 1$, $\gamma \in \Delta$ and it is not hard to see that $G_\gamma = \{e\}$. So, $t_\gamma = l$ and then $\frac{ht_\gamma}{l} \in \mathbb{Z}$. By Proposition 7.4, $\dim(V^\phi_\gamma(C)) = 4$. Thus, if $V^\phi_\gamma(C)$ has an o-basis, then there exist distinct $1 \le i_1, i_2, i_3, i_4 \le t_\gamma$ such that $\{e^\phi_{\gamma\sigma_{i_1}}, e^\phi_{\gamma\sigma_{i_2}}, e^\phi_{\gamma\sigma_{i_3}}, e^\phi_{\gamma\sigma_{i_4}},\}$ forms an o-basis. Since h is odd and $4 \mid l$, $\nu(\frac{2h}{l}) = -k$, for some positive integer k. Hence, if there are at least three of i_1, i_2, i_3, i_4 which are all even or all odd, say i_1, i_2, i_3, then, by (7.2), there must exist odd integers o_1, o_2, o_3 such that

$$i_1 - i_2 = o_1 \cdot 2^{k-1}, \;\; i_1 - i_3 = o_2 \cdot 2^{k-1}, \;\text{ and }\; i_2 - i_3 = o_3 \cdot 2^{k-1}.$$

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This implies that $o_2 - o_3 = o_1$, which is a contradiction. If there are exactly two of i_1, i_2, i_3, i_4 which are all odd, say i_1, i_2, then, by (7.2), there must exist integer s such that $i_1 - i_3 = s \cdot 2^k$. This implies that $i_1 = i_3 + s \cdot 2^k$ is even (because i_3 is even), which is a contradiction. Therefore, $V_\gamma^\phi(C)$ does not have an o-basis and by Proposition 3.2, we finish the proof for the first statement. The second statements is a consequence of Proposition 7.4 and a direct calculation as in Proposition 4.4. □

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