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# BLOCK IMBEDDING AND INTERLACING RESULTS FOR NORMAL MATRICES*

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**Abstract.** A pair of matrices is said to be imbeddable precisely when one is an isometric projection of the other on a suitable subspace. The concept of imbedding has been the subject of extensive study. Particular emphasis has been placed on relating the spectra of the matrices involved, especially when both matrices are Hermitian or normal. In this paper, the notion of block imbedding is introduced and shown to be intimately connected to an extension of interlacing for eigenvalues of normal matrices. Thus, a generalization of a classic Theorem of K. Fan and G. Pall is obtained, which is then applied to yield bounds on the number of eigenvalues of a block imbeddable pair in a closed, convex set. Moreover, a wide class of normal matrices, for which block imbedding applies, is indicated. Finally, comments and links on the necessary imbedding conditions of D. Carlson and E.M. de Sa, and J.P. Queiro and A.L. Duarte are provided.

**Key words.** Normal matrix, Eigenvalues, Interlacing.

**AMS subject classifications.** 15A18, 15A42, 15A57, 47A20, 47B20.

**1. Introduction.** Imbedding holds for a pair of matrices precisely when one is an isometric projection of the other on a suitable subspace. More precisely, given $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$ $(1 \leq k < n)$, we shall say that $B$ is *imbeddable* in $A$, or a *compression* of $A$, if there exists an isometry $V \in \mathbb{C}^{n\times(n-k)}$ $(V^*V = I_{n-k})$, such that $V^*AV = B$. The concept of imbedding has been the subject of extensive study. For instance, in the context of iterative methods for eigenvalue computation [13, 12], the eigenvalues of the imbeddable matrix $B = V^*AV$ are referred to as *Ritz values* of $A$ with respect to the range space of the isometry $V$ and play an important role both as eigenvalue estimates and as the roots of restart polynomials. Hence, for the convergence analysis of such methods, it is of interest to estimate their location for a given matrix. An immediate observation is that the *spectrum* $\sigma(B)$ is contained in the *numerical range* of $A$, which is defined as the closed and convex subset of the complex plane

$$w(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\|_2 = 1\}.$$

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Notice that when $A$ normal, then $w(A)$ coincides with the convex hull of its eigenvalues, i.e., $w(A) = \mathrm{co}\,\{\sigma(A)\}$. An improved geometric inclusion would be highly desirable. Since the eigenvalues of a diagonal matrix are revealed along its main diagonal, some results related to the construction of isometries $V$, such that $V^*AV$ is diagonal, can be found in [8, 9, 10].

A fundamental result is the interlacing property of the eigenvalues in the *Hermitian* case. More specifically, if $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$ are Hermitian matrices with eigenvalues $\sigma(A) = \{\lambda_i\}_{i=1}^{n}$, $\sigma(B) = \{\mu_i\}_{i=1}^{n-k}$ in non-decreasing order respectively, then [2, Thm. 1] $B$ is imbeddable in $A$ *if and only if*

$$\mu_i \in [\lambda_i, \lambda_{i+k}], \quad \text{for } i = 1, \ldots, n-k. \tag{1.1}$$

Relations (1.1) were known to be necessary for Hermitian matrices earlier [3] and are usually referred to as *Cauchy interlacing inequalities* or the inclusion principle [4, Thm. 4.3.15]. There are also several generalizations of the interlacing property to normal matrices. The case $A$, $B$ are both normal and $k = 1$ has been handled by Fan and Pall in [2, Thm. 2]. Their statement is the following:

THEOREM 1.1. (Fan and Pall, [2]) *Let the normal matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-1)\times(n-1)}$ with spectra $\sigma(A) = \{\lambda_i\}_{i=1}^{n}$ and $\sigma(B) = \{\mu_i\}_{i=1}^{n-1}$ respectively. The matrix $B$ is imbeddable in $A$ if and only if their eigenvalues $\{\lambda_i\}_{i=1}^{q}$ and $\{\mu_i\}_{i=1}^{q-1}$ ($q \in \{1, \ldots, n\}$) are distinct and lain on a straight line $\mathcal{L}$ of $\mathbb{C}$, such that $\mu$'s separate $\lambda$'s, while the remaining eigenvalues are common, i.e., $\lambda_i = \mu_{i-1}$, for $i = q+1, \ldots, n$.*

Hence, by Theorem 1.1, it becomes apparent that the imbeddable normal pair $B$, $A$ emerges as an *affine* transformation (i.e., a rotation followed by a translation) of some corresponding Hermitian pair $Q \in \mathbb{C}^{(q-1)\times(q-1)}$, $H \in \mathbb{C}^{q\times q}$, with distinct eigenvalues and $Q$ imbeddable in $H$ ($q \in \{1, \ldots, n\}$), possibly expanded (when $q \neq n$) via a direct sum with an arbitrary diagonal matrix.

Additionally, for $k > 1$, a necessary interlacing condition involving the arguments of the eigenvalues of $A$ and $B$ has been proved by Carlson and de Sa in [1], while in [11], Queiro and Duarte have shown that these are interlacing with respect to the lexicographic orders in $\mathbb{C}$.

In this paper, we investigate the case $k > 1$ further and introduce the concept of *block imbedding*, which allows for an extension of Theorem 1.1 for larger $k$. More precisely, block imbedding involves specific restrictions on the form of the normal imbeddable pair $A$, $B$ and is shown then to be equivalent to the allocation of their noncommon eigenvalues on collinear segments. Recognizing the fact that the imbedding of $B$ in $A$ is equivalent to $B$ being the leading submatrix of the unitarily similar to $A$ matrix $\begin{bmatrix} V & V^{\perp} \end{bmatrix}^* A \begin{bmatrix} V & V^{\perp} \end{bmatrix} = \begin{bmatrix} B & D \\ C^* & F \end{bmatrix}$, block imbedding involves certain additional restrictions on the blocks $C$, $D$ and $F$. Thus, this notion is from another

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perspective intimately associated to another line of research concerning the expansion of matrices of the form $\left[\begin{smallmatrix} B \\ C^* \end{smallmatrix}\right]$, with $B \in \mathbb{C}^{(n-k)\times(n-k)}$ normal and $C \in \mathbb{C}^{(n-k)\times k}$, so that the resulting matrix $A = \left[\begin{smallmatrix} B & D \\ C^* & F \end{smallmatrix}\right] \in \mathbb{C}^{n\times n}$ is normal as well. Formulations for the remaining submatrices $C$, $D$ and $F$ in the cases $k = 1, 2$ have been determined by Ikramov and Elsner [5], while sufficient conditions on $C$ for the above extension to be possible when $k$ is larger have been obtained by Jiang and Kuo in [6].

In the following section, we introduce the notion of block imbedding and investigate the implications of the corresponding definition. It is proved that the non-common eigenvalues of an imbeddable normal matrix pair $A$, $B$ are interlacing on collinear segments precisely when block imbedding holds, resulting thus to an extension of Theorem 1.1 for $k > 1$. Moreover, a wide class of normal matrices for which block imbedding applies is presented. We conclude this section with upper and lower bounds on the number of eigenvalues of the imbedded matrix $B$ inside a convex set, that also contains eigenvalues of $A$.

In Section 3, we review some necessary imbedding conditions for normal matrices that have appeared in the literature, providing links and connections thereon. More precisely, we consider the Carlson and de Sa, and Queiro and Duarte conditions mentioned earlier [1, 11]. The first ensures eigenvalue interlacing with respect to sectors, while the latter involves $\vartheta$-interlacing. (For more details, see Theorem 3.1.) One of the open problems posed by the authors in [11] inquires which more specific geometric restrictions does the $\vartheta$-interlacing condition for all $\vartheta$ impose on the eigenvalue configurations. Here, we provide an answer in the case the spectrum of $A$ happens to be convexly independent. In this direction, we characterize polygons in the complex plane through $\vartheta$-interlacing and apply our result to show that $\vartheta$-interlacing for all $\vartheta$ implies that every $(k+1)$-polygon with vertices in $\sigma(A)$ contains at least one eigenvalue of $B$. Finally, we investigate the interrelation of the Carlson and de Sa, and Queiro and Duarte imbedding conditions for $k = 1$ and conclude that both are equivalent to the Fan and Pall criterion (Theorem 1.1).

**2. Block imbedding for normal matrices.** Considering a pair of normal matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$ with $k > 1$ and non-collinear eigenvalues, it is not in general easy to determine whether $B$ is imbeddable in $A$. Nevertheless, if parts of the spectra of $A$ and $B$ form groups of collinear and interlacing eigenvalues, then it turns out that imbeddability holds. Toward this direction, we introduce the following definition concerning the form of $A$:

DEFINITION 2.1. Let the normal matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$. The matrix $B$ is called *block imbeddable* in $A$, when for some straight lines $\mathcal{L}_j$, $j = 1, \ldots, r\,(\leq n-k)$, of $\mathbb{C}$, the matrix $A$ is unitarily similar to a normal extension $\left[\begin{smallmatrix} B & D \\ C^* & F \end{smallmatrix}\right]$ of $B$ satisfying the following conditions:

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1. $F = \oplus_{j=1}^{r} \mathrm{diag}\left\{ f_i^j \right\}_{i=1}^{k_j}$, *where* $f_i^j \in \mathcal{L}_j$ $(i = 1, \ldots, k_j)$ *and* $\sum_{j=1}^{r} k_j = k$.
2. $C$ *and* $D$ *are related by the equation* $D = CU$, *where* $U = \oplus_{j=1}^{r} e^{2\mathbf{i}\vartheta_j} I_{k_j}$ *with* $\vartheta_j \in [0, \pi)$ *the slopes of the lines* $\mathcal{L}_j$ $(j = 1, \ldots, r)$.
3. *Partitioning* $C = \begin{bmatrix} C_1 & \cdots & C_r \end{bmatrix}$, *with* $C_j \in \mathbb{C}^{(n-k)\times k_j}$, *the blocks* $C_j$ *are pairwise orthogonal, i.e.,* $C_i^* C_j = 0$ *for* $i \neq j \in \{1, \ldots r\}$.

In this way, conditions 1, 2 and 3 should be called "conditions of *block imbeddability*".

Some remarks concerning the *independence* of the conditions of block imbeddability in the definition above are in order. By the normality of the extension of $B$ and the diagonal form of $F$ in condition 1, it is straightforward to check that the equations

$$CC^* = DD^*, \tag{2.1}$$

$$BC + DF^* = B^*D + CF \tag{2.2}$$

hold. Using (2.1) and considering the singular value decompositions $C = M^*\Sigma N$ and $D = M^*\Sigma\hat{N}$, it is readily established that $D = CU$, where $U = N^*\hat{N} \in \mathbb{C}^{k\times k}$ is unitary. However, we cannot say that condition 2 holds, since $U$ is not necessarily of the form stated therein. In condition 2, $\vartheta_j$ is the slope of $\mathcal{L}_j$ $(j = 1, \ldots, r)$, whereas $U$ in general remains undetermined.

Moreover, note that conditions 1 and 2 of block imbeddability necessarily imply a special structure for the columns of $C$, $D$. In the following, we denote by $E_\lambda(\cdot)$ the eigenspace of a matrix associated with its eigenvalue $\lambda \in \sigma(\cdot)$ and, for any line $\mathcal{L}$ on the complex plane, let $E_{\mathcal{L}}(\cdot) = \oplus_{\lambda\in\sigma(\cdot)\cap\mathcal{L}} E_\lambda(\cdot)$.

PROPOSITION 2.2. *Conditions 1 and 2 of block imbeddability imply that the columns of blocks* $C_j$ *belong in*

$$E_{\mathcal{L}_j}(B) = \bigoplus_{\mu\in\sigma(B)\cap\mathcal{L}_j} E_\mu(B), \quad \text{for } j \in \{1, \ldots, r\}.$$

*Proof.* Retaining the notation in Definition 2, we consider $B \in \mathbb{C}^{(n-k)\times(n-k)}$ and its normal extension $\left[\begin{smallmatrix} B & D \\ C^* & F \end{smallmatrix}\right] \in \mathbb{C}^{n\times n}$ satisfying conditions 1 and 2 of block imbeddability. Combining equation (2.2) with $D = CU$, we obtain

$$BC - B^*CU = CF - CUF^*. \tag{2.3}$$

Denoting the submatrix $C_j = \begin{bmatrix} c_1^j & \cdots & c_{k_j}^j \end{bmatrix} \in \mathbb{C}^{(n-k)\times k_j}$ $(j = 1, \ldots, r)$ and taking the forms of $F$ and $U$ in conditions 1 and 2 into account, equation (2.3) yields

$$Bc_i^j - e^{2\mathbf{i}\vartheta_j} B^* c_i^j = c_i^j \left( f_i^j - e^{2\mathbf{i}\vartheta_j} \overline{f_i^j} \right), \quad \text{for } i = 1, \ldots, k_j \text{ and } j = 1, \ldots, r. \tag{2.4}$$

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Multiplying (2.4) by $e^{-\mathbf{i}\vartheta_j}$ shows that the columns of $C_j$ are eigenvectors of the skew-Hermitian part of $e^{-\mathbf{i}\vartheta_j}B$ corresponding to the eigenvalues $\left\{\mathrm{Im}\left(e^{-\mathbf{i}\vartheta_j}f_i^j\right)\right\}_{i=1}^{k_j}$. Recalling $B$ is normal and the inclusion $\left\{f_i^j\right\}_{i=1}^{k_j} \subset \mathcal{L}_j$ from Definition 2, it is immediate that the columns of $C_j$ also span an eigenspace of $B$ with corresponding eigenvalues lying on the line $\mathcal{L}_j$, whereby the assertion follows. □

Hence, in view of Proposition 2.2, properties 1 and 2 of block imbedding necessarily *imply* property 3 when the intersections of the lines $\mathcal{L}_j$ are not eigenvalues of $B$.

REMARK 2.3. It is of interest to note that the conclusion of Proposition 2.2 is related to one of the sufficient conditions investigated by Jiang and Kuo in [6], upon which $\left[\begin{smallmatrix} B \\ C^* \end{smallmatrix}\right]$ with $B$ normal may have a normal extension $\left[\begin{smallmatrix} B & D \\ C^* & F \end{smallmatrix}\right]$.

An important consequence of block imbeddability is the special restrictions it imposes upon the spectra of $A$, $B$. Indeed, we may exploit the pairwise orthogonality of the block columns of $C$ from Proposition 2.2 to reach the following result, showing in fact that their noncommon eigenvalues necessarily form collinear and interlacing groups on the lines $\mathcal{L}_j$. Thus, Theorem 1.1 is extended to the case of groups of collinear eigenvalues. In the following, we use the notation $|\cdot|$ to denote the cardinality of a set and $\overline{zw}$ for the line segment in $\mathbb{C}$ with endpoints $z$, $w$.

THEOREM 2.4. *Let the normal matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$. Then, $B$ is block imbeddable in $A$ for some straight lines $\mathcal{L}_j$ $(j = 1, \ldots, r\,(\leq n-k))$ of $\mathbb{C}$ if and only if the distinct eigenvalues of $A$ and $B$ are interlacing on $\{\mathcal{L}_j\}_{j=1}^r$, i.e., their spectra are partitioned in the sets*

$$\sigma_j(A) = [\sigma(A)\backslash(\sigma(A)\cap\sigma(B))]\cap\mathcal{L}_j \equiv \left\{\lambda_i^j\right\}_{i=1}^{n_j} \quad \textit{and}$$
$$\sigma_j(B) = [\sigma(B)\backslash(\sigma(A)\cap\sigma(B))]\cap\mathcal{L}_j \equiv \left\{\mu_i^j\right\}_{i=1}^{n_j-k_j}, \ j = 1, \ldots, r,$$

*with $\sum_{j=1}^r n_j = n - s$, $\sum_{j=1}^r k_j = k$, $s = |\sigma(A)\cap\sigma(B)|$, and may be ordered so that*

$$\mu_i^j \in \overline{\lambda_i^j\lambda_{i+k_j}^j}, \ \textit{for } i = 1, \ldots, n_j - k_j. \tag{2.5}$$

*Proof.* Due to the fact the spectrum of a matrix is preserved under unitary similarity, it is clearly enough to consider for $A$ the special formulation $\left[\begin{smallmatrix} B & D \\ C^* & F \end{smallmatrix}\right]$. We will show that its leading submatrix $B$ is block imbeddable in $A$ for some straight lines $\mathcal{L}_j$ $(j = 1, \ldots, r)$ precisely when their noncommon eigenvalues are interlacing on $\mathcal{L}_j$. Denoting $|\sigma(A)\cap\sigma(B)| = s$, we may partition the remaining eigenvalues in $\sigma(B)$

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according to their configuration on the lines $\mathcal{L}_j$ as

$$\sigma_j(B) = [\sigma(B)\backslash(\sigma(A)\cap\sigma(B))]\cap\mathcal{L}_j \quad (j=1,\ldots,r)$$

with $|\sigma_j(B)| \equiv n_j - k_j$ for some $n_j$ and $k_j$ as in the definition of block imbeddability. In the case the intersection of some lines $\mathcal{L}_\sigma \cap \mathcal{L}_\tau$, with $\sigma,\tau \in \{1,\ldots,r\}$, is an eigenvalue of B, then this is assumed to belong in $\sigma_\sigma(B)$ and $\sigma_\tau(B)$ with respective multiplicities adding up to its multiplicity as an element of $\sigma(B)$. Notice that $\sum_{j=1}^r (n_j - k_j) = \sum_{j=1}^r n_j - k \leq |\sigma(B)\backslash(\sigma(A)\cap\sigma(B))| = (n-k) - s$, where the difference

$$n - s - \sum_{j=1}^{r} n_j \geq 0 \tag{2.6}$$

enumerates the total number of eigenvalues in $\sigma(B)\backslash(\sigma(A)\cap\sigma(B))$ which are not allocated on the lines $\mathcal{L}_j$. Consider $A = WD_AW^*$, $B = VD_BV^*$ the respective diagonalizations of $A$ and $B$, where $\oplus_{j=1}^r E_j \equiv \left(\oplus_{j=1}^r \mathrm{diag}\,\{\sigma_j(B)\}\right)$ should be the leading $\left(\sum_{j=1}^r n_j - k\right) \times \left(\sum_{j=1}^r n_j - k\right)$ submatrix of $D_B$. Our previous assumption regarding the intersections $\mathcal{L}_\sigma \cap \mathcal{L}_\tau$ ensures that the spaces $E_{\mathcal{L}_1}(B)$, $E_{\mathcal{L}_2}(B)$, $\ldots$, $E_{\mathcal{L}_r}(B)$, as have been defined in Proposition 2.2, are pairwise orthogonal. Hence, the unitary diagonalizing $B$ may be partitioned as $V = \begin{bmatrix} V_1 & \cdots & V_r & V_{r+1} \end{bmatrix} \in \mathbb{C}^{(n-k)\times(n-k)}$, where, according to the order of the diagonal entries of $D_B$, $V_j \in \mathbb{C}^{(n-k)\times(n_j-k_j)}$ corresponds to $\sigma_j(B)$ $(j=1,\ldots,r)$ and $V_{r+1} \in \mathbb{C}^{(n-k)\times\left(n-\sum_{j=1}^r n_j\right)}$ to its remaining eigenvalues. Recalling from property 3 in Definition 2 that $C_j$ $(j = 1,\ldots,r)$ are the block columns of $C$, Proposition 2.2 implies that $C_j^* V_i = 0$ for $i \neq j$. Consequently, considering the diagonalizations of $A$, $B$ and using the form of $D$ in condition 2 of Definition 2, a computation shows that

(2.7)

$$D_A = R^* \begin{bmatrix} D_B & V^*D \\ C^*V & F \end{bmatrix} R = R^* \begin{bmatrix} \left(\oplus_{j=1}^r E_j\right) \oplus V_{r+1}^* B V_{r+1} & \begin{bmatrix} \oplus_{j=1}^r \left(e^{2i\theta_j} V_j^* C_j\right) \\ 0_{t\times k} \end{bmatrix} \\ \begin{bmatrix} \oplus_{j=1}^r \left(C_j^* V_j\right) & 0_{k\times t} \end{bmatrix} & F \end{bmatrix} R,$$

for $R = \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} W$ and $t \equiv n - \sum_{j=1}^r n_j$. Moreover, taking the formulation of $F$ in condition 1 of Definition 2 into account, the expression (2.7) implies that the matrix $D_A$ (hence, $A$ itself) is permutationally (unitarily) similar to the direct sum

$$\left(\oplus_{j=1}^r \begin{bmatrix} E_j & e^{2\mathbf{i}\vartheta_j} V_j^* C_j \\ C_j^* V_j & \mathrm{diag}\left\{f_i^j\right\}_{i=1}^{k_j} \end{bmatrix}\right) \oplus V_{r+1}^* B V_{r+1} \equiv \left(\oplus_{j=1}^r A_j\right) \oplus V_{r+1}^* B V_{r+1}. \tag{2.8}$$

This property reveals that all entries of $V_{r+1}^* B V_{r+1} \in \mathbb{C}^{t\times t}$ constitute common eigenvalues of $A$ and $B$. Therefore, $t \leq s$ and then (2.6) reduces in fact to equality, whereby it is verified that there do not exist eigenvalues of $\sigma(B)\backslash(\sigma(A)\cap\sigma(B))$ lying in $\mathbb{C}\backslash\left(\cup_{j=1}^r \mathcal{L}_j\right)$. On the other hand, it becomes immediately apparent that

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$\sigma(A)\backslash(\sigma(A)\cap\sigma(B)) = \cup_{j=1}^{r}\sigma_j(A)$, where $\sigma_j(A)\equiv\sigma(A_j)$. Recalling that the diagonal entries of each of the first $r$ summands in (2.8) are collinear on $\mathcal{L}_j$, notice that each $A_j$ $(j=1,\ldots,r)$ corresponds to a translated and rotated Hermitian matrix. Indeed, we may determine $c_j\in\mathcal{L}_j$ and real parameters $\left\{m_i^j\right\}_{i=1}^{n_j-k_j}$, $\left\{z_i^j\right\}_{i=1}^{k_j}\subset\mathbb{R}$ to express these as

$$A_j=\begin{bmatrix} E_j & e^{2\mathbf{i}\vartheta_j}V_j^*C_j \\ C_j^*V_j & \operatorname{diag}\left\{f_i^j\right\}_{i=1}^{k_j}\end{bmatrix} = c_jI_{n_j}+e^{\mathbf{i}\vartheta_j}\begin{bmatrix}\operatorname{diag}\left\{m_i^j\right\}_{i=1}^{n_j-k_j} & e^{\mathbf{i}\vartheta_j}V_j^*C_j \\ e^{-\mathbf{i}\vartheta_j}C_j^*V_j & \operatorname{diag}\left\{z_i^j\right\}_{i=1}^{k_j}\end{bmatrix}$$
$$\equiv c_jI_{n_j}+e^{\mathbf{i}\vartheta_j}L_j.$$

Hence, combining with (2.8), the above analysis reveals that the matrix $B$ is block imbeddable in $A$ precisely when $A$ is permutationally similar to the direct sum

$$\left[\oplus_{j=1}^{r}\left(c_jI_{n_j}+e^{\mathbf{i}\vartheta_j}L_j\right)\right]\oplus V_{r+1}^*BV_{r+1},$$

with each of the first $r$ summands being a translated and rotated Hermitian. Finally, the asserted equivalent interlacing condition on the collinear sets

$$\sigma_j(A)=\sigma(A_j)=c_j+e^{\mathbf{i}\vartheta_j}\sigma(L_j)\subset\mathcal{L}_j \text{ and}$$
$$\sigma_j(B)=\sigma(E_j)=c_j+e^{-\mathbf{i}\vartheta_j}\left\{m_i^j\right\}_{i=1}^{n_j-k_j}$$

follows in view of (1.1), i.e., the corresponding interlacing result for the Hermitian case. □

As can be seen from the proof above, condition 1 in Definition 2 is a crucial assumption. Since $F$ has to be normal for block imbeddability, the reduction to interlacing for Hermitian matrices is allowed. Hence, by Theorem 2.4, if one can determine lines $\{\mathcal{L}_j\}_{j=1}^{r}$ on which the noncommon eigenvalues of $A$,$B$ are interlacing, then $B$ is block imbeddable in $A$ and vice versa. We illustrate with the following:

Example 2.5. Consider

$$B=\operatorname{diag}\left\{\frac{1}{10}+\frac{1}{2}\mathbf{i},\frac{1}{2}+\frac{1}{2}\mathbf{i},\frac{9}{10}+\frac{1}{2}\mathbf{i},\frac{2}{5}+\frac{4}{5}\mathbf{i},\frac{1}{10}+\frac{1}{10}\mathbf{i},1,\mathbf{i}\right\}\in\mathbb{C}^{7\times7}$$

and its $10\times10$ normal extension

$$A=\begin{bmatrix} B & \begin{bmatrix}\begin{bmatrix}\frac{3}{40}\sqrt{\frac{7}{2}}\\ \frac{3}{8}\\ \frac{3}{40}\sqrt{\frac{7}{2}}\end{bmatrix}\oplus\left[\frac{2}{5}\mathbf{i}\right]\oplus\left[\frac{3}{10}(1+\mathbf{i})\right]\\ 0_{2,3}\end{bmatrix} \\ \begin{bmatrix}\begin{bmatrix}\frac{3}{40}\sqrt{\frac{7}{2}}\\ \frac{3}{8}\\ \frac{3}{40}\sqrt{\frac{7}{2}}\end{bmatrix}^T\oplus\left[\frac{2}{5}\mathbf{i}\right]\oplus\left[\frac{3}{10}(1+\mathbf{i})\right] & 0_{3,2}\end{bmatrix} & \operatorname{diag}\left\{\frac{1}{2}+\frac{1}{2}\mathbf{i},\frac{2}{5}+\frac{1}{5}\mathbf{i},\frac{9}{10}+\frac{9}{10}\mathbf{i}\right\}\end{bmatrix},$$

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where the additional blocks in this partition are, as previously, denoted by $C$, $D$ and $F$. Introducing the lines $\mathcal{L}_1 = \frac{1}{2}\mathbf{i}\mathbb{R}$, $\mathcal{L}_2 = \left\{\frac{2}{5} + t\mathbf{i} : t \in \mathbb{R}\right\}$ and $\mathcal{L}_3 = \{t(1+\mathbf{i}) : t \in \mathbb{R}\}$, it is straightforward to check that the properties of block imbedding hold for the pair $(A, B)$. Indeed, recalling the notation for the diagonal entries of $F$ therein, we have $k_j = 1$ and $f_1^j \in \mathcal{L}_j$, for $j = 1, 2, 3$. Moreover, $D = CU$, where $U = \operatorname{diag}\left\{1, e^{\mathbf{i}\pi}, e^{\mathbf{i}\frac{\pi}{2}}\right\}$. Therefore, $B$ is block imbeddable in $A$. A computation shows that $\sigma(A) = \cup_{j=1}^{3}\sigma_j(A) \cup (\sigma(A) \cap \sigma(B))$, where $\sigma(A) \cap \sigma(B) = \{1, \mathbf{i}\}$ and the pairs of subsets $\sigma_1(A) = \left\{\frac{1}{2}\mathbf{i}, \frac{1}{5} + \frac{1}{2}\mathbf{i}, \frac{4}{5} + \frac{1}{2}\mathbf{i}, 1 + \frac{1}{2}\mathbf{i}\right\}$, $\sigma_1(B) = \left\{\frac{1}{10} + \frac{1}{2}\mathbf{i}, \frac{1}{2} + \frac{1}{2}\mathbf{i}, \frac{9}{10} + \frac{1}{2}\mathbf{i}\right\}$ are interlacing on $\mathcal{L}_1$, similarly $\sigma_2(A) = \left\{\frac{2}{5}, \frac{2}{5} + \mathbf{i}\right\}$, $\sigma_2(B) = \left\{\frac{2}{5} + \frac{4}{5}\mathbf{i}\right\}$ are interlacing on $\mathcal{L}_2$ and $\sigma_3(A) = \{0, 1 + \mathbf{i}\}$, $\sigma_3(B) = \left\{\frac{1}{10} + \frac{1}{10}\mathbf{i}\right\}$ on $\mathcal{L}_3$, verifying thus Theorem 2.4. Here $n_1 = 4$, $n_2 = n_3 = 2$ and $s = 2$. Noticing that $\frac{1}{2} + \frac{1}{2}\mathbf{i} \in \sigma(B) \cap \mathcal{L}_1 \cap \mathcal{L}_3$, we consider $\frac{1}{2} + \frac{1}{2}\mathbf{i} \in \sigma_1(B)$ and not in $\sigma_3(B)$, due to interlacing on each line.

REMARK 2.6. If $\sigma(A) \cap \sigma(B) = \emptyset$, the matrix $F$ is block imbeddable in $A$, precisely when $B$ is block imbeddable in $A$, considering the same lines $\mathcal{L}_j$, $j = 1, \ldots, r$. In the complementary case $\sigma(A) \cap \sigma(B) \neq \emptyset$, considering the diagonalization $B = V\left(\oplus_{j=1}^{r+1}E_j\right)V^*$, where $\sigma(E_{r+1}) = \sigma(A) \cap \sigma(B)$, as in the proof of Theorem 2.4, the matrix $(V^* \oplus I_k)\,A\,(V \oplus I_k)$ is similar via permutation matrices to

$$\mathcal{A} = \begin{bmatrix} \oplus_{j=1}^{r}E_j & \oplus_{j=1}^{r}e^{2\mathbf{i}\vartheta_j}V_j^*C_j \\ \oplus_{j=1}^{r}C_j^*V_j & F \end{bmatrix} \oplus E_{r+1}.$$

Since $\cup_{j=1}^{r}\sigma(E_j) \cap \sigma(A) = \emptyset$, application of the previous case shows that $B$ is block imbeddable in $A$ if and only if $F \oplus E_{r+1}$ is block imbeddable in $\mathcal{A}$ (hence, in $A$).

REMARK 2.7. According to Theorem 2.4, the lines $\mathcal{L}_j$ defining block imbeddability are determined by the distribution of spectra of the normal $A$ and $B$ on collinear segments. Therefore, concerning the definition of block imbeddability, the open problem "under which conditions the distribution of the noncommon interlacing eigenvalues of $A$ and $B$ on some lines $\mathcal{L}_j$ is unique" arises. Then clearly, this leads to the uniqueness of the lines in the definition of block imbeddability and this case is presented in Example 2.5.

In the case $C = D$, notice that the normality of $\left[\begin{smallmatrix} B & C \\ C^* & F \end{smallmatrix}\right] \in \mathbb{C}^{n\times n}$ leads to the normality of both $B$, $F$ and equation (2.2) yields the homogeneous Lyapunov equation

$$(B - B^*)C - C(F - F^*) = 0_{(n-k)\times k}, \tag{2.9}$$

which has a nontrivial solution in the case where $\beta_j = \varphi_j$ holds for some pairs of real eigenvalues $\beta_j \in \sigma(\mathrm{S}(B))$, $\varphi_j \in \sigma(\mathrm{S}(F))$ $(j = 1, \ldots, r)$. Hence, considering for each of these pairs the horizontal line $\mathcal{L}_j \equiv \{z \in \mathbb{C} : \operatorname{Im} z = \varphi_j\}$ $(j = 1, \ldots, r)$, it is clear that both $\sigma(B) \cap \mathcal{L}_j$ and $\sigma(F) \cap \mathcal{L}_j$ are nonempty. Taking $F$ diagonal and in permuted form $F = \tilde{P}(F_1 \oplus F_2)\tilde{P}^T$, where $F_i$ $(i = 1, 2)$ are such that $\sigma(F_1) \subset \cup_{j=1}^{r}\mathcal{L}_j$ and the

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sets $\sigma(F_2)$ and $\cup_{j=1}^{r}\mathcal{L}_j$ do not intersect, then according to the above, in this case we have:

PROPOSITION 2.8. *Let the pair of normal matrices* $A = \left[\begin{smallmatrix} B & C \\ C^* & F \end{smallmatrix}\right] \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{(n-k)\times(n-k)}$. *Then the normal matrix* $B \oplus F_2$ *is block imbeddable in* $A$ *for the horizontal lines* $\mathcal{L}_j = \{z \in \mathbb{C} : \operatorname{Im} z = \varphi_j\}$ $(j = 1, \ldots, r)$ *defined by the pairs* $\beta_j \in \sigma(\mathrm{S}(B))$, $\varphi_j \in \sigma(\mathrm{S}(F))$ *for which* $\beta_j = \varphi_j$.

*Proof.* Henceforth, we let $\sigma(\mathrm{S}(B)) = \{\beta_1, \ldots, \beta_\sigma\}$ and $\sigma(\mathrm{S}(F)) = \{\varphi_1, \ldots, \varphi_\rho\}$ with algebraic multiplicities $\psi_j$ $(j = 1, \ldots, \sigma)$ and $\chi_j$ $(j = 1, \ldots, \rho)$ respectively. Clearly, $\sum_{j=1}^{\sigma}\psi_j = n - k$ and $\sum_{j=1}^{\rho}\chi_j = k$. For simplicity of notation, we adopt the convention that $\beta_j = \varphi_j$, for $j = 1, \ldots, r \le \min\{\sigma, \rho\}$. Application of equation (2.9) for the normal matrix $\mathcal{A} \equiv \left(I \oplus \tilde{P}^T\right) A \left(I \oplus \tilde{P}\right) = \left[\begin{smallmatrix} B & C\tilde{P} \\ \tilde{P}^T C^* & F_1 \oplus F_2 \end{smallmatrix}\right]$ yields

$$(B - B^*)\mathcal{C} - \mathcal{C}\left((F_1 \oplus F_2) - (F_1 \oplus F_2)^*\right) = 0_{(n-k)\times k}, \tag{2.10}$$

where $\mathcal{C} \equiv C\tilde{P}$. Then, applying the procedure in [7] to solve (2.10), we obtain

$$\mathcal{C} = V\left\{\left(\oplus_{j=1}^{r}C_j\right) \oplus 0_{\left(\sum_{j=r+1}^{\sigma}\psi_j\right)\times\left(\sum_{j=r+1}^{\rho}\chi_j\right)}\right\} \equiv V\tilde{C},$$

for suitable $C_j \in \mathbb{C}^{\psi_j\times\chi_j}$ and $V \in \mathbb{C}^{(n-k)\times(n-k)}$ the unitary diagonalizing $B$. Hence, the matrix $\mathcal{A}$ takes the form

$$\mathcal{A} = \begin{bmatrix} B & \mathcal{C} \\ \mathcal{C}^* & F_1 \oplus F_2 \end{bmatrix} = \begin{bmatrix} VD_BV^* & V\tilde{C} \\ \tilde{C}^*V^* & F_1 \oplus F_2 \end{bmatrix} = (V \oplus I_k)\begin{bmatrix} D_B & \tilde{C} \\ \tilde{C}^* & F_1 \oplus F_2 \end{bmatrix}(V^* \oplus I_k),$$

whereby the assertion is easily verified. □

Proposition 2.8 provides us with a wide class of normal matrices, for which block imbeddability applies. In particular, Theorem 2.4 implies interlacing of noncommon eigenvalues of $A$ and $B$ on each of the lines $\mathcal{L}_j$ $(j = 1, \ldots, r)$ defined above.

Having presented the intimate relation between block imbedding and interlacing on multiple lines in Theorem 2.4, we may invoke the inclusions (2.5) repeatedly to derive bounds on the number of eigenvalues of $A$ and $B$ inside a closed, convex region $\mathcal{D}$.

PROPOSITION 2.9. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{(n-k)\times(n-k)}$ *be normal matrices, such that* $B$ *is block imbeddable in* $A$ *and let* $\mathcal{D}$ *be a closed, convex region intersecting all of the straight lines* $\mathcal{L}_j$ $(j = 1, \ldots, r)$, *on which the spectra of* $A$, $B$ *are distributed, so that the set* $(\sigma(A) \cup \sigma(B)) \setminus (\sigma(A) \cap \sigma(B))$ *is contained in* $\left(\cup_{j=1}^{r}\mathcal{L}_j\right) \setminus \left(\cap_{j=1}^{r}\mathcal{L}_j\right)$.

**I.** *If* $|\sigma(A) \cap \mathcal{D}| = p$ $(\ge k + 1)$, *then*

$$p - k \le |\sigma(B) \cap \mathcal{D}| \le p + k.$$

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**II.** *If* $|\sigma(B)\cap\mathcal{D}|=s\ (\geq k+1)$*, then*

$$s-k\leq|\sigma(A)\cap\mathcal{D}|\leq s+k.$$

*Proof.* **I.** Letting $|\sigma(A)\cap\sigma(B)\cap\mathcal{D}|=t\ (\leq p)$ be the number of eigenvalues that $A$ and $B$ have in common inside the convex set $\mathcal{D}$ and $|(\sigma(A)\backslash\sigma(B))\cap\mathcal{L}_j\cap\mathcal{D}|=p_j$, clearly $p=t+\sum_{j=1}^{r}p_j$. Then, by Theorem 2.4, the sets $(\sigma(A)\backslash\sigma(B))$ and $(\sigma(B)\backslash\sigma(A))$ $\left(\subset\cup_{j=1}^{r}\mathcal{L}_j\right)$ are partitioned to the disjoint sets $\sigma_j(A)$, $\sigma_j(B)$ consisting of some of the interlacing eigenvalues on each line. Recalling the notation in Theorem 2.4, notice that at least $p_j-k_j$ eigenvalues of $B$ lie in $\mathcal{L}_j\cap\mathcal{D}$, since $n_j-k_j$ eigenvalues of $B$ are interlacing with $n_j$ eigenvalues of $A$ on $\mathcal{L}_j$. Therefore,

$$|\sigma(B)\cap\mathcal{D}|=|\sigma(B)\cap\sigma(A)\cap\mathcal{D}|+\sum_{j=1}^{r}|\sigma_j(B)\cap\mathcal{D}|\geq t+\sum_{j=1}^{r}(p_j-k_j)=p-k.$$

If for an index $q$ we have $\left\{\mu_q^j,\mu_{q+1}^j,\ldots,\mu_{q+p_j+k_j}^j\right\}\subset\sigma_j(B)\cap\mathcal{D}$ (i.e., $|\sigma_j(B)\cap\mathcal{D}|\geq p_j+k_j+1$), then by Theorem 2.4, $\mu_i^j$ interlace with elements of $\sigma_j(A)$:

$$\mu_i^j\in\overline{\lambda_i^j\lambda_{i+k_j}^j},\ \text{ for } i=q,\ldots,q+p_j+k_j.$$

In particular, notice that the extremal eigenvalues satisfy the inclusions $\mu_q^j\in\overline{\lambda_q^j\lambda_{q+k_j}^j}$ and $\mu_{q+p_j+k_j}^j\in\overline{\lambda_{q+p_j+k_j}^j\lambda_{q+p_j+2k_j}^j}$, respectively, showing that

$$\overline{\lambda_{q+k_j}^j\lambda_{q+p_j+k_j}^j}\subseteq\overline{\mu_q^j\mu_{q+p_j+k_j}^j}\subset\mathcal{D}.$$

Hence, the line segment $\overline{\lambda_{q+k_j}^j\lambda_{q+p_j+k_j}^j}$ includes $(q+p_j+k_j)-(q+k_j)+1=p_j+1$ eigenvalues of $A$, whereupon we have the contradiction $|\sigma_j(A)\cap\mathcal{D}|\geq p_j+1$. Therefore, $\mathcal{L}_j\cap\mathcal{D}$ may contain at most $p_j+k_j$ eigenvalues of $B$ and in this way we derive

$$|\sigma(B)\cap\mathcal{D}|\leq t+\sum_{j=1}^{r}(p_j+k_j)=p+k.$$

**II.** Follows directly from the previous statement, by contradiction. □

For $k=1$, block imbeddability holds precisely when a *unique* line exists upon which the noncommon eigenvalues of $A$ and $B$ interlace. In this case, necessarily

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$r = 1$. Therefore, block imbeddability reduces in this case to imbeddability in the usual sense and we conclude the following special interlacing statement.

COROLLARY 2.10. *Let the normal matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-1)\times(n-1)}$ with $B$ imbeddable in $A$ and also $\mathcal{D} \subset \mathbb{C}$ be a given closed convex set that is not a point.*

**I.** *If $|\sigma(A) \cap \mathcal{D}| = p$ $(\geq 2)$, then*

$$p - 1 \leq |\sigma(B) \cap \mathcal{D}| \leq p + 1.$$

**II.** *If $|\sigma(B) \cap \mathcal{D}| = s$ $(\geq 2)$, then*

$$s - 1 \leq |\sigma(A) \cap \mathcal{D}| \leq s + 1.$$

This extends a result by R. Horn presented as Problem 42-4 in the bulletin of ILAS "*Image*", issue 43, p. 40 (2009), where only the lower bounds in Corollary 2.10 were proved using a different technique. We remark that Corollary 2.10 refers to the case $\mathcal{D} \cap \mathcal{L} \neq \emptyset$, where $\mathcal{L}$ is the straight line on which the distinct eigenvalues of $A$, $B$ lie. In the trivial case $\mathcal{D} \cap \mathcal{L} = \emptyset$, clearly $|\sigma(A) \cap \mathcal{D}| = |\sigma(B) \cap \mathcal{D}|$.

**3. Links on imbedding conditions for normal matrices.** Moving away from collinearity of eigenvalues, several authors have investigated imbedding for normal matrices when $k > 1$ and provided related results. Interlacing of eigenvalues of an imbeddable normal pair $A$ and $B$ has been verified with respect to suitable orders; one approach [1] exploits the order of their arguments, while another [11] introduces the lexicographic orders in $\mathbb{C}$. Regarding the latter, we denote

$$\mathcal{J} = \{a + \mathbf{i}b : a > 0, \text{ or } a = 0 \text{ and } b > 0\}.$$

its positive cone and write $w <_0 z$ precisely when $w$, $z$ are ordered *lexicographically,* i.e., if and only if $z - w \in \mathcal{J}$ and $w \leq_0 z$ when $z - w \in \mathcal{J} \cup \{0\}$. More generally for arbitrary $\vartheta \in [0, 2\pi)$, the total order $\leq_\vartheta$ in $\mathbb{C}$ is defined by the positive cone $e^{\mathbf{i}\vartheta}\mathcal{J}$, where

$$w \leq_\vartheta z \;\Leftrightarrow\; e^{-\mathbf{i}\vartheta} w \leq_0 e^{-\mathbf{i}\vartheta} z \qquad (\textit{lexicographic order}).$$

The inequality $z_1 \leq_0 w \leq_0 z_2$ is clearly satisfied for all $w \in \mathbb{C}$ in the semiclosed zone depicted in Figure 3.1.

Denoting for $x \leq_\vartheta y$ the zone

$$\mathcal{Z}_\vartheta(x, y) = \{z \in \mathbb{C} : x \leq_\vartheta z \leq_\vartheta y\}$$

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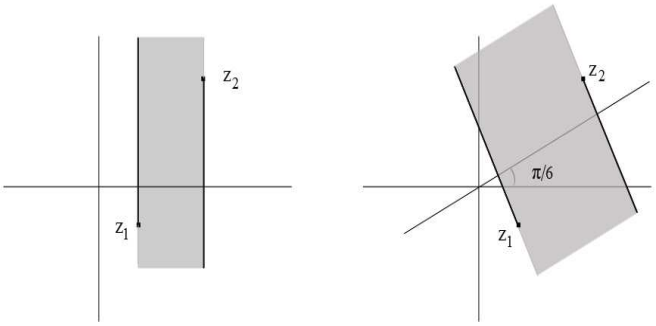


FIG. 3.1. *Illustrating the sets* $\{w : z_1 \leq_0 w \leq_0 z_2\}$ *(left) and* $\{w : z_1 \leq_{\pi/6} w \leq_{\pi/6} z_2\}$ *(right).*

and reindexing the pair of points $z_1$, $z_2$ in non-decreasing $\vartheta$-lexicographic order as $\left\{z_{i(\vartheta)}\right\}_{i=1}^{2}$, where $i(\vartheta)$ signifies the dependence of the indices on $\vartheta$, then it is readily verified that

$$\bigcap_{\vartheta} \mathcal{Z}_{\vartheta}(z_{1(\vartheta)}, z_{2(\vartheta)}) \equiv \overline{z_1 z_2}. \tag{3.1}$$

The following theorem reviews a pair of *necessary* (but not *sufficient*) interlacing conditions, which have appeared in the literature:

THEOREM 3.1. *Let the normal matrices* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{(n-k)\times(n-k)}$*, where* $B$ *is imbeddable in* $A$ *with eigenvalues* $\{\lambda_i\}_{i=1}^{n}$ *and* $\{\mu_i\}_{i=1}^{n-k}$ *respectively.*

**I.** (Carlson and de Sa, [1]) *Let* $z \notin w(A)$ *and suppose that, for some* $\varphi \in \mathbb{R}$*, we have the order*

$$\varphi \leq \arg(\lambda_{i(z)} - z) \leq \arg(\lambda_{(i+1)(z)} - z) < \varphi + \pi; \quad i = 1, \ldots, n-1, \tag{3.2}$$

$$\varphi \leq \arg(\mu_{i(z)} - z) \leq \arg(\mu_{(i+1)(z)} - z) < \varphi + \pi; \quad i = 1, \ldots, n-k-1, \tag{3.3}$$

*where* $i(z)$ *signifies the dependence of the indices on the point* $z$*. Then,*

$$\mu_{i(z)} \in \mathcal{S}_z(\lambda_{i(z)}, \lambda_{(i+k)(z)}) \cap w(A), \quad \textit{for } i = 1, \ldots, n-k,$$

*where* $\mathcal{S}_z(\lambda_{i(z)}, \lambda_{(i+k)(z)})$ denotes the sector

$$\mathcal{S}_z(\lambda_{i(z)}, \lambda_{(i+k)(z)}) = \left\{w \in \mathbb{C} : \arg(\lambda_{i(z)} - z) \leq \arg(w - z) \leq \arg(\lambda_{(i+k)(z)} - z)\right\}.$$

**II.** (Queiro and Duarte, [11]) *If the spectra of* $A$*,* $B$ *are indexed in nondecreasing* $\vartheta$*-lexicographic order for some* $\vartheta \in [0, 2\pi)$*, then*

$$\mu_{i(\vartheta)} \in \mathcal{Z}_{\vartheta}(\lambda_{i(\vartheta)}, \lambda_{(i+k)(\vartheta)}) \cap w(A), \quad i = 1, \ldots, n-k. \tag{3.4}$$

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Both statements are proved as consequences of corresponding min-max theorems, following thoughts completely analogue to the ones in the proof for the necessity of (1.1). Moreover, note that Theorem 3.1 presents somewhat modified versions of the original statements in [1] and [11], respectively. The interlacing of $\sigma(A)$, $\sigma(B)$ with respect to sectors in Theorem 3.1.**I** has been presented in [1] for the special case $z = 0$ and $A$ invertible, while both statements **I** and **II** take the inclusion $\sigma(B) \subset w(A)$ into account.

Theorem 3.1.**II** yields for each $\vartheta \in [0, 2\pi)$ different inclusion zones for the eigenvalues of $B$. Therefore, the question emerges "*what restrictions do the* $\leq_\vartheta$ *interlacing conditions in* (3.4) *impose on the eigenvalue configurations?*". This problem was posed in [11] and here we give an answer in the case where $\sigma(A)$ is *convexly independent*, i.e., when $\lambda_j \notin \operatorname{co}\{\lambda_i\}_{i=1,i\neq j}^n$ for any $j$. Convexly independent spectrum implies for a normal matrix $A$ that $w(A) = \operatorname{co}\{\sigma(A)\}$ is a convex $n$-polygon.

In this direction, given any two distinct eigenvalues $\lambda_i$, $\lambda_j$ of $A$, the set

$$\begin{aligned}\mathcal{H}(\lambda_i, \lambda_j) &= \left\{z \in \mathbb{C} : \operatorname{Im}\left((\bar{\lambda}_j - \bar{\lambda}_i)(z - \lambda_i)\right) \geq 0\right\} \\ &= \left\{z \in \mathbb{C} : \operatorname{Re}\left(e^{-\mathbf{i}\varphi} z\right) \leq \operatorname{Re}\left(e^{-\mathbf{i}\varphi}\lambda_i\right), \text{ where } \varphi = \arg(\lambda_j - \lambda_i) - \frac{\pi}{2} \pmod{2\pi}\right\}\end{aligned}$$

defines the left closed half-plane with boundary the line passing through $\lambda_i$ and $\lambda_j$. Clearly for $\varphi = \arg(\lambda_j - \lambda_i) - \frac{\pi}{2} \pmod{2\pi}$ and $\lambda_i \equiv \lambda_{\tau(\varphi)}$ in non-decreasing $\varphi$-lexicographic order, the zones $\mathcal{Z}_\varphi(\lambda_\ell, \lambda_{\tau(\varphi)}) \subset \mathcal{H}(\lambda_i, \lambda_j)$ for all indices $\ell$ such that $\lambda_\ell \leq_\varphi \lambda_{\tau(\varphi)}$. Denoting moreover by $m(\vartheta)$ and $M(\vartheta)$ respectively the minimum and maximum of the indices of elements of $\Gamma \subset \sigma(A)$ according to the $\vartheta$-lexicographic ordering, obviously $M(\vartheta) - m(\vartheta) \geq |\Gamma| - 1$ and the equality holds only if all the indices of elements of $\Gamma$ are successive. We will use the following characterization of the polygon $\operatorname{co}\Gamma$ as an intersection of zones, which generalizes the expression (3.1).

LEMMA 3.2. *Let the normal matrix* $A \in \mathbb{C}^{n\times n}$ *with* $\sigma(A) = \{\lambda_j\}_{j=1}^n$.

**I.** If $\Gamma \subseteq \sigma(A)$, then

$$\operatorname{co}\Gamma = \bigcap_{\vartheta\in[0,2\pi)} \mathcal{Z}_\vartheta(\lambda_{m(\vartheta)}, \lambda_{M(\vartheta)}).$$

**II.** *If* $\Gamma \subset \sigma(A)$, *with* $|\Gamma| = \ell$, *consists of consecutive elements of* $\sigma(A)$ *on the boundary* $\partial w(A)$, *then there exists* $\varphi \in [0, 2\pi)$ *such that* $\operatorname{co}\Gamma = \mathcal{Z}_\varphi(\lambda_{1(\varphi)}, \lambda_{\ell(\varphi)}) \cap w(A)$, *where* $\lambda_{1(\varphi)}$ *and* $\lambda_{\ell(\varphi)}$ *denote the smallest and* $\ell$*-th smallest eigenvalues of* $A$ *respectively, according to their reordering in non-decreasing* $\varphi$*-lexicographic order. In particular, if* $\Gamma = \{\lambda_i, \ldots, \lambda_{i+\ell-1}\}$ *with elements indexed counterclockwise on* $\partial w(A)$ *and* $\lambda_l \equiv \lambda_{l-n}$ *for* $l > n$, *then* $\varphi = \arg(\lambda_{i+\ell-1} - \lambda_i) - \frac{\pi}{2} \pmod{2\pi}$.

*Proof.* **I.** For $\vartheta \in [0, 2\pi)$, the definitions of $m(\vartheta)$ and $M(\vartheta)$ clearly imply the

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relation $\operatorname{co}\Gamma \subseteq \mathcal{Z}_\vartheta(\lambda_{m(\vartheta)}, \lambda_{M(\vartheta)})$, whereby we get

$$\operatorname{co}\Gamma \subseteq \bigcap_{\vartheta\in[0,2\pi)} \mathcal{Z}_\vartheta(\lambda_{m(\vartheta)}, \lambda_{M(\vartheta)}).$$

For the converse inclusion, since $\Gamma$ may not be convexly independent, we choose a maximal convexly independent subset $E \subset \Gamma$, say $E = \{\lambda_{i_1}, \ldots, \lambda_{i_\ell}\}$, where $\ell \leq |\Gamma|$ and the indices are such that its elements are arranged counterclockwise on $\operatorname{co} E$ $(= \operatorname{co}\Gamma)$. Hence, $\operatorname{co} E$ may be expressed as intersection of the halfplanes $\big\{\mathcal{H}(\lambda_{i_j}, \lambda_{i_{j+1}})\big\}_{j=1}^{\ell}$, with $\lambda_{i_{\ell+1}} \equiv \lambda_{i_1}$ and obviously the inclusions

$$\mathcal{Z}_{\vartheta_j}(\lambda_{m(\vartheta_j)}, \lambda_{M(\vartheta_j)}) \subset \mathcal{H}(\lambda_{i_j}, \lambda_{i_{j+1}})$$

hold for $j = 1, \ldots, \ell$, where $\vartheta_j = \arg\big(\lambda_{i_{j+1}} - \lambda_{i_j}\big) - \frac{\pi}{2} \pmod{2\pi}$. Therefore, we have

$$\bigcap_{\vartheta\in[0,2\pi)} \mathcal{Z}_\vartheta(\lambda_{m(\vartheta)}, \lambda_{M(\vartheta)}) \subseteq \bigcap_{j=1}^{\ell} \mathcal{Z}_{\vartheta_j}(\lambda_{m(\vartheta_j)}, \lambda_{M(\vartheta_j)}) \subseteq \bigcap_{j=1}^{\ell} \mathcal{H}(\lambda_{i_j}, \lambda_{i_{j+1}}) = \operatorname{co} E.$$

**II.** If $\Gamma = \{\lambda_1, \ldots, \lambda_\ell\}$ consists of elements of $\sigma(A)$ that are consecutive on the boundary $\partial w(A)$ and counterclockwise, then $\overline{\lambda_1\lambda_\ell}$ is the only side of $\operatorname{co}\Gamma$ that does not lie on $\partial w(A)$. Then for $\varphi = \arg(\lambda_1 - \lambda_\ell) - \frac{\pi}{2} \pmod{2\pi}$ the elements of $\Gamma$ are reindexed according to the $\varphi$-lexicographic order as $\Gamma = \big\{\lambda_{1(\varphi)}, \ldots, \lambda_{\ell(\varphi)}\big\}$ and clearly $\operatorname{co}\Gamma = \mathcal{Z}_\varphi(\lambda_{1(\varphi)}, \lambda_{\ell(\varphi)}) \cap w(A)$. □

Now we are ready to give a geometric description of the relative location of eigenvalues of $A$ and $B$ on the complex plane, independently of the $\vartheta$-interlacing condition (3.4), in the case of convexly independent spectrum.

PROPOSITION 3.3. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$ be normal matrices with $B$ imbeddable in $A$ and $k \geq 2$. If $\sigma(A)$ is convexly independent, then the convex hull of every subset $\Gamma \subset \sigma(A)$, with $|\Gamma| = k+1$, contains at least one element of $\sigma(B)$.*

*Proof.* Denoting $\sigma(A) = \{\lambda_1, \ldots \lambda_n\}$ and $\sigma(B) = \{\mu_1, \ldots, \mu_{n-k}\}$ as before, consider the subset $\Gamma = \{\lambda_i, \ldots, \lambda_{i+k}\}$ for $i \in \{1, \ldots, n\}$ and $\lambda_l \equiv \lambda_{l-n}$ for $l > n$, where the $\lambda$'s are consecutive on the boundary $\partial w(A)$. Clearly, the side $\overline{\lambda_i\lambda_{i+k}}$ of $\operatorname{co}\Gamma$ does not lie on $\partial w(A)$. If $\vartheta_0 = \arg(\lambda_{i+k} - \lambda_i) - \frac{\pi}{2} (\operatorname{mod} 2\pi)$, we may reindex the elements of $\sigma(A)$, $\sigma(B)$ in nondecreasing $\vartheta_0$-lexicographic order and then $\Gamma = \big\{\lambda_{1(\vartheta_0)}, \ldots, \lambda_{(1+k)(\vartheta_0)}\big\}$. Thus, (3.4) and Lemma 3.2.**II** clearly imply

$$\mu_{1(\vartheta_0)} \in \mathcal{Z}_{\vartheta_0}(\lambda_{1(\vartheta_0)}, \lambda_{(1+k)(\vartheta_0)}) \cap w(A) = \operatorname{co}\big\{\lambda_{1(\vartheta_0)}, \ldots, \lambda_{(1+k)(\vartheta_0)}\big\} = \operatorname{co}\Gamma,$$

which proves the assertion in this case.

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If the elements of $\Gamma = \{\lambda_1, \lambda_{i_1}, \ldots, \lambda_{i_k}\}$ are arranged in counterclockwise order but not consecutively on $\partial w(A)$, due to $\sigma(A)$ being convexly independent, the elements of $\sigma(A)\backslash\Gamma$ are partitioned in groups $P_1, \ldots, P_{k+1}$ of consecutive vertices, say $P_\tau = \{\lambda_{i_{\tau-1}+1}, \ldots, \lambda_{i_\tau-1}\}$ $(\tau = 1, \ldots, k)$ and $P_{k+1} = \{\lambda_{i_k+1}, \ldots, \lambda_n\}$, where $i_0 \equiv 1$. We note that $|P_\tau| \equiv p_\tau = i_\tau - i_{\tau-1} - 1$ and $|P_{k+1}| \equiv p_{k+1} = n - i_k$, so clearly $\sum_{\tau=1}^{k+1} p_\tau + (1+k) = n$. Moreover, $P_\tau = \emptyset$, if and only if the points $\lambda_{i_{\tau-1}}$ and $\lambda_{i_\tau}$ form successive edges on $\partial w(A)$. Denoting $\Pi_\tau = \operatorname{co}\left(P_\tau \cup \{\lambda_{i_\tau}, \lambda_{i_{\tau+1}}\}\right) \setminus \overline{\lambda_{i_\tau}\lambda_{i_{\tau+1}}}$, the elements of $\sigma(A)\backslash P_\tau$ according to $\vartheta_\tau = \arg\left(\lambda_{i_\tau} - \lambda_{i_{\tau-1}}\right) - \frac{\pi}{2} \pmod{2\pi}$-lexicographic order are consecutive, i.e., $\sigma(A)\backslash P_\tau = \{\lambda_{1(\vartheta_\tau)}, \ldots, \lambda_{(n-p_\tau)(\vartheta_\tau)}\}$. Hence, by (3.4) the inclusions

$$\begin{aligned}\mu_{i(\vartheta_\tau)} \in \mathcal{Z}_{\vartheta_\tau}(\lambda_{i(\vartheta_\tau)}, \lambda_{(i+k)(\vartheta_\tau)}) \cap w(A) &\subset \mathcal{Z}_{\vartheta_\tau}(\lambda_{1(\vartheta_\tau)}, \lambda_{(n-p_\tau)(\vartheta_\tau)}) \cap w(A) \\ &= \operatorname{co}\{\lambda_{1(\vartheta_\tau)}, \ldots, \lambda_{(n-p_\tau)(\vartheta_\tau)}\} = w(A)\backslash\Pi_\tau,\end{aligned}$$

hold for all indices $i = 1, \ldots, n - p_\tau - k$. Then by $\{\mu_{1(\vartheta_\tau)}, \ldots, \mu_{(n-p_\tau-k)(\vartheta_\tau)}\} \subset w(A)\backslash\Pi_\tau$ we have that in $\Pi_\tau$ belong at most $p_\tau$ eigenvalues of $B$ and consequently, for the polygon $\operatorname{co}\Gamma = w(A)\backslash\left(\bigcup_{\tau=1}^{k+1}\Pi_\tau\right)$ we conclude

$$|\sigma(B) \cap \operatorname{co}\Gamma| = |\sigma(B)| - \sum_{\tau=1}^{k+1} |\sigma(B) \cap \Pi_\tau| \geq (n-k) - \sum_{\tau=1}^{k+1} p_\tau = 1,$$

as the Proposition asserts. □

Remark 3.4. Note that the conclusion of Proposition 3.3 remains trivially valid even for $k = 1$, in which case the sets $\sigma(A)$, $\sigma(B)$ are necessarily collinear and interlacing by Theorem 1.1. On the contrary, we illustrate an application of Proposition 3.3 for the minimum applicable pair $(n, k) = (4, 2)$ in the case of convexly independent spectrum $\sigma(A)$. Thus, for $A = \operatorname{diag}(0, 1, \mathbf{i}, 1+\mathbf{i})$ and $B = \operatorname{diag}\left(\frac{5+8\mathbf{i}}{10}, \frac{5+2\mathbf{i}}{10}\right)$ notice that $B$ is imbeddable in $A$ via the isometry $V = \frac{1}{\sqrt{10}}\left[\begin{smallmatrix} 1 & 1 & 2 & 2 \\ 2 & -2 & -1 & 1 \end{smallmatrix}\right]^T$. Clearly the sets $\sigma(A)$ and $\sigma(B)$ satisfy $\vartheta$-interlacing for all $\vartheta$, i.e., the interior of any triangle formed with edges in $\sigma(A)$ contains at least one eigenvalue of $B$, as Proposition 3.3 asserts. Evenly, note that Proposition 3.3 is trivialized for $\Gamma = \sigma(A)$. In this case $k = n-1$, so $B$ is a complex scalar and since it is imbeddable in $A$, there exists a unit vector $v \in \mathbb{C}^n$ such that $B = v^*Av$, i.e., $B \in w(A) = \operatorname{co}\Gamma$. Finally, the conclusion of Proposition 3.3 is not valid for $|\Gamma| < k+1$. Indeed, for $A$ and $B$ as above, there does not exist any eigenvalue of $B$ in the interior of any line segment with endpoints eigenvalues of $A$, i.e., when $|\Gamma| = 2$.

It is not known if for any values of $n$ and $k$ imbeddability is possible for $n$ independently convex $\lambda$'s. A lower bound for $k$, such that an imbedding is always possible is $k \geq \frac{2(n-1)}{3}$ [8].

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In the remainder of this section, we turn our attention to the case $k = 1$, where, as verified by Theorem 1.1, imbeddability imposes a severe condition on the $\lambda$'s; namely, not only must they be convexly dependent but, in fact, collinear. Thus, it is of interest to relate the correspondingly specialized imbedding conditions by Carlson and de Sa, and Queiro and Duarte in Theorem 3.1 for $k = 1$ to the Fan and Pall criterion. The following two results are concerned with the geometry of complex plane, hence are stated for interlacing sets of complex numbers with respect to sectors or zones respectively and are presented without explicit references to normal imbeddings. We consider the Carlson and de Sa condition for $k = 1$ first:

PROPOSITION 3.5. *Let the sets $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(B) = \{\mu_1, \ldots, \mu_{n-1}\}$ of complex numbers. The following are equivalent:*

**I.** *The sets $\sigma(A)$, $\sigma(B)$ are interlacing in sectors for all $z \notin \operatorname{co}\{\sigma(A)\}$, i.e.*

$$\mu_{i(z)} \in \mathcal{S}_z(\lambda_{i(z)}, \lambda_{(i+1)(z)}), \quad \text{for } i = 1, \ldots, n-1, \tag{3.5}$$

*where $\{\lambda_{i(z)} - z\}_{i=1}^{n}$ and $\{\mu_{i(z)} - z\}_{i=1}^{n-1}$ are indexed in order of non-decreasing argument.*

**II.** *Every straight line defined by a pair of elements in $\sigma(A)\setminus(\sigma(A) \cap \sigma(B))$ contains at least one element of $\sigma(B)\setminus(\sigma(A) \cap \sigma(B))$.*

**III.** *The non-common elements of $\sigma(A)$, $\sigma(B)$ are collinear and interlacing on a common line $\mathcal{L}$.*

*Proof.* [**I.** $\Rightarrow$ **II.**] Letting $\{\lambda_i\}_{i=1}^{q}$, $\{\mu_i\}_{i=1}^{q-1}$ the noncommon elements of $\sigma(A)$ and $\sigma(B)$, then for $\lambda_r$, $\lambda_s$ $(1 \le r < s \le q)$ define the line $\mathcal{L}_{rs}$ on $\mathbb{C}$ passing through these points. Reindexing $\sigma(A)$, $\sigma(B)$ according to the order in (3.2) defined by a point $z_{rs} \in \mathcal{L}_{rs}\setminus\operatorname{co}\{\sigma(A)\}$, the elements $\lambda_r$, $\lambda_s$ are reindexed as $\lambda_{\rho(z_{rs})}$ and $\lambda_{\sigma(z_{rs})}$ (say $\rho(z_{rs}) < \sigma(z_{rs})$). Noticing $\arg(\lambda_{\rho(z_{rs})} - z_{rs}) = \arg(\lambda_{\sigma(z_{rs})} - z_{rs})$, it becomes apparent that the sector $\mathcal{S}_{z_{rs}}(\lambda_{\rho(z_{rs})}, \lambda_{\sigma(z_{rs})})$ reduces to the line $\mathcal{L}_{rs}$. Hence, the assumption (3.5) ensures $\sigma(B) \cap \mathcal{S}_{z_{rs}}(\lambda_{\rho(z_{rs})}, \lambda_{\sigma(z_{rs})}) = \sigma(B) \cap \mathcal{L}_{rs} \neq \emptyset$. As a final step, we will in fact show that $\{\mu_i\}_{i=1}^{q-1} \cap \mathcal{L}_{rs} \neq \emptyset$. In this direction, assume that $r\,(\ge 1)$ in total elements of $\sigma(A) \cap \sigma(B)$ lie on $\mathcal{L}_{rs}$; namely, $\mu_q = \lambda_{q+1}$, $\ldots$, $\mu_{q+r-1} = \lambda_{q+r}$. Then, reindexing the elements in $\sigma(A) \cap \mathcal{L}_{rs} = \{\lambda_r, \lambda_s, \lambda_{q+1}, \ldots, \lambda_{q+r}\}$, as in (3.2), these are induced $(r+2)$ consecutive indices and then (3.5) implies that $(r+1)$ elements in $\sigma(B)$ lie on $\mathcal{L}_{rs}$. Hence, at least one of these is non-common point, i.e., $\{\mu_i\}_{i=1}^{q-1} \cap \mathcal{L}_{rs} \neq \emptyset$, as asserted.

[**II.** $\Rightarrow$ **III.**] We proceed to show that $\{\lambda_i\}_{i=1}^{q}$ are in fact collinear. For $q \ge 3$ if this is not the case, then there exists $\lambda_k$ $(k \in \{1, \ldots, q\})$ that lies outside the line $\mathcal{L}$ where the remaining distinct elements of $\sigma(A)\setminus(\sigma(A) \cap \sigma(B)) = \{\lambda_i\}_{i=1}^{q}$ stand. Noticing that $\{\lambda_k\} = \cap_{i=1,\ i\neq k}^{q} \mathcal{L}_{ki}$ is not a common element of $\sigma(A)$ and $\sigma(B)$, then

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the statement in **II** ensures that each of the $q-1$ lines $\mathcal{L}_{ki}$ $(i \in \{1,\dots,q\}\setminus\{k\})$, as well as $\mathcal{L}$, contains at least one of $\{\mu_i\}_{i=1}^{q-1}$. Moreover, the $(q-1)$ elements of $\{\mu_i\}_{i=1}^{q-1}$ on $\cup_{i=1,\ i\neq k}^{q}\mathcal{L}_{ki}$ are clearly distinct from the ones on $\mathcal{L}$, due to $\left(\cup_{i=1,\ i\neq k}^{q}\mathcal{L}_{ki}\right)\cap\mathcal{L} = \{\lambda_i\}_{i=1,\ i\neq k}^{q}$. Hence, the contradiction

$$\begin{aligned}|\sigma(B)\setminus(\sigma(A)\cap\sigma(B))| &\geq \sum_{\substack{i=1\\ i\neq k}}^{q}\left|\{\mu_i\}_{i=1}^{q-1}\cap\mathcal{L}_{ki}\right| + \left|\{\mu_i\}_{i=1}^{q-1}\cap\mathcal{L}\right| \\ &\geq (q-1)+1 = q > q-1.\end{aligned}$$

As a next step, we will show that $\{\mu_i\}_{i=1}^{q-1}$ are also collinear with $\{\lambda_i\}_{i=1}^{q}$ on a common line $\mathcal{L}$. Choosing $z\in\mathcal{L}\setminus\mathrm{co}\,\{\sigma(\mathcal{A})\}$ and letting $|(\sigma(A)\cap\sigma(B))\cap\mathcal{L}| \equiv s(\leq n-q)$, the $q+s$ elements of $\sigma(A)$ on $\mathcal{L}$ are induced according to the order in (3.2) consecutive indices $i(z), i(z)+1,\dots,i(z)+q+s-1$. As before, the inclusion of $(q+s-1)$ elements of $\sigma(B)$ in $\mathcal{L}$ is verified. This means that, along with $s$ elements of $\sigma(A)\cap\sigma(B)$, all non-common points $\{\mu_i\}_{i=1}^{q-1}$ lie on $\mathcal{L}$. Finally, it is straightforward to see that $\{\mu_i\}_{i=1}^{q-1}$ and $\{\lambda_i\}_{i=1}^{q}$ are interlacing on $\mathcal{L}$. Indeed, choosing any $z\notin\mathcal{L}\cup\mathrm{co}\,\{\sigma(\mathcal{A})\}$, then $z$ lies in a semiplane defined by $\mathcal{L}$, so either the inequalities $\arg(\lambda_{i(z)}-z) < \arg(\mu_{i(z)}-z) < \arg(\lambda_{(i+1)(z)}-z)$ or $\arg(\lambda_{i(z)}-z) > \arg(\mu_{i(z)}-z) > \arg(\lambda_{(i+1)(z)}-z)$ hold for $i=1,\dots,q-1$, thus verifying interlacing on $\mathcal{L}$.

[**III.** $\Rightarrow$ **I.**] Suppose that $\{\lambda_i\}_{i=1}^{q}$ and $\{\mu_i\}_{i=1}^{q-1}$ are disjoint, collinear and interlacing on $\mathcal{L}$, while $\lambda_i=\mu_{i-1}$ for $i=q+1,\dots,n$. If we consider a point $z\in\mathcal{L}\setminus\mathrm{co}\,\{\sigma(\mathcal{A})\}$, then all distinct points $\{\lambda_i-z\}_{i=1}^{q}$ and $\{\mu_i-z\}_{i=1}^{q-1}$ lie on the line $\mathcal{L}$ passing through the origin $z$ of a translated system of coordinates. Hence, their arguments coincide and the assertion holds trivially. In the case $z\notin(\mathrm{co}\,\{\sigma(A)\}\cup\mathcal{L})$, as before, it is immediate that the non-common elements are interlacing in sectors. Indexing common and collinear eigenvalues together, according to (3.2), (3.3) yields the desired inequalities. □

Note that the presence of the arbitrary point $z$ in the statement Theorem 3.1.**I** has been instrumental in enabling the proof of the equivalences in Proposition 3.5. Proceeding similarly, we note an analogue implication involving the Queiro and Duarte condition for $k=1$.

COROLLARY 3.6. *Let the sets $\sigma(A)=\{\lambda_1,\dots,\lambda_n\}$ and $\sigma(B)=\{\mu_1,\dots,\mu_{n-1}\}$ of complex numbers. The following are equivalent:*

**I.** *$\vartheta$-interlacing holds for all $\vartheta$, i.e.,*

$$\mu_{i(\vartheta)}\in\mathcal{Z}_\vartheta(\lambda_{i(\vartheta)},\lambda_{(i+1)(\vartheta)}),\quad \textit{for } i=1,\dots,n-1,$$

*where the $\lambda$'s and $\mu$'s are in nondecreasing $\vartheta$-lexicographic order.*

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**II.** *The convex hull of every subset* $\Gamma \subset \sigma(A)\setminus(\sigma(A)\cap\sigma(B))$*, with* $|\Gamma| = 2$ *contains at least one element of* $\sigma(B)\setminus(\sigma(A)\cap\sigma(B))$*.*

*Proof.* [**I.** $\Rightarrow$ **II.**] Consider the points $\lambda_i$ and $\lambda_j$ and let $\vartheta_0 = \arg(\lambda_i - \lambda_j) - \frac{\pi}{2} \pmod{2\pi}$. The indices $m(\vartheta_0)$, $M(\vartheta_0)$ of $\lambda_i$ and $\lambda_j$ by the $\vartheta_0$-ordering are adjacent at most and then the assumption ensures that at least one element of $\sigma(B)$ belongs in the zone $\mathcal{Z}_{\vartheta_0}(\lambda_{m(\vartheta_0)}, \lambda_{M(\vartheta_0)})$. Noting that for $\vartheta_0$ as above, this zone degenerates into the line segment $\mathcal{Z}_{\vartheta_0}(\lambda_{m(\vartheta_0)}, \lambda_{M(\vartheta_0)}) = \overline{\lambda_i\lambda_j}$, we immediately obtain $\sigma(B)\cap\operatorname{co}\Gamma \neq \emptyset$. To complete the proof, we continue similarly as in the proof of the first implication of Proposition 3.5.

[**II.** $\Rightarrow$ **I.**] If $\sigma(A)\setminus(\sigma(A)\cap\sigma(B)) = \{\lambda_i\}_{i=1}^{n-q}$ in non-decreasing $\vartheta$-lexicographic order, the interior of each of the segments $\overline{\lambda_i\lambda_{i+1}}$ for $i \in \{1,\dots,n-1\}$ includes at least one element of $\sigma(B)\setminus(\sigma(A)\cap\sigma(B))$, say $\mu_i$. Thus, $\mu_i \in \overline{\lambda_i\lambda_{i+1}} \subset \mathcal{Z}_\vartheta(\lambda_i, \lambda_{i+1})$ $(i = 1,\dots,n-q-1)$. Indexing common and non-common elements of $\sigma(A)$ and $\sigma(B)$ in non-decreasing $\vartheta$-lexicographic order yields the result. □

Notice that Corollary 3.6 is an analogue statement to Proposition 3.3 for $k = 1$. Moreover, the fact that Corollary 3.6.**II** also implies collinearity and interlacing of non-common elements of $\sigma(A)$, $\sigma(B)$, as in Proposition 3.5.**III**, has been observed in [11, Cor. 1]. The connections between the necessity part of the Theorem of Fan and Pall (Theorem 1.1), Queiro and Duarte (Theorem 3.1.**II**), Carlson and de Sa (Theorem 3.1.**I**) and Corollary 2.10 are depicted in Figure 3.2.

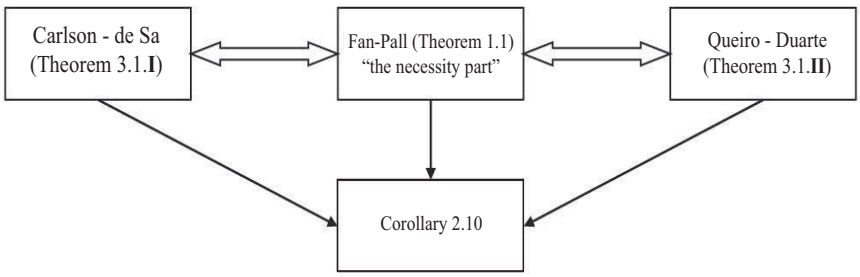


FIG. 3.2. *Links on imbedding conditions in the case* $k = 1$.

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