

SIGNED GRAPHS WITH SMALL POSITIVE INDEX OF INERTIA*

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Abstract. In this paper, the signed graphs with one positive eigenvalue are characterized, and the signed graphs with pendant vertices having exactly two positive eigenvalues are determined. As a consequence, the signed trees, the signed unicyclic graphs and the signed bicyclic graphs having one or two positive eigenvalues are characterized.

Key words. Signed graph, Adjacency matrix, Positive index of inertia.

AMS subject classifications. 05C50, 05C20, 05C75.

1. Introduction. All graphs considered in this paper are connected, simple and undirected. Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $A(G)$ of a graph G of order n is the symmetric 0-1 matrix $(a_{ij})_{n \times n}$ such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. Denote by P_n , C_n , K_n , $K_{1,n-1}$ a path, a cycle, a complete graph and a star, respectively, all of which are simple graphs on n vertices. Sometimes $K_{1,n-1}$ is written as S_n . $K_{n_1, n_2, \dots, n_r}$ represents a complete r -partite graph with part sizes $n_1, n_2, \dots, n_r$. A graph is called *trivial* if it has one vertex and no edges.

A signed graph $\Gamma = (G, \sigma)$ consists of a simple graph $G = (V, E)$, referred to as its underlying graph, and a sign function $\sigma : E \rightarrow \{+, -\}$. Sometimes Γ is written as G^σ . The number of vertices in Γ is sometimes denoted by $|\Gamma|$. The adjacency matrix of Γ is $A(\Gamma) = (a_{ij}^\sigma)$ with $a_{ij}^\sigma = \sigma(v_i v_j) a_{ij}$ where a_{ij} is an element in the adjacency matrix of the underlying graph G . If all edges are signed positive, the adjacency matrix $A(G, \sigma)$ is exactly the ordinary adjacency matrix $A(G)$. We write G^+ for the signed graph with all positive edges. Similarly G^- represents the signed graph with all negative

*Received by the editors on October 24, 2014. Accepted for publication on March 31, 2016.
 Handling Editor: Leslie Hogben.

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edges. The *inertia* of Γ is defined to be the triplet $\text{In}(\Gamma) = (i_+(\Gamma), i_-(\Gamma), i_0(\Gamma))$, where $i_+(\Gamma), i_-(\Gamma), i_0(\Gamma)$ are the numbers of the positive, negative and zero eigenvalues of the adjacency matrix $A(\Gamma)$ including multiplicities, respectively. Traditionally $i_+(\Gamma)$ (resp., $i_-(\Gamma)$) is called the *positive* (resp., *negative*) *index of inertia* (abbreviated *positive* (resp., *negative*) *index*) of Γ . The number $i_0(\Gamma)$ is called the *nullity* of Γ , usually denoted by $\eta(\Gamma)$. The *rank* $r(\Gamma)$ of Γ is defined to be the rank of $A(\Gamma)$. Obviously $i_+(\Gamma) + i_-(\Gamma) = r(\Gamma) = n - \eta(\Gamma)$ if Γ has n vertices.

Let C be a cycle of Γ . The sign of C , denoted by $\text{sgn}(C)$, is the product of the signs of all edges. A cycle C is said to be *positive* or *negative* if $\text{sgn}(C) = +$ or $\text{sgn}(C) = -$. A signed graph is said to be *balanced* if all its cycles are positive, or equivalently, all its cycles have an even number of negative edges; otherwise it is called *unbalanced*. A *switching function* is a function $\theta : V \rightarrow \{+, -\}$. Suppose that θ is a switching function. Then Γ is transformed by θ to a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$ such that the underlying graph remains the same and the sign function is defined on the edge uv by $\sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v)$. Note that switching does not change the signs of cycles in Γ . Two signed graphs Γ_1, Γ_2 are said to be *switching equivalent*, denoted by $\Gamma_1 \sim \Gamma_2$, if there exists a switching function θ such that $\Gamma_2 = \Gamma_1^\theta$, or there exists a diagonal matrix $D^\theta = \text{diag}(\theta(v_1), \theta(v_2), \dots, \theta(v_n))$ such that $A(\Gamma_2) = D^\theta A(\Gamma_1) D^\theta$. There is the following result which relates balance to switching equivalence of signed graphs. It will play a pivotal role in the next section and, consequently, throughout the rest of the paper.

THEOREM 1.1. [5] *Let Γ be a signed graph. Then Γ is balanced if and only if $\Gamma = (G, \sigma)$ is switching equivalent to G^+ .*

A signed graph is called *acyclic* (resp., *unicyclic*) if its underlying graph is acyclic (resp., unicyclic). The *degree* of a vertex $u \in V(\Gamma)$ is the number of edges incident to u . A vertex $v \in V(\Gamma)$ is called a *pendant vertex* if its degree is 1. A graph Γ' is called an *induced subgraph* of Γ on the vertices of Γ' including the signs of edges. For $V_1 \subseteq V(\Gamma)$, we write $\Gamma - V_1$ for the graph obtained from Γ by removing all vertices in V_1 together with all edges incident to them. If $V_1 = \{v\}$, we write $\Gamma - v$ instead of $\Gamma - \{v\}$. Sometimes we use the notation $\Gamma - \Gamma_0$ instead of $\Gamma - V(\Gamma_0)$ if Γ_0 is a subgraph of Γ .

Recently, the nullity of signed graphs was well studied, see [1, 2, 7, 12]. Yu et al. investigated the positive index of unicyclic signed graphs [12, 13]. More results on inertia of graphs can be found in [10, 11].

This paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we characterize all signed graphs with one positive eigenvalue and characterize the signed graphs with pendant vertices having two positive eigenvalues. Moreover, we determine all signed trees, all unicyclic signed graphs having one or two

positive eigenvalues. In Section 4, we characterize all bicyclic signed graphs with one or two positive eigenvalues.

2. Preliminary results. In order to present our main results, the following lemmas are needed.

LEMMA 2.1. [8] *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

LEMMA 2.2. [3, 9] *A complete signed graph is balanced if and only if all triangles are positive.*

LEMMA 2.3. [12, 13] *Let C_n^σ , P_n^σ be a signed cycle, a signed path on n vertices, respectively. Let k be an integer. Then the following hold:*

- (1) *If C_n^σ is balanced, then $i_+(C_n^\sigma) = 2\lfloor \frac{n-1}{4} \rfloor + 1 = 2k + 1$ for $n = 4k + 1, \dots, 4k + 4$.*
- (2) *If C_n^σ is unbalanced, then $i_+(C_n^\sigma) = 2\lfloor \frac{n+1}{4} \rfloor = 2k$ for $n = 4k - 1, \dots, 4k + 2$.*
- (3) *$i_+(P_n^\sigma) = \lfloor \frac{n}{2} \rfloor$.*

COROLLARY 2.4. *If C_n^σ be an unbalanced cycle on n vertices, then $i_+(C_n^\sigma) \geq 2$.*

LEMMA 2.5. [12, 13] *Let Γ be a signed graph containing a pendant vertex v with its unique neighbor u . Then $i_+(\Gamma) = i_+(\Gamma - u - v) + 1$.*

The following results are obvious and we omit their proofs.

LEMMA 2.6. *The following statements hold:*

- (1) *Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_t$, where $\Gamma_1, \Gamma_2, \dots, \Gamma_t$ are the components of Γ . Then $i_+(\Gamma) = \sum_{i=1}^t i_+(\Gamma_i)$.*
- (2) *Let Γ be a signed graph. Then $i_+(\Gamma) = 0$ if and only if Γ is a graph without edges.*
- (3) *Let Γ^* be an induced subgraph of Γ . Then $i_+(\Gamma^*) \leq i_+(\Gamma)$.*

DEFINITION 2.7. Let M be a real symmetric matrix. The three types of elementary congruence matrix operations of M are defined as follows:

- (1) interchanging the i th and j th rows of M and interchanging the i th and j th columns of M ;
- (2) multiplying the i th row and column of M by a nonzero scalar k ;
- (3) adding the i th row of M multiplied by a nonzero scalar k to the j th row and adding the i th column of M multiplied by k to the j th column.

LEMMA 2.8. (Sylvester's law of inertia, [4]) *Let M be an $n \times n$ real symmetric matrix and P be an $n \times n$ nonsingular matrix. Then*

$$i_+(PMP^T) = i_+(M) \quad \text{and} \quad i_-(PMP^T) = i_-(M).$$

By Sylvester's law of inertia, elementary congruence matrix operations do not change the inertia of a real symmetric matrix.

3. Signed graphs with one or two positive eigenvalues. We are now ready to present the main results of this paper.

THEOREM 3.1. *Let $\Gamma = (G, \sigma)$ be a connected signed graph. Then $i_+(\Gamma) = 1$ if and only if Γ is a balanced complete multipartite signed graph.*

Proof. Sufficiency: Assume that Γ is a balanced complete multipartite signed graph. By Theorem 1.1, Γ is switching equivalent to its underlying graph. So $i_+(\Gamma) = 1$ from Lemma 2.1.

Necessity: Since $i_+(P_4) = i_+(\text{paw}) = 2$ where the paw is a graph obtained by adding a pendant edge to one vertex of C_3 , G has no P_4 or paw as an induced subgraph. Suppose that G has an edge e and a vertex u which is not incident to e . Since G is connected, there exists a path P from the u to e ; we may assume that P is the shortest path from the u to e . The length of P is at most two, for otherwise G would have a P_4 as an induced subgraph. Since u is not incident to the e , P must have length two. Let v be the vertex on P incident to the e . If v is incident to exactly one end of e , then G has an induced P_4 . If v is incident to both ends of e , then G has an induced paw. These contradictions show that G has no induced $K_2 \cup K_1$, i.e., G is a complete multipartite graph.

Assume that Γ is unbalanced. Then Γ must contain an unbalanced signed cycle as an induced subgraph. By Corollary 2.4 and Lemma 2.6, $i_+(\Gamma) \geq 2$. This is also a contradiction.

Combining the above discussion, Γ is a balanced complete multipartite signed graph if $i_+(\Gamma) = 1$. $\square$

THEOREM 3.2. *Let Γ be a signed graph with pendant vertices. Then $i_+(\Gamma) = 2$ if and only if Γ is a graph obtained by inserting some edges with arbitrary signs between the center of a signed star and some or all vertices (maybe partial or all) of a balanced complete multipartite signed graph.*

Proof. Assume that Γ has two positive eigenvalues. Let x be a pendant vertex in Γ and $N(x) = y$. Suppose that $\Gamma - x - y = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_t$ where $\Gamma_1, \Gamma_2, \dots, \Gamma_t$ are the components of $\Gamma - x - y$. If each Γ_i ($i = 1, 2, \dots, t$) is trivial, then $\Gamma - x - y$ is a signed star and $i_+(\Gamma) = 1$. This is impossible. Next we shall verify that there exists exactly one nontrivial component in $\Gamma - x - y$. Assume that $\Gamma - x - y$ has two

nontrivial components, say Γ_1, Γ_2 . By Lemma 2.5, we have

$$\begin{aligned} i_+(\Gamma) &= 1 + i_+(\Gamma - x - y) \\ &= 1 + \sum_{j=1}^2 i_+(\Gamma_j) \\ &\geq 1 + \sum_{j=1}^2 1 \quad (\text{since } i_+(\Gamma_j) \geq 1) \\ &= 3. \end{aligned}$$

This is a contradiction.

Therefore, there exists exactly one nontrivial component in $\Gamma - x - y$, say Γ_1 . So $\Gamma - x - y = \Gamma_1 \cup (n - |\Gamma_1| - 2)K_1$. Hence, $i_+(\Gamma) = i_+(\Gamma_1) + 1 \geq 2$ with the equality holding if and only if $i_+(\Gamma_1) = 1$. It is evident that the subgraph induced by x, y and all isolated vertices in $\Gamma - x - y$ is a signed star $K_{1,|\Gamma|-|\Gamma_1|}$. So Γ can be obtained by inserting some edges with arbitrary signs between the center of $K_{1,k-1}$ and some or all vertices of a balanced complete multipartite signed graph with $|\Gamma| - k$ vertices, where $k \geq 2$ is any positive integer. $\square$

As consequences of Theorems 3.1 and 3.2, we determine the signed trees and unicyclic signed graphs on n vertices having one or two positive eigenvalues as follows.

THEOREM 3.3. *Let T be a signed tree on n vertices. Then the following statements hold:*

- (1) $i_+(T) = 1$ if and only if T is $K_{1,n-1}$.
- (2) $i_+(T) = 2$ if and only if T is T_1 or T_2 (as depicted in Figure 3.1).



FIG. 3.1. Two trees in Theorem 3.3.

Proof. (1) If a tree is a complete bipartite graph, then it should be a star. So this result is obvious from Theorem 3.1.

(2) Assume that T is a signed tree and $i_+(T) = 2$. By Theorem 3.2, T is obtained by inserting an edge with arbitrary sign between the center of signed star and one vertex of another signed star. This implies the result. $\square$

THEOREM 3.4. *Let U be a unicyclic signed graph on n vertices. Then the following statements hold:*

- (1) $i_+(U) = 1$ if and only if U is the balanced cycle C_3 or the balanced cycle C_4 .
 (2) $i_+(U) = 2$ if and only if U is one of the following graphs:
- (a) The unbalanced cycle C_3 , the unbalanced cycle C_4 , the unbalanced cycle C_5 or the unbalanced cycle C_6 ;
 - (b) The signed unicyclic graphs with $U_1^{r,s}$ or $U_2^{p,q}$ (as depicted in Figure 3.2) as the underlying graph;
 - (c) The balanced signed unicyclic graphs with U_3^{n-4} (as depicted in Figure 3.2) as the underlying graph;
 - (d) The balanced signed unicyclic graphs with U_4^{n-5} (as depicted in Figure 3.2) as the underlying graph.

Proof. (1). The result can be derived from Theorem 3.1.

(2). Assume that U has no pendant vertices. By Lemma 2.3, $i_+(U) = 2$ if and only if U is the unbalanced cycle C_3 , the unbalanced cycle C_4 , the unbalanced cycle C_5 or the unbalanced cycle C_6 .

Assume that U is a signed unicyclic graph with pendant vertices and $i_+(U) = 2$. It is obvious that the necessity holds from Lemmas 2.3 and 2.5. Next we shall verify the sufficiency. By Theorem 3.2, U is obtained by inserting some edges with arbitrary signs between the center of signed star $K_{1,k-1}^\sigma$ and some vertices (maybe partial or all) of a balanced complete t -partite signed graph $K_{n_1,n_2,\dots,n_t}^\sigma$, where $n_1 + n_2 + \dots + n_t = n - k$. Since U is unicyclic, $t = 2$ or 3 . In the following we shall divide into two cases.

Case 1: $t = 2$.

In this case, K_{n_1,n_2} is isomorphic to a star or C_4 since U is unicyclic. If $K_{n_1,n_2} \cong K_{1,n-k-1}$, by Theorem 3.2 U is one of signed graphs with $U_1^{r,s}$ or $U_2^{p,q}$ (as depicted in Figure 3.2) as the underlying graph, where each edge has arbitrary sign. If $K_{n_1,n_2} \cong C_4$, by Theorem 3.2 U is a balanced signed graph with U_4^{n-5} (as depicted in Figure 3.2) as the underlying graph.

Case 2: $t = 3$.

In this case, $K_{n_1,n_2,n_3} \cong C_3$ since U is unicyclic. By Theorem 3.2 U is a balanced signed graph with U_3^{n-4} (as depicted in Figure 3.2) as the underlying graph. $\square$

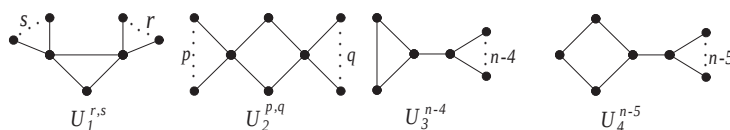


FIG. 3.2. Four unicyclic graphs in Theorem 3.4.

4. Bicyclic signed graphs with one or two positive eigenvalues. Let G be a bicyclic graph. The *base* of G , denoted by $\widehat{G}$, is the unique bicyclic subgraph of G containing no pendant vertices. Thus, G can be obtained from $\widehat{G}$ by attaching trees to some vertices of $\widehat{G}$. It is well known that (see, for example [6]) there are two types of bases of bicyclic graphs, as described next.

Let C_p ($p \geq 3$) and C_q ($q \geq 3$) be two cycles and $P_l = v_1 v_2 \cdots v_l$ ($l \geq 1$) be a path. Assume that $v \in V(C_p)$ and $u \in V(C_q)$. Denote by $\infty(p, l, q)$ (as depicted in Figure 4.1) the graph obtained from C_p, C_q, P_l by identifying v with v_1 , u with v_l , respectively. The bicyclic graph containing $\infty(p, l, q)$ as its base is called an ∞ -graph. Note that in an ∞ -graph the two cycles share a vertex if P_l has $l = 1$.

Let $P_{p+2}, P_{l+2}, P_{q+2}$ be three paths with $\min\{p, l, q\} \geq 0$, where at most one of p, l, q is 0. Denote by $\theta(p, l, q)$ (as depicted in Figure 4.1) the graph obtained from $P_{p+2}, P_{l+2}, P_{q+2}$ by identifying the three initial vertices and terminal vertices. The bicyclic graph containing $\theta(p, l, q)$ as its base is called a θ -graph.

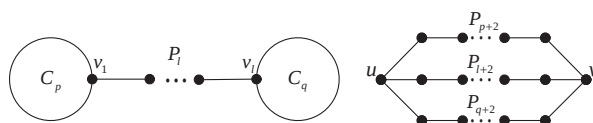


FIG. 4.1. $\infty(p, l, q)$ and $\theta(p, l, q)$.

THEOREM 4.1. *Let B be a bicyclic signed graph. Then $i_+(B) = 1$ if and only if B is the balanced signed graph with $K_{1,1,2}$ or $K_{2,3}$ as the underlying graph.*

Proof. By Theorem 3.1, the underlying graph of B is a complete multipartite graph. Since B is bicyclic, $|B|$ is bipartite or tripartite. Since $K_{2,3}$ is the unique complete bipartite graph which is a bicyclic graph and $K_{1,1,2}$ is the unique complete tripartite graph which is also a bicyclic graph, the result can be derived from Theorem 3.1. $\square$

THEOREM 4.2. *Let B be a bicyclic signed graph with at least one pendant vertex. Then $i_+(B) = 2$ if and only if B is one of the following graphs:*

- (1) *The signed graphs with B_1 (as depicted in Figure 4.2) as the underlying graph such that the induced subgraph on v_1, v_2, v_3 is balanced.*
- (2) *The signed graphs with B_2 or B_3 (as depicted in Figure 4.2) as the underlying graph;*
- (3) *The signed graphs with B_4 or B_5 (as depicted in Figure 4.2) as the underlying graph such that the induced subgraph on u_1, u_2, u_3, u_4 is balanced;*
- (4) *The balanced signed graphs with one of the B_i 's ($i = 6, 7, 8, 9$) (as depicted in Figure 4.2) as the underlying graph.*

Proof. Assume that B is a signed graph of order n with at least one pendant vertex and $i_+(B) = 2$. The sufficiency can be derived from Lemmas 2.3 and 2.5. Next we shall verify the necessity. By Theorem 3.2, B is obtained by inserting some edges with arbitrary signs between the center of signed star $K_{1,k-1}^\sigma$ and some vertices (maybe partial or all) of a balanced complete t -partite signed graph $K_{n_1, n_2, \dots, n_t}^\sigma$, where $n_1 + n_2 + \dots + n_t = n - k$. Since B is bicyclic, $t = 2$ or 3 .

Assume that $t = 2$. K_{n_1, n_2} is isomorphic to C_4 , $K_{1, n-k-1}$ or $K_{2,3}$ since B is bicyclic. If $K_{n_1, n_2} \cong C_4$, by Theorem 3.2 B is one of signed graphs with B_4 or B_5 (as depicted in Figure 4.2) as the underlying graph such that the induced subgraph on u_1, u_2, u_3, u_4 is balanced. If $K_{n_1, n_2} \cong K_{1, n-k-1}$, by Theorem 3.2 B is one of signed graphs with B_2 or B_3 (as depicted in Figure 4.2) as the underlying graph. If $K_{n_1, n_2} \cong K_{2,3}$, by Theorem 3.2 B is one of the balanced signed graphs with B_6 or B_7 (as depicted in Figure 4.2) as the underlying graph.

Assume that $t = 3$. K_{n_1, n_2, n_3} is isomorphic to C_3 or $K_{1,1,2}$ since B is bicyclic. If $K_{n_1, n_2, n_3} \cong C_3$, by Theorem 3.2 B is one of signed graph with B_1 (as depicted in Figure 4.2) as the underlying graph such that the induced subgraph on v_1, v_2, v_3 is balanced. If $K_{n_1, n_2} \cong K_{1,1,2}$, by Theorem 3.2 B is one of the balanced signed graphs with B_8 or B_9 (as depicted in Figure 4.2) as the underlying graph. $\square$

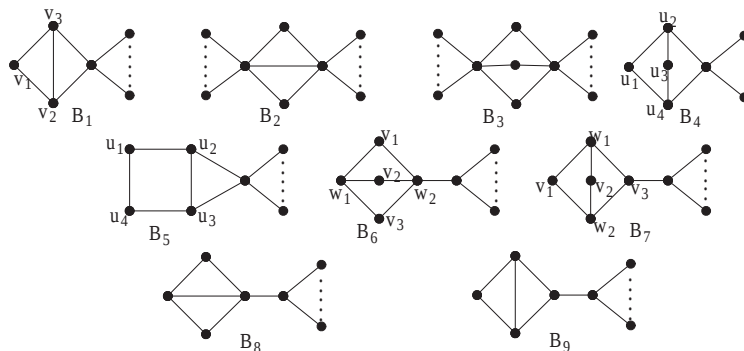


FIG. 4.2. Nine unsigned bicyclic graphs in Theorem 4.2.

In what follows, we shall determine the bicyclic signed graphs without pendant vertices having exactly two positive eigenvalues.

THEOREM 4.3. *Let B be a bicyclic signed graph without pendant vertices. Then $i_+(B) = 2$ if and only if B is one of the following graphs:*

- (1) *The signed graphs with $\infty(3, 1, 3)$ as the underlying graph such that at least one of the cycles is balanced;*
- (2) *The balanced signed graphs with $\infty(3, 2, 3)$, $\infty(3, 1, 4)$ or $\infty(4, 1, 4)$ as the under-*

lying graph;

(3) The unbalanced signed graphs with $\theta(1, 0, 1)$ or $\theta(1, 1, 1)$ as the underlying graph;

Proof. Let B be a bicyclic graph without pendant vertices. Then B is a base. Next we shall divide into two cases to verify our results.

Case 1: B is of type ∞ -graph.

Assume that $p + l + q - 4 > 5$, i.e., $p + l + q > 9$. Then B must be contain P_6 as an induced subgraph. By Lemma 2.6, $i_+(B) \geq 3$ which is a contradiction. So $p + l + q \leq 9$. Note that $p + l + q \geq 7$, therefore $7 \leq p + l + q \leq 9$. In the sequel we distinguish three subcases.

Subcase 1.1: $p + l + q = 7$.

In this subcase, B is the signed graph with $\infty(3, 1, 3)$ as its underlying graph. With appropriate ordering of vertices, the adjacency matrix of B can be expressed as

$$A(\infty(3, 1, 3)^\sigma) = \begin{pmatrix} 0 & a_{12}^\sigma & a_{13}^\sigma & 0 & 0 \\ a_{12}^\sigma & 0 & a_{23}^\sigma & 0 & 0 \\ a_{13}^\sigma & a_{23}^\sigma & 0 & a_{34}^\sigma & a_{35}^\sigma \\ 0 & 0 & a_{34}^\sigma & 0 & a_{45}^\sigma \\ 0 & 0 & a_{35}^\sigma & a_{45}^\sigma & 0 \end{pmatrix}.$$

Applying elementary congruence matrix operations on $A(\infty(3, 1, 3)^\sigma)$, we have

$$i_+(\infty(3, 1, 3)^\sigma) = i_+ \begin{pmatrix} 0 & a_{12}^\sigma & 0 & 0 & 0 \\ a_{12}^\sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\frac{a_{13}^\sigma a_{23}^\sigma}{a_{12}^\sigma} - 2\frac{a_{34}^\sigma a_{35}^\sigma}{a_{45}^\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45}^\sigma \\ 0 & 0 & 0 & a_{45}^\sigma & 0 \end{pmatrix}.$$

This implies that $i_+(\infty(3, 1, 3)^\sigma) = 2$ if and only if $\frac{a_{13}^\sigma a_{23}^\sigma}{a_{12}^\sigma} + \frac{a_{34}^\sigma a_{35}^\sigma}{a_{45}^\sigma} = 0$ or 2 , i.e., at least one of the two cycles in $\infty(3, 1, 3)^\sigma$ is balanced.

Subcase 1.2: $p + l + q = 8$.

In this subcase, B is the signed graphs with $\infty(3, 2, 3)$ or $\infty(3, 1, 4)$ as its underlying graph. With appropriate ordering of vertices, the adjacency matrices of

$\infty(3, 2, 3)^\sigma$, $\infty(3, 1, 4)^\sigma$ can be expressed as

$$A(\infty(3, 2, 3)^\sigma) = \begin{pmatrix} 0 & a_{12}^\sigma & a_{13}^\sigma & 0 & 0 & 0 \\ a_{12}^\sigma & 0 & a_{23}^\sigma & 0 & 0 & 0 \\ a_{13}^\sigma & a_{23}^\sigma & 0 & a_{34}^\sigma & 0 & 0 \\ 0 & 0 & a_{34}^\sigma & 0 & a_{45}^\sigma & a_{46}^\sigma \\ 0 & 0 & 0 & a_{45}^\sigma & 0 & a_{56}^\sigma \\ 0 & 0 & 0 & a_{46}^\sigma & a_{56}^\sigma & 0 \end{pmatrix},$$

$$A(\infty(3, 1, 4)^\sigma) = \begin{pmatrix} 0 & a_{12}^\sigma & a_{13}^\sigma & 0 & 0 & 0 \\ a_{12}^\sigma & 0 & a_{23}^\sigma & 0 & 0 & 0 \\ a_{13}^\sigma & a_{23}^\sigma & 0 & a_{34}^\sigma & 0 & a_{36}^\sigma \\ 0 & 0 & a_{34}^\sigma & 0 & a_{45}^\sigma & 0 \\ 0 & 0 & 0 & a_{45}^\sigma & 0 & a_{56}^\sigma \\ 0 & 0 & a_{36}^\sigma & 0 & a_{56}^\sigma & 0 \end{pmatrix}.$$

By elementary congruence matrix operations, we have

$$i_+(\infty(3, 2, 3)^\sigma) = 2 + i_+(M_1) \quad \text{and} \quad i_+(\infty(3, 1, 4)^\sigma) = 2 + i_+(M_2),$$

$$\text{where } M_1 = \begin{pmatrix} -2\frac{a_{13}^\sigma a_{23}^\sigma}{a_{12}^\sigma} & a_{34}^\sigma \\ a_{34}^\sigma & -2\frac{a_{45}^\sigma a_{46}^\sigma}{a_{56}^\sigma} \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} -2\frac{a_{13}^\sigma a_{23}^\sigma}{a_{12}^\sigma} & a_{34}^\sigma - \frac{a_{45}^\sigma a_{36}^\sigma}{a_{56}^\sigma} \\ a_{34}^\sigma - \frac{a_{45}^\sigma a_{36}^\sigma}{a_{56}^\sigma} & 0 \end{pmatrix}.$$

It can be verified that $i_+(M_1) = 0$ if and only if $\frac{a_{13}^\sigma a_{23}^\sigma}{a_{12}^\sigma} = 1$ and $\frac{a_{45}^\sigma a_{46}^\sigma}{a_{56}^\sigma} = 1$, which implies that $\infty(3, 2, 3)^\sigma$ is balanced. Similarly, $i_+(M_2) = 0$ if and only if $\frac{a_{13}^\sigma a_{23}^\sigma}{a_{12}^\sigma} = 1$ and $a_{34}^\sigma - \frac{a_{45}^\sigma a_{36}^\sigma}{a_{56}^\sigma} = 0$, which implies that $\infty(3, 1, 4)^\sigma$ is balanced.

So $i_+(\infty(3, 2, 3)^\sigma) = 2$ (resp., $i_+(\infty(3, 1, 4)^\sigma) = 2$) if and only if $\infty(3, 2, 3)^\sigma$ (resp., $\infty(3, 1, 4)^\sigma$) is balanced.

Subcase 1.3: $p + l + q = 9$.

Note that B must contain one of $\infty(3, 3, 3)^\sigma$, $\infty(3, 2, 4)^\sigma$, $\infty(3, 1, 5)^\sigma$, $\infty(4, 1, 4)^\sigma$ as its underlying graph. Note that the signed graph with $\infty(3, 3, 3)^\sigma$, $\infty(3, 2, 4)^\sigma$, or $\infty(3, 1, 5)^\sigma$ as the underlying graph has at least three positive eigenvalues since its underlying graph contains B_1 (as depicted in Figure 4.3) as an induced subgraph.

If there exists one unbalanced cycle in $\infty(4, 1, 4)^\sigma$, then $\infty(4, 1, 4)^\sigma$ is unbalanced and $\infty(4, 1, 4)^\sigma$ must contain an unbalanced B_2^σ as an induced subgraph, where B_2 is depicted in Figure 4.3. So $i_+(\infty(4, 1, 4)^\sigma) \geq i_+(B_2^\sigma) \geq 3$ which is a contradiction. If $\infty(4, 1, 4)^\sigma$ is balanced, then $\infty(4, 1, 4)^\sigma$ and its underlying graph have the same spectrum by Lemma 1.1. So $i_+(\infty(4, 1, 4)^\sigma) = i_+(\infty(4, 1, 4)) = 2$ from simple calculations.

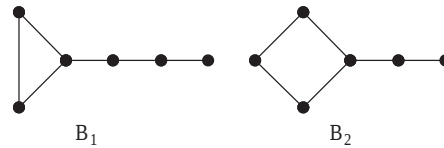


FIG. 4.3. Two graphs in Theorem 4.3.

Case 2: B is of type θ -graph.

Without loss of generality, we assume that $l \leq p \leq q$.

Subcase 2.1: $l > 0$.

Note that if $p + q + 2 \geq 5$, then B contains C_k ($k \geq 5$) as an induced subgraph. So $i_+(B) \geq i_+(C_k^\sigma) \geq 3$. Therefore, $p + q + 2 \leq 4$, i.e., $p + q \leq 2$. Hence, $p + q = 2$ since $p + q \geq 2$. Then B contains $\theta(1, 1, 1)$ as its underlying graph. As in the above discussion, we have

$$i_+(\theta(1, 1, 1)^\sigma) = 1 + i_+ \begin{pmatrix} 0 & a_{34}^\sigma - \frac{a_{14}^\sigma}{a_{12}^\sigma} a_{23}^\sigma & a_{35}^\sigma - \frac{a_{15}^\sigma}{a_{12}^\sigma} a_{23}^\sigma \\ a_{34}^\sigma - \frac{a_{14}^\sigma}{a_{12}^\sigma} a_{23}^\sigma & 0 & 0 \\ a_{35}^\sigma - \frac{a_{15}^\sigma}{a_{12}^\sigma} a_{23}^\sigma & 0 & 0 \end{pmatrix}.$$

So, $i_+(\theta(1, 1, 1)^\sigma) = 2$ if and only if $a_{34}^\sigma - \frac{a_{14}^\sigma}{a_{12}^\sigma} a_{23}^\sigma \neq 0$ or $a_{35}^\sigma - \frac{a_{15}^\sigma}{a_{12}^\sigma} a_{23}^\sigma \neq 0$, which implies that $\theta(1, 1, 1)^\sigma$ is unbalanced.

Subcase 2.2: $l = 0$.

Assume that $p + q \geq 5$. Then B contains P_6^σ as an induced subgraph and $i_+(B) \geq 3$. So $p + q \leq 4$. Note that $p + q \geq 2$. Hence, $2 \leq p + q \leq 4$.

If $p + q = 2$, then B contains $\theta(1, 0, 1)$ as its underlying graph. By Theorem 4.1, $\theta(1, 0, 1)^\sigma$ has two eigenvalues if and only if $\theta(1, 0, 1)^\sigma$ is unbalanced.

If $p + q = 3$, then B contains $\theta(1, 0, 2)$ as its underlying graph. As in the above discussion, we get

$$i_+(\theta(1, 0, 2)^\sigma) = 1 + i_+ \begin{pmatrix} 0 & a_{34}^\sigma & 0 \\ a_{34}^\sigma & 0 & 0 \\ 0 & 0 & -2\frac{a_{15}^\sigma a_{25}^\sigma}{a_{12}^\sigma} + 2\frac{a_{15}^\sigma a_{23}^\sigma a_{45}^\sigma}{a_{12}^\sigma a_{34}^\sigma} \end{pmatrix}.$$

Note that $-2\frac{a_{15}^\sigma a_{25}^\sigma}{a_{12}^\sigma} + 2\frac{a_{15}^\sigma a_{23}^\sigma a_{45}^\sigma}{a_{12}^\sigma a_{34}^\sigma} = 0, -4$ or 4 . So $i_+(\theta(1, 0, 2)^\sigma) = 2$ if and only if $-2\frac{a_{15}^\sigma a_{25}^\sigma}{a_{12}^\sigma} + 2\frac{a_{15}^\sigma a_{23}^\sigma a_{45}^\sigma}{a_{12}^\sigma a_{34}^\sigma} = 0$, or -4 , which implies that C_3 is balanced or C_5 is unbalanced in $\theta(1, 0, 2)^\sigma$.

If $p + q = 4$, then B contains $\theta(1, 0, 3)$ or $\theta(2, 0, 2)$ as its underlying graph. It can be verify that

$$i_+(\theta(2, 0, 2)^\sigma) = 2 + i_+ \begin{pmatrix} 0 & a_{56}^\sigma - \frac{a_{16}^\sigma a_{25}^\sigma}{a_{12}^\sigma} + \frac{a_{45}^\sigma}{a_{34}^\sigma} \cdot \frac{a_{16}^\sigma a_{23}^\sigma}{a_{12}^\sigma} \\ a_{56}^\sigma - \frac{a_{16}^\sigma a_{25}^\sigma}{a_{12}^\sigma} + \frac{a_{45}^\sigma}{a_{34}^\sigma} \cdot \frac{a_{16}^\sigma a_{23}^\sigma}{a_{12}^\sigma} & 0 \end{pmatrix},$$

$$i_+(\theta(1, 0, 3)^\sigma) = 2 + i_+ \begin{pmatrix} 0 & -\frac{a_{16}^\sigma a_{23}^\sigma}{a_{12}^\sigma} - \frac{a_{34}^\sigma a_{56}^\sigma}{a_{45}^\sigma} \\ -\frac{a_{16}^\sigma a_{23}^\sigma}{a_{12}^\sigma} - \frac{a_{34}^\sigma a_{56}^\sigma}{a_{45}^\sigma} & -2 \frac{a_{16}^\sigma a_{26}^\sigma}{a_{12}^\sigma} \end{pmatrix}.$$

Note that $a_{56}^\sigma - \frac{a_{16}^\sigma a_{25}^\sigma}{a_{12}^\sigma} + \frac{a_{45}^\sigma}{a_{34}^\sigma} \cdot \frac{a_{16}^\sigma a_{23}^\sigma}{a_{12}^\sigma} \neq 0$, so $i_+(\theta(2, 0, 2)^\sigma) = 3$. It can be verify that $i_+(\theta(1, 0, 3)^\sigma) = 2$ if and only if $-\frac{a_{16}^\sigma a_{23}^\sigma}{a_{12}^\sigma} - \frac{a_{34}^\sigma a_{56}^\sigma}{a_{45}^\sigma} = 0$ and $\frac{a_{16}^\sigma a_{26}^\sigma}{a_{12}^\sigma} > 0$, which implies that C_3^σ is balanced and C_6^σ is unbalanced in $\theta(1, 0, 3)^\sigma$. $\square$

Acknowledgement. The authors are grateful to anonymous referees for many helpful comments to an earlier version of this paper.

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