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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

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ON THE PRINCIPAL PERMANENT RANK CHARACTERISTIC SEQUENCES OF GRAPHS AND DIGRAPHS*

KEIVAN HASSANI MONFARED†, PAUL HORN‡, FRANKLIN H.J. KENTER§, KATHLEEN NOWAK¶, JOHN SINKOVIC‖, AND JOSH TOBIN**

Abstract. The principal permanent rank characteristic sequence is a binary sequence $r_0r_1 \cdots r_n$ where $r_k = 1$ if there exists a principal square submatrix of size k with nonzero permanent and $r_k = 0$ otherwise, and $r_0 = 1$ if there is a zero diagonal entry.

A characterization is provided for all principal permanent rank sequences obtainable by the family of nonnegative matrices as well as the family of nonnegative symmetric matrices. Constructions for all realizable sequences are provided. Results for skew-symmetric matrices are also included.

Key words. Symmetric matrix, Skew-symmetric matrix, Permanent rank, Principal permanent rank characteristic sequence, Generalized cycle, Matching, Minor.

AMS subject classifications. 15A15, 15A03, 15B57, 05C50.

1. Introduction.

The *principal rank characteristic sequence problem* which was introduced by Brualdi, Deaett, Olesky and van den Driessche asks [2]:

> Given a binary sequence $r_0r_1 \cdots r_n$ of length $n+1$, is there an $n \times n$ matrix A such that $r_k = 1$ if and only if there is a principal submatrix of rank k?

This problem is a simplified form of the more general *principal assignment problem* (see for example [5]).

*Received by the editors on September 22, 2015. Accepted for publication on February 23, 2016. Handling Editor: Leslie Hogben.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

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Recently, several groups have studied the principal rank characteristic sequence problem with different variations. For real matrices, Brualdi et al. characterize all realizable sequences with $n \leq 6$ and all realizable sequences beginning $010\cdots$ for $7 \leq n \leq 10$ [2]. They also provide several forbidden subsequences. Barrett et al. characterize all allowable sequences over fields with characteristic 2 and also provide additional results for other fields [1]. Additionally, in [4], the authors study a variation, the *enhanced principal rank sequence*, which differentiates whether "some" or "all" of the principal minors of order k have rank k where they characterize all such realizable sequences for real matrices of order $n \leq 5$.

Our focus will be the permanent, $\operatorname{per}(A)$, instead of the rank or determinant. Recall the definition of the permanent:

Definition 1.1 ([7]). The *permanent* of an $n \times n$ matrix A is defined to be the sum of all diagonal products of A. That is,

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\,\sigma(i)} \right).$$

Recall that

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)} \right),$$

where $\operatorname{sgn}(\cdot)$ is the sign of the permutation. Hence, in some sense, the permanent can be viewed as a variation of the determinant.

Note that determining whether there is a principal submatrix of rank k is equivalent to seeing if there is a principal submatrix of size k with nonzero determinant (see [2]). Therefore, in a similar fashion, one can define the *permanent rank*:

Definition 1.2 ([9]). The *permanent rank* of a matrix A, denoted $\operatorname{perrank}(A)$ is defined to be the size of the largest square submatrix of A with nonzero permanent.

We study the principal permanent rank characteristic sequence defined as follows.

Definition 1.3. Given an $n \times n$ matrix A, the *principal characteristic permanent rank sequence* of A (abbreviated ppr-sequence of A or $\operatorname{ppr}(A)$) is defined as $r_0 r_1 r_2 \cdots r_n$, where for $1 \leq k \leq n$,

$$r_k = \begin{cases} 1 & \text{if } A \text{ has a principal submatrix of size } k \text{ with nonzero permanent, and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if A has a zero on its main diagonal.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

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Naturally, in this paper, we introduce the *principal permanent rank sequence problem*:

> Given a binary sequence $r_0r_1\cdots r_n$, when is there an $n\times n$ matrix A such that $ppr(A) = r_0r_1\cdots r_n$?

Our contribution is to answer this question and to fully characterize which sequences can be realized for various families of real matrices including

- nonnegative matrices (Section 3, Theorem 3.1),
- symmetric nonnegative matrices (Section 4, Theorem 4.5), and
- skew-symmetric matrices whose underlying graph is a tree (Section 5, Theorem 5.2).

Additionally, for each characterization, we provide a construction that produces a realization for any realizable sequence.

2. Preliminaries. Our main approach is to exploit the duality between matrices and graphs. Throughout, we will consider graphs, both directed and undirected and with or without loops. However, we will not consider graphs with multiple edges (see Proposition 2.1).

Let $[n] = \{1, \ldots, n\}$. For a (directed) graph G on n vertices, $V(G) = [n]$, and if $\alpha \subseteq [n]$, the graph $G[\alpha]$ is the induced subgraph of G on vertices in α. For an $n\times n$ matrix A and $\alpha \subseteq [n]$, $A[\alpha]$ denotes the principal submatrix of A formed from the rows and columns indexed by α. The zero–nonzero pattern of A is a $(0, 1)$-matrix B of the same order where $B_{ij} = 1$ if and only if $A_{ij} \neq 0$. Also, the *underlying graph* of a matrix A is the graph G whose adjacency matrix is the zero–nonzero pattern of A. Note that G is undirected if and only if the zero–nonzero pattern of A is symmetric.

The following proposition shows that the ppr-sequence of a nonnegative matrix and its zero–nonzero pattern are one and the same. Thus, for a nonnegative matrix, we will focus our attention on its underlying graph.

PROPOSITION 2.1. *Let B be the zero–nonzero pattern of an $n\times n$ nonnegative matrix A. Then* $\mathrm{ppr}(A) = \mathrm{ppr}(B)$.

Proof. The proof follows immediately from the definition of permanent. □

It is well known that various graph properties are captured by the permanent rank of matrices describing the graph. Such properties include the size of a largest *generalized cycle* and the size of the largest perfect matching in the graph (see [8, page 54] and [3, pages 112 and 247]). Let us formally define a generalized cycle.

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A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

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Definition 2.2. A *generalized cycle* of size k is a permutation, π_C, on a subset of k vertices, C, such that $i\pi_C(i)$ is a directed edge (or a loop if $i = \pi_C(i)$) for all $i \in C$.

Observe that for a (directed) graph G, $C \subset V(G)$ supports a generalized cycle if there is a collection of edges within $G[C]$, such that every component of the subgraph induced on those edges has a Hamiltonian cycle. A generalized cycle can be viewed as both a permutation or a subset of edges. Here, a bi-directed edge (or undirected edge) can be seen as a 2-cycle. With this clear bijection, we will refer to such a collection of cycles also as a "generalized cycle."

Further, a generalized cycle of order $|G|$ is said to be *spanning*. Next, recall that a *matching* is a collection of disjoint edges. Since a matching in an undirected graph can be viewed as a disjoint collection of directed 2-cycles, every matching forms a generalized cycle. The set of all generalized cycles of order k of a (directed) graph G is denoted by $\text{cyc}_k(G)$.

The connection between generalized cycles and permanent ranks is made formal by the following proposition.

Proposition 2.3. *Let G be the underlying (directed) graph of the nonnegative matrix A and let $\text{ppr}(A) = r_0r_1\cdots r_n$. For $k \geq 1$, $r_k = 1$ if and only if G has a (directed) generalized cycle of order k.*

Proof. Let $\alpha \subseteq [n]$ with $|\alpha| = k$. Then

$$\text{per}(A[\alpha]) = \sum_{\pi \in S_\alpha} \prod_{j=1}^{k} a_{i_j,\pi(i_j)},$$

where $\alpha = \{i_1, \dots, i_k\}$. A term of the sum above is nonzero (and positive) if and only if $\pi \in \text{cyc}_k(G)$. □

We say a binary sequence $r_0r_1\cdots r_n$ is *realizable*, if there is a matrix whose ppr-sequence is $r_0r_1\cdots r_n$.

3. General nonnegative matrices. In this section, we characterize the principal permanent rank sequences of nonnegative matrices. We prove:

Theorem 3.1. *The binary sequence $r_0r_1\cdots r_n$ is realizable as a ppr-sequence of a nonnegative matrix if and only if*

- $r_0 = 0$ *and* $r_i = 1$ *for* $i = 1, 2, \dots, n$, *or*
- $r_0 = 1$.

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A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

First, let us prove the following lemma.

LEMMA 3.2. *Let A be a nonnegative $n \times n$ matrix with* ppr*-sequence $r_0r_1 \cdots r_n$. If $r_0 = 0$, then $r_i = 1$ for all $i = 1, 2, \ldots, n$.*

Proof. Recall $r_0 = 0$ if and only if there is a loop on every vertex in the underlying graph G. Thus, for all $k \in [n]$, G has a generalized cycle of order k consisting of k loops. Therefore, by Proposition 2.3, $r_k = 1$ for all $k \in [n]$. Lastly, any sequence of the form $r_0r_1 \cdots r_n = 011 \cdots 1$ is realized by I_n, the identity matrix of order n. □

Proof of Theorem 3.1. The case when $r_0 = 0$ is covered by Lemma 3.2. Hence, we can assume $r_0 = 1$. We will construct a directed graph, G, as follows. Start with the directed path $v_1 \to v_2 \to \cdots \to v_n$. Next, for each $k \in [n]$, add a directed edge from v_k to v_1 if and only if $r_k = 1$ (see Figure 3.1).

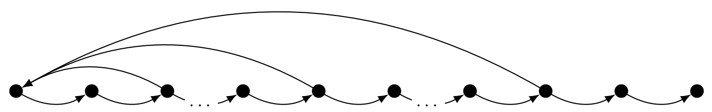

FIG. 3.1. *An illustration of the construction in Theorem 3.1.*

Let A be the adjacency matrix of G and $\text{ppr}(A) = q_0q_1 \cdots q_n$. Note that there may be a loop at vertex v_1, in which case we take $a_{11} = 1$. We claim that $q_i = r_i$ for each $i \in [n]$. First note that $r_0 = 1$, because $a_{22} = 0$. Now consider r_k, for $k \geq 1$. If $r_k = 1$, then there is an edge from v_k to v_1. Hence, $C = (v_1, \ldots, v_k)$ is a directed generalized cycle of order k in G. Thus, by Proposition 2.3, $q_k = 1$.

Now suppose that $r_k = 0$ and consider a subset S of k vertices. If $v_1 \notin S$, then $G[S]$ is a disjoint union of directed paths and thus has no spanning generalized cycle. Now assume that $v_1 \in S$. If $v_j \in S$ for some $j > k$, then $v_i \notin S$ for some $1 < i \leq k$. Thus, $G[S]$ has no generalized cycle containing v_j. Finally, if $S = \{v_1, v_2, \ldots, v_k\}$, $G[S]$ is a graph on k vertices with a pendent vertex v_k. Therefore, $G[S]$ has no spanning generalized cycle. Hence, by Proposition 2.3, $q_k = 0$. □

4. Nonnegative symmetric matrices. In this section, we consider the principal permanent rank characteristic sequences of nonnegative symmetric matrices. In contrast to general nonnegative matrices, the set of allowable sequences is more restrictive. The key difference between the symmetric and general case is that in the symmetric case a single graph edge always counts as a 2-cycle. For example, $r_2 = 1$ if the underlying graph has an edge. Moreover, since every even cycle contains a perfect matching, this implies that we may always choose a generalized cycle where all even cycles are 2-cycles. That is, if G contains a generalized cycle of order k, then

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

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one realization consists of a (possibility empty) matching and a (possibility empty) collection of odd cycles.

Ultimately, in Theorem 4.5, we fully characterize which ppr-sequences are realizable by nonnegative symmetric matrices. First, we provide some necessary conditions for a binary sequence to be realizable in Lemmas 4.1–4.4.

The following lemma shows if there is no generalized cycle of an even order $2k$, then every generalized cycle of the graph is smaller than $2k$.

LEMMA 4.1. *Suppose A is a nonnegative $n \times n$ symmetric matrix, and let $ppr(A) = r_0 r_1 r_2 \cdots r_n$. If $r_{2k} = 0$ for some $k > 0$, then $r_j = 0$ for all $j \geq 2k$.*

Proof. Recall that Proposition 2.3 asserts that $r_j = 1$ if and only if the underlying graph, G, has a generalized cycle on j vertices. First, suppose $r_j = 1$ for some odd $j = 2t + 1$. Then there exists a generalized cycle of G consisting of at least one odd cycle, along with (possibly) a matching. If this odd cycle consists of a single loop, removing this loop yields a generalized cycle on $j - 1$ vertices, and so $r_{j-1} = 1$. Otherwise, it is an odd cycle of length at least three, and discarding an arbitrary vertex from this odd cycle results in a path on an even number of vertices. This path contains a spanning matching. When this matching is considered with the other components of the original odd generalized cycle, we have a generalized cycle on $j - 1$ vertices. Thus, $r_{j-1} = 1$.

Now, suppose that $r_j = 1$ for some $j = 2t$. Then there is a generalized cycle C of order j consisting of a (possibly empty) collection of odd cycles plus a (possibly empty) matching. If C contains a matching edge, then discarding it yields a generalized cycle on $2t - 2$ vertices. Hence, $r_{j-2} = r_{2t-2} = 1$. Otherwise, C consists solely of an even number of odd cycles. Discarding one vertex each from two different odd cycles, and noting again that the remaining even paths contain a spanning matching, yields a generalized cycle on $2t - 2$ vertices. That is, $r_{2t-2} = 1$.

Therefore, if $r_{2t+2} = 1$, or $r_{2t+1} = 1$, we have that $r_{2t} = 1$, and this implies the lemma. ◻

The proof of Lemma 4.1 further demonstrates that if an even generalized cycle exists, then there is a generalized cycle for all smaller even orders. Thus, we are left to study the restriction that odd cycles impose on the ppr-sequence. In Lemma 4.3, we show that the odd indices i for which $r_i = 1$ must be sequential; however, first we make a few structural observations.

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FACT 1. Suppose A is a nonnegative $n \times n$ symmetric matrix, and let $ppr(A) = r_0r_1r_2 \cdots r_n$. If $2\ell + 1$ is the length of the shortest odd cycle of G, then $r_{2\ell+1} = 1$, and $r_t = 0$ for all odd $t < 2\ell + 1$.

LEMMA 4.2. *Suppose A is a nonnegative $n \times n$ symmetric matrix, and let $ppr(A) = r_0r_1r_2 \cdots r_n$. If $r_{2k-1} = 0$ and $r_{2k+1} = 1$, then every generalized cycle on $2k+1$ vertices is a $2k+1$ cycle, and the vertex set of that generalized cycle induces a cycle with no chords.*

Proof. Consider a generalized cycle on $2k + 1$ vertices. As noted before, we can choose a generalized cycle consisting of a collection of odd cycles plus a matching. If there is an edge in the matching, however, discarding it yields a generalized cycle on $2k - 1$ vertices, that is, $r_{2k-1} = 1$. Thus, the generalized cycle is a collection of odd cycles. If there is more than one odd cycle in the collection, one vertex can be discarded from two different cycles, and a perfect matching can be taken from the resulting even paths to find a generalized cycle on $2k - 1$ vertices. Thus, assuming $r_{2k-1} = 0$, the generalized cycle is a single cycle.

Now let V be the vertex set for some generalized cycle of order $2k + 1$. The set V induces a $2k + 1$ cycle, along with potentially some chords. If there is a chord, however, $G[V]$ also consists of a smaller odd cycle (containing the chord) plus a path on the remaining even number of vertices containing at least one edge. Converting this path to a matching and discarding an edge would yield a generalized cycle on $2k - 1$ vertices, completing the proof of the lemma. ∎

LEMMA 4.3. *Suppose A is a nonnegative $n \times n$ symmetric matrix, and let $ppr(A) = r_0r_1r_2 \cdots r_n$. If $r_{2i+1} = r_{2k+1} = 1$ for some integer $i < k$, then $r_t = 1$ for all $2i + 1 \leq t \leq 2k + 1$.*

Proof. By Lemma 4.1 it suffices to just consider r_t for odd t.

It also suffices to show that $r_{2k-1} = 1$. Suppose to the contrary that $r_{2k-1} = 0$. By Lemma 4.2, every generalized cycle of size $2k+1$ is an (induced) $2k+1$ cycle with no chords. Fix such a generalized cycle on vertex set V. Suppose $j < k$ is minimum with the property that $r_{2j-1} = 1$. Again, fix a generalized cycle with size $2j-1$. This is also an (induced) cycle on a vertex set V'.

If $V' \cap V = \emptyset$, then we are done; discarding a vertex from V, we have a path on $2k$ vertices, and a cycle on $2j - 1$ vertices. This path can be treated as a matching, and discarding sufficiently many edges in the matching yields a generalized cycle of size $2k - 1$. Otherwise, we may assume that the cycle on V' follows along the cycle on V on s contiguous segments sharing a total of ℓ vertices. Immediately following each segment there must be at least one vertex in $V' \setminus V$, since by Lemma 4.2, the $2k + 1$ cycle has no chords; so $s + \ell \leq 2j - 1$. The vertices in V *not* in V' lie on s

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A publication of the International Linear Algebra Society
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segments and there are $2k+1-\ell$ such vertices. For parity reasons, we may have to delete one vertex from each segment, but we can then obtain a matching on at least

$$2k+1-\ell-s$$

vertices. Combining this matching with the cycle on $2j-1$ vertices, we have a generalized cycle on

$$2k+1+(2j-1)-\ell-s \geq 2k+1$$

vertices comprised of a cycle on $2j-1$ vertices plus a matching. Discarding sufficiently many edges of the matching, we again obtain a generalized cycle of size $2k-1$. ◻

Finally, the following lemma shows that the largest odd generalized cycle of a graph strongly constrains the largest even generalized cycle.

LEMMA 4.4. *Suppose A is a nonnegative $n \times n$ symmetric matrix, and let $ppr(A) = r_0r_1r_2\cdots r_n$. If m and M are respectively the smallest and largest odd integers so that $r_m = r_M = 1$, then $r_{m+M+2} = 0$, assuming $m+M+2 \leq n$.*

Proof. Let t be the largest even number so that $r_t = 1$, and consider a generalized cycle C_1 on t vertices. Suppose $t > M+1$. If the generalized cycle on t vertices contains an odd cycle, then a single vertex can be discarded from an odd cycle to obtain a generalized cycle on $t-1$ vertices, and hence $t-1 \leq M$, which is a contradiction.

Thus, we may assume that the generalized cycle C_1, on t vertices, consists of $\frac{t}{2}$ disjoint edges. Consider a generalized cycle C_2 on m vertices. Notice that the vertices of C_2 intersect at most m of the disjoint edges of C_1. Consider C_3 to be the union of C_2 with all of the edges of C_1 that are not adjacent to C_2. Now we count how many vertices are in C_3; there are m vertices from C_2 and at least $2(\frac{t}{2}-m)$ vertices from C_1. Hence, C_3 has at least $m+2(\frac{t}{2}-m) = t-m$ vertices. Since C_3 is an odd generalized cycle, $t-m \leq M$. Rearranging, we get $t \leq m+M$, which proves the lemma. ◻

The following theorem shows that the above necessary conditions on the ppr-sequence of a nonnegative symmetric matrix are indeed sufficient. That is, if a binary sequence $r_0r_1\cdots r_n$ satisfies the conditions of Lemmas 4.1–4.4, then there is a nonnegative symmetric matrix whose ppr-sequence is $r_0r_1\cdots r_n$.

THEOREM 4.5. *Any binary sequence not discounted by Lemmas 3.2–4.4 is realizable by a symmetric nonnegative matrix.*

That is, $r_0r_1\cdots r_n$ is realizable as a ppr-sequence of a nonnegative symmetric matrix if and only if

Case 1: $r_0 = 0$ *and* $r_i = 1$ *for* $i = 1, 2, \ldots, n$*; or*

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Case 2: *there are nonnegative integers* ℓ, k, h *with* $\ell \leq k \leq h \leq \ell + k + 1$ *where*
a) $r_{2j+1} = 0$ *for any* $j < \ell$*,*
b) $r_{2j+1} = 1$ *for any* j *with* $\ell \leq j \leq k$*,*
c) $r_{2j+1} = 0$ *for any* j *with* $k < j \leq \frac{n-1}{2}$*,*
d) $r_{2j} = 1$ *for any* $0 \leq j \leq h$*, and*
e) $r_{2j} = 0$ *for any* $h < j \leq \frac{n-1}{2}$*; or*

Case 3: $r_0 = 1$*,* $r_i = 0$ *for all odd* $i \leq n$*,* $r_i = 1$ *for all even* $i \leq 2h$ *and* $r_i = 0$ *for all even* $i > 2h$ *for some nonnegative* $h \leq \lceil \frac{n-1}{2} \rceil$*.*

Proof. We now proceed in proving the theorem by cases. For each case, we prove necessity using the previous lemmas then provide a construction.

Case 1: A sequence with $r_0 = 0$ is realizable as a ppr-sequence if and only if $r_i = 1$ for $i = 1, 2, \ldots, n$.

Necessity for this case is covered by Lemma 3.2 where the identity matrix, I_n, realizes the sequence.

Case 2: A sequence with $r_0 = 0$ and $r_i = 1$ for some odd i is realizable as a ppr-sequence if and only if Cases 2a–2e are met.

Let us demonstrate the necessity of Cases 2a–2e. We first show that $h \leq k+\ell+1$. Using Lemma 4.4 with $m = 2\ell + 1$ and $M = (2\ell + 1) + 2(k - \ell) = 2k + 1$, we have $r_{m+M+2} = r_{2(\ell+k+2)} = 0$. By Lemma 2, it follows that the maximum index i for which $r_i = 1$ obeys $i < \ell + k + 1$. Hence, $h \leq \ell + k + 1$. Fact 1 implies that $r_t = 0$ for any odd t where t is less than the length of the shortest odd cycle of the graph. This implies Case 2a is necessary. Similarly, Lemma 4.3 implies for any odd t between the length of the shortest odd cycle and the length of the largest odd generalized cycle of the graph, $r_t = 1$. This shows the necessity of Case 2b and Case 2c. Lemma 3.2 asserts $r_0 = 1$, and Lemma 4.1 implies for even numbers t no more than a fixed number, $r_t = 1$, and for even numbers t more than that fixed number, $r_t = 0$. This implies Case 2d and Case 2e are necessary. It now suffices to construct a matrix for the three following subcases depending upon the equality/inequality among ℓ, k and h.

<u>*(all equal to 0)*</u>: $0 = \ell = k = h$ (i.e., $r_0 = r_1 = 1$ and $r_i = 0$ for $i = 2, 3, \ldots, n$).

Consider the graph with n isolated vertices where one vertex has a loop. The adjacency matrix has $r_0 = r_1 = 1$ and $r_i = 0$ otherwise.

<u>*(all equal but nonzero)*</u>: $r_0 = 1$ and $0 < \ell = k = h$.

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A publication of the International Linear Algebra Society
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Consider a cycle on $2\ell+1$ vertices and $n-2\ell-1$ isolated vertices. The only odd generalized cycle is on $2\ell+1$ vertices, and there is a matching on the cycle for all even $2j$ for $j \le \ell$.

(not all equal): $r_0 = 1$ and ℓ, k, h are not all equal.

We construct a graph as follows. Construct an odd cycle on vertices $1, 2, \dots, 2\ell+1$ (if $\ell = 0$, take the odd cycle to be a loop on a single vertex), and a path on vertices $2\ell+1, 2\ell+2, \dots, 2k+1$. Add $2(h-k)-1$ vertices and connect each of them to one of the vertices $1, 2, \dots, 2(h-k)-1$ such that no pair of them is connected to the same vertex. Finally, add $n-2h$ vertices and connect all of them to vertex 1. See Figure 4.1.

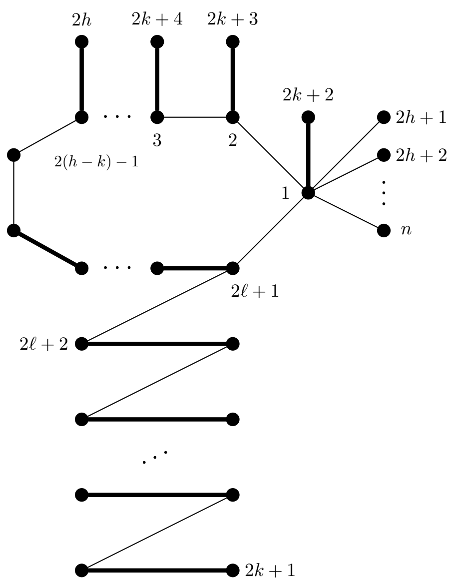

FIG. 4.1. *An illustration of the construction of Case 2 (not all equal) in the proof of Theorem 4.5.*

For the non-equal case, we now verify that Cases 2a–2e hold. Cases 2a–2c assert that the smallest and the largest odd generalized cycles of the graph are to be of sizes $2\ell+1$ and $2k+1$, respectively. Also, Case 2d and 2e assert that the graph has to have a maximum matching of size $2h$. This matching is also a generalized cycle of size $2h$.

The smallest odd cycle of G is of size $2\ell+1$, hence

a) $r_{2j+1} = 0$ for $j < \ell$.

Now, consider the $2\ell+1$ cycle joint with (possibly zero) disjoint edges from the path. This shows there are generalized cycles of length $2j+1$ for $\ell \le j \le k$. That is,

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b) $r_{2j+1} = 1$ for $\ell \leq j \leq k$, and
c) $r_{2j+1} = 0$ for any j with $k < j$.

Note that the graph has a maximum matching of size h. This is obtained by taking the edges that connect each of the vertices $1, 2, \ldots, 2(h-k)-1$ to the pendent vertex adjacent to them ($2(h-k)-1$ edges), every other edge in the rest of the $2\ell+1$ cycle ($\frac{2\ell+1-2(h-k)+1}{2}$ edges), and the maximum matching from the path ($k-\ell$ edges). Thus,

d) $r_{2j} = 1$ for any $1 \leq j \leq h$, and
e) $r_{2j} = 0$ for any $h < j$.

Case 3: A sequence with $r_0 = 0$ and $r_i = 1$ for no odd i is realizable as a ppr-sequence if and only if $r_i = 1$ for all even $i \leq 2h$ and $r_i = 0$ for all even $i > 2h$ for some nonnegative $h \leq \lceil \frac{n-1}{2} \rceil$.

Necessity follows from Lemmas 3.2 and 4.1 as in the proof for Cases 2d and 2e. This case is obtained with a graph with h disjoint edges and $n - 2h$ isolated vertices. □

5. Skew-symmetric matrices. Previously, we only considered nonnegative matrices. This consideration benefited the analysis as every contribution to the permanent was necessarily positive.

In this section, we consider *skew-symmetric* matrices. Recall that a real matrix A is *skew-symmetric* if $A_{ji} = -A_{ij}$. First note that the odd positions in the ppr-sequence of a skew-symmetric matrix have to be all zero, as shown in the following lemma.

LEMMA 5.1. *Let A be a skew-symmetric matrix with* $\text{ppr}(A) = r_0 r_1 \cdots r_n$. *Then* $r_{2i+1} = 0$ *for all integers i with* $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

Proof. The proof follows from Section 2.11 of [6]. Indeed, if B is an odd principal submatrix, $\text{per}(B) = \text{per}(B^T) = \text{per}(-B)$. Since B is of odd order, $\text{per}(-B) = -\text{per}(B)$. Hence, $\text{per}(B) = 0$. □

Now, the question is to characterize the patterns of zeros and ones in the even positions of this sequences. Concentrating on the even positions, several examples of small size are checked and it is observed that there are no gaps between the ones in the even positions. It is easy to see that this property holds for trees. For any graph G, let $\mu(G)$ denote the number of edges in the largest matching of the graph G. In the following theorem, we will characterize $\text{ppr}(A)$ for all skew-symmetric matrices whose underlying graph is a tree.

THEOREM 5.2. *Let A be a skew-symmetric matrix whose underlying graph is a tree T. Then, the principal permanent rank sequence* $\text{ppr}(A) = r_0 r_1 \cdots r_n$ *has* $r_k = 1$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 187-199, April 2016

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if and only if k is even and $k \leq 2\mu(T)$. Furthermore, any such sequence is realizable by a skew-symmetric matrix whose underlying graph is a tree.

Proof. If $k \leq n$ is odd, then $r_k = 0$ by Lemma 5.1. Choose k even and $\alpha \subset [n]$ with $|\alpha| = k$. Let $B = A[\alpha] = (b_{i,j})$. We will show that the permanent of B is nonzero if $k \leq 2\mu(T)$ and 0 otherwise.

$$\begin{aligned}
\operatorname{per}(B) &= \sum_{\sigma \in S_k} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) \\
&= \sum_{\sigma \in M_{k/2}} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) + \sum_{\sigma \in D_k \setminus M_{k/2}} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) \\
&\quad + \sum_{\sigma \in S_k \setminus (D_k \cup M_{k/2})} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right),
\end{aligned}$$

where $M_{k/2}$ is the set of permutations corresponding to the maximum matchings of $T[\alpha]$ (i.e., a disjoint product of transpositions) and D_k is the set of all derangements on α. Observe that for $\sigma \in D_k \setminus M_{k/2}$, σ must have a cycle of size 3 or more; however, T is a tree, so no such σ contributes to its sum. Similarly, any permutation $\sigma \notin D_k$ also contributes 0. Therefore, we have

$$\operatorname{per}(B) = \sum_{\sigma \in M_{k/2}} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) = (-1)^{k/2} \sum_{m \in M_{k/2}} \left(\prod_{\{i,j\} \in m} b_{ij}^2 \right).$$

where the final line considers the matchings as a collection of edges. Since for the last term, $b_{i,j}^2 > 0$, the final sum is nonzero so long as the sum is not empty. The sum is empty only when $k > 2\mu(T)$.

Now, we construct a skew-symmetric matrix A whose underlying graph is a tree T and $\operatorname{ppr}(A) = r_0 r_1 \cdots r_n$, where $r_j = 1$ if and only if j is even and $1 \leq j \leq 2m$, for some $m \leq n/2$. Consider a path of length $2m$ on vertices $1, 2, \ldots, 2m$. Add $n - 2m$ vertices and connect all of them to vertex $2m - 1$. Let B be the adjacency matrix of this graph, and A be the matrix obtained from B by negating all the lower-diagonal entries. Since T does not have any cycles, all the nonzero terms in the permanent of a principal submatrix of A come from a matching of T. Hence, $\operatorname{ppr}(A) = r_0 r_1 \cdots r_n$, with $r_j = 1$ if and only if j is even and $1 \leq j \leq 2m$. ☐

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A publication of the International Linear Algebra Society
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Note that for the matrix A below whose graph is not a tree (it is C_4, a cycle on four vertices) the $\mathrm{ppr}(A) = 10100$. That is, $r_4 = 0$ even though $\mu(C_4) = 2$:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}.$$

This shows that Theorem 5.2 is not necessarily true for skew-symmetric matrices whose graph is not a tree. An interesting question is the following:

QUESTION. Let A be an $n \times n$ skew-symmetric matrix with $\mathrm{ppr}(A) = r_0 r_1 \cdots r_n$. Are there $i < j \leq \lfloor \frac{n}{2} \rfloor$ such that $r_{2i} = 0$ and $r_{2j} = 1$?

Acknowledgement. Research supported in part by NSF grants DMS-1427526, "The Rocky Mountain – Great Plains Graduate Research Workshop in Combinatorics," CMMI-1300477 and CMMI-1404864.

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