

NOTES ON TWO RECENT RESULTS OF AUDENAERT*

YAZHOU HAN[†] AND JINGJING SHAO[‡]

Abstract. In this article, some inequalities for τ -measurable operators which are related to two recent results of Audenaert are proved.

Key words. Noncommutative L_p spaces, Von Neumann algebras, τ -Measurable operators.

AMS subject classifications. 47A63, 46L52.

1. Introduction. Let \mathbb{M}_n be the space of $n \times n$ complex matrices. A norm $\|\cdot\|$ on \mathbb{M}_n is called unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$.

Let \mathbb{M}_n^+ be the positive part of \mathbb{M}_n . In [1], Audenaert proved that if $A_i, B_i \in \mathbb{M}_n^+$ ($i = 1, 2, \dots, n$), such that $A_i B_i = B_i A_i$, then

$$(1.1) \quad \left\| \sum_{i=1}^n A_i B_i \right\| \leq \left\| \left(\sum_{i=1}^n A_i^{\frac{1}{2}} B_i^{\frac{1}{2}} \right)^2 \right\| \leq \left\| \left(\sum_{i=1}^n A_i \right) \left(\sum_{i=1}^n B_i \right) \right\|.$$

A special case of inequality (1.1) confirms a conjecture of Hayajneh and Kittaneh in [9] and answers a question of Bourin.

In another paper [2], Audenaert proved that for $X, Y \in \mathbb{M}_n$ and $0 \leq q \leq 1$,

$$(1.2) \quad \|XY^*\|^2 \leq \|qX^*X + (1-q)Y^*Y\| \|(1-q)X^*X + qY^*Y\|.$$

As is explained in [2], inequality (1.2) interpolates between the Arithmetic-Geometric mean and Cauchy-Schwarz matrix norm inequalities. Very recently Lin [12] gave another proof of inequality (1.1) and (1.2).

Using the notion of the generalized singular numbers studied by Fack and Kosaki [7], we show that the inequality (1.1) and (1.2) hold for the norm on noncommutative L_p spaces. Our idea of proof follows the one given in [12].

*Received by the editors on December 24, 2015. Accepted for publication on March 10, 2016.
 Handling Editor: Harm Bart.

[†]College of Mathematics and System Sciences, Xinjiang University Urumqi 830046, China (hyz0080@aliyun.com). Supported by the National Natural Science Foundation of China No. 11401507 and the Natural Science Foundation of Xinjiang University (Starting Fund for Doctors, Grant No. BS150202).

[‡]College of Information Science and Technology, Xiamen University Tan Kah Kee College, Xiamen 363123, China (jingjing.shao86@yahoo.com).

2. Preliminaries. Unless stated otherwise, \mathcal{M} will always denote a semifinite von Neumann algebra acting on the Hilbert space \mathcal{H} , with a normal faithful finite normalized trace τ . We refer to [14] for noncommutative integration. We denote the identity of \mathcal{M} by 1. A closed densely defined linear operator x in \mathcal{H} with domain $D(x) \subseteq \mathcal{H}$ is said to be affiliated with \mathcal{M} if $u^*xu = x$ for all unitary operators u which belong to the commutant \mathcal{M}' of \mathcal{M} . If x is affiliated with \mathcal{M} , we define its distribution function by $\lambda_s(x) = \tau(e_s^\perp(|x|))$ and x will be called τ -measurable if and only if $\lambda_s(x) < \infty$ for some $s > 0$, where $e_s^\perp(|x|) = e_{(s, \infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞) . The set of all τ -measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$ -algebra with sum and product being the respective closures of the algebraic sum and product. The measure topology in $L_0(\mathcal{M})$ is the vector space topology defined via the neighbourhood base $\{N(\varepsilon, \delta) : \varepsilon, \delta > 0\}$, where $N(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \tau(e_{(\varepsilon, \infty)}(|x|)) \leq \delta\}$ and $e_{(\varepsilon, \infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (ε, ∞) . With respect to the measure topology, $L_0(\mathcal{M})$ is a complete topological $*$ -algebra.

DEFINITION 2.1. Let $x \in L_0(\mathcal{M})$ and $t > 0$. The t -th singular number (or generalized singular number) of x , $\mu_t(x)$, is defined by

$$\mu_t(x) = \inf \{ \|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^\perp) \leq t \}.$$

If $x, y \in L_0(\mathcal{M})$, then we say that x is submajorized by y and write $x \prec y$ if and only if

$$\int_0^a \mu_t(x) dt \leq \int_0^a \mu_t(y) dt \quad \text{for all } a \geq 0.$$

We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \rightarrow \lambda_t(x)$ and $t \rightarrow \mu_t(x)$, respectively. For $0 < p < \infty$, $L^p(\mathcal{M})$ is defined as the set of all densely-defined closed operators x affiliated with \mathcal{M} such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} = \left(\int_0^\infty \mu_t(x)^p dt \right)^{\frac{1}{p}} < \infty.$$

As usual, we put $L^\infty(\mathcal{M}; \tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty (= \|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ ($1 \leq p \leq \infty$). For every $x \in L_0(\mathcal{M})$, there is a unique polar decomposition $x = u|x|$ where $|x| \in L_0(\mathcal{M})^+$ (the positive part of $L_0(\mathcal{M})$) and u is a partial isometry operator. Let $r(x) = u^*u$ and $l(x) = uu^*$. We call $r(x)$ and $l(x)$ the right and left supports of x , respectively. Note that $l(x)$ (resp., $r(x)$) is the least projection e of $\mathcal{B}(\mathcal{H})$ such that $ex = x$ (resp., $xe = x$). If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. Let $\mathcal{M}^+ = \{x \in \mathcal{M} : x \geq 0\}$ (i.e., the

positive part of \mathcal{M}). We write $S(\mathcal{M})^+ = \{x \in \mathcal{M}^+ : \tau(s(x)) < \infty\}$. Let $S(\mathcal{M})$ be the linear span of $S(\mathcal{M})^+$. It is well known that $(S(\mathcal{M}), \|\cdot\|_p)$ is dense on $L^p(\mathcal{M})$. For further results about noncommutative L_p spaces, the reader is referred to [7, 14].

Given $x, y \in L_0(\mathcal{M})$ and $0 < p < \infty$, from Theorem 4.2 of [7], we have

$$\int_0^t \mu_s(xy)^p ds \leq \int_0^t \mu_s(y)^p \mu_s(x)^p ds, \quad t > 0.$$

Let $0 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. If $x \in L^q(\mathcal{M})$ and $y \in L^r(\mathcal{M})$, then the usual Hölder inequality implies that

$$\left(\int_0^\infty \mu_s(xy)^p ds \right)^{\frac{1}{p}} \leq \left(\int_0^\infty (\mu_s(y) \mu_s(x))^p ds \right)^{\frac{1}{p}} \leq \left(\int_0^\infty \mu_s(x)^q ds \right)^{\frac{1}{q}} \left(\int_0^\infty \mu_s(y)^r ds \right)^{\frac{1}{r}}.$$

That is

$$(2.1) \quad \|xy\|_p \leq \|x\|_q \|y\|_r.$$

3. Main result. We start this section by two simple lemmas.

LEMMA 3.1. Let $x_1, x_2, \dots, x_n \in L_0(\mathcal{M})^+$ and $1 \leq p < \infty$.

(1) If f is any nonnegative convex function on $[0, \infty)$ with $f(0) = 0$, then

$$(3.1) \quad \left\| \sum_{i=1}^n f(x_i) \right\|_p \leq \left\| f \left(\sum_{i=1}^n x_i \right) \right\|_p.$$

(2) If f is any nonnegative concave function on $[0, \infty)$, then

$$(3.2) \quad \left\| f \left(\sum_{i=1}^n x_i \right) \right\|_p \leq \left\| \sum_{i=1}^n f(x_i) \right\|_p.$$

Proof. (1) If $f(\sum_{i=1}^n x_i) \notin L^p(\mathcal{M})$, then $\|f(\sum_{i=1}^n x_i)\|_p = \infty$. Thus, the inequality (3.1) is clear. If $f(\sum_{i=1}^n x_i) \in L^p(\mathcal{M})$, it follows from Theorem 5.3 (i) of [6] that

$$\int_0^t \mu_s \left(\sum_{i=1}^n f(x_i) \right) ds \leq \int_0^t \mu_s \left(f \left(\sum_{i=1}^n x_i \right) \right) ds, \quad t > 0.$$

Then Theorem 2.1 of [4] tells us that

$$\int_0^t \mu_s \left(\sum_{i=1}^n f(x_i) \right)^p ds \leq \int_0^t \mu_s \left(f \left(\sum_{i=1}^n x_i \right) \right)^p ds, \quad t > 0.$$

Hence, $\|\sum_{i=1}^n f(x_i)\|_p \leq \|f(\sum_{i=1}^n x_i)\|_p$.

(2) The proof can be done similarly to (1) by using Theorem 5.3(ii) of [6]. The details are omitted. \square

The matrix version of Lemma 3.1 appears in [11].

Let $x, y, z \in L_0(\mathcal{M})$. The block matrix $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is positive partial transpose (i.e., PPT) if $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$ and $\begin{pmatrix} x & z^* \\ z & y \end{pmatrix} \geq 0$.

LEMMA 3.2. Let $x, y \in S(\mathcal{M})^+$ and $z \in S(\mathcal{M})$. If $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT and $1 \leq p < \infty$, then $\|z^* z\|_p \leq \|xy\|_p$.

Proof. From the fact $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT, we deduce $z = x^{\frac{1}{2}} k_1 y^{\frac{1}{2}}$ and $z^* = x^{\frac{1}{2}} k_2 y^{\frac{1}{2}}$, where k_1, k_2 are contraction operators. By Lemma 2 in [3] and Lemma 2.5 in [7], we have

$$\begin{aligned} \int_0^t \mu_s(z^* z)^p ds &= \int_0^t \mu_s \left(x^{\frac{1}{2}} k_2 y^{\frac{1}{2}} x^{\frac{1}{2}} k_1 y^{\frac{1}{2}} \right)^p ds \\ &= \int_0^t \mu_s \left(y^{\frac{1}{2}} x^{\frac{1}{2}} k_2 y^{\frac{1}{2}} x^{\frac{1}{2}} k_1 \right)^p ds \\ &\leq \int_0^t \mu_s \left(y^{\frac{1}{2}} x^{\frac{1}{2}} k_2 y^{\frac{1}{2}} x^{\frac{1}{2}} \right)^p ds \\ &\leq \int_0^t \mu_s \left(\left(y^{\frac{1}{2}} x^{\frac{1}{2}} \right)^2 \right)^p ds, \quad t > 0. \end{aligned}$$

It follows from Theorem 2 of [10] that

$$\begin{aligned} \int_0^t \mu_s(z^* z)^p ds &\leq \int_0^t \mu_s \left(\left(y^{\frac{1}{2}} x^{\frac{1}{2}} \right)^2 \right)^p ds \\ &\leq \int_0^t \mu_s(yx)^p ds \\ &= \int_0^t \mu_s(xy)^p ds, \quad t > 0. \end{aligned}$$

This completes the proof. \square

The matrix version of Lemma 3.2 appears in [13].

Now, using Lemma 3.1 and Lemma 3.2, we get our first main result of this note.

THEOREM 3.3. *Let $1 \leq p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. For $i = 1, 2, \dots, n$, let $x_i \in L^q(\mathcal{M})^+$, $y_i \in L^r(\mathcal{M})^+$ such that $x_i y_i = y_i x_i$. Then*

$$\left\| \sum_{i=1}^n x_i y_i \right\|_p \leq \left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \leq \left\| \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) \right\|_p.$$

Proof. By inequality (2.1), we obtain

$$\sum_{i=1}^n x_i y_i \in L^p(\mathcal{M}), \quad \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) \in L^p(\mathcal{M}).$$

Since $x_i y_i = y_i x_i$, we have $x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} = y_i^{\frac{1}{2}} x_i^{\frac{1}{2}}$. Thus, $x_i y_i \geq 0$ and $x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \geq 0$. From Lemma 3.1, we deduce

$$\left\| \sum_{i=1}^n x_i y_i \right\|_p = \left\| \sum_{i=1}^n \left(x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \leq \left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p.$$

Let $0 < s \leq \infty$. It is well known that $S(\mathcal{M})$ is dense in $L^s(\mathcal{M})$. Hence, there exist

$$\{x_{i,k}\}_{k=1}^\infty, \{y_{i,k}\}_{k=1}^\infty, \{z_{i,k}\}_{k=1}^\infty \subseteq S(\mathcal{M})$$

such that

$$\|x_i - x_{i,k}\|_q \rightarrow 0, \quad \|y_i - y_{i,k}\|_r \rightarrow 0, \quad \|z_i - z_{i,k}\|_{2p} \rightarrow 0, \quad k \rightarrow \infty.$$

Note that

$$\begin{pmatrix} x_{i,k} & x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \\ y_{i,k}^{\frac{1}{2}} x_{i,k}^{\frac{1}{2}} & y_{i,k} \end{pmatrix} = \begin{pmatrix} x_{i,k}^{\frac{1}{2}} & y_{i,k}^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} x_{i,k}^{\frac{1}{2}} & y_{i,k}^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \geq 0.$$

It is easy to see that $\begin{pmatrix} x_{i,k} & x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \\ y_{i,k}^{\frac{1}{2}} x_{i,k}^{\frac{1}{2}} & y_{i,k} \end{pmatrix}$ is PPT. Hence,

$$\begin{pmatrix} \sum_{i=1}^n x_{i,k} & \sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \\ \sum_{i=1}^n y_{i,k}^{\frac{1}{2}} x_{i,k}^{\frac{1}{2}} & \sum_{i=1}^n y_{i,k} \end{pmatrix}$$

is PPT. It follows from Lemma 3.2 that

$$(3.3) \quad \left\| \left(\sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right)^2 \right\|_p \leq \left\| \left(\sum_{i=1}^n x_{i,k} \right) \left(\sum_{i=1}^n y_{i,k} \right) \right\|_p.$$

On the other hand, inequality (2.1) implies that

$$\begin{aligned} \left\| x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} &\leq \left\| x_i^{\frac{1}{2}} \left(y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right) \right\|_{2p} + \left\| \left(x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right) y_i^{\frac{1}{2}} \right\|_{2p} \\ &\leq \left\| x_i^{\frac{1}{2}} \right\|_{2q} \left\| y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right\|_{2r} + \left\| x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right\|_{2q} \left\| y_i^{\frac{1}{2}} \right\|_{2r} \\ &= \left\| x_i \right\|_q^{\frac{1}{2}} \left\| y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right\|_{2r} + \left\| x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right\|_{2q} \left\| y_i \right\|_r^{\frac{1}{2}}. \end{aligned}$$

Put $g(t) = t^{\frac{1}{2}}$, then g is nonnegative and operator monotone. According to Theorem 1.1 in [5], we obtain

$$\int_0^t \mu_s \left(x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right) ds \leq \int_0^t \mu_s \left(|x_i - x_{i,k}|^{\frac{1}{2}} \right) ds, \quad t > 0.$$

From the fact $2q > 1$ and Theorem 2.1 of [4] and Lemma 2.5(iv) of [7], we get

$$\int_0^t \mu_s \left(x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right)^{2q} ds \leq \int_0^t \mu_s (x_i - x_{i,k})^q ds, \quad t > 0.$$

This implies that

$$\left\| x_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} \right\|_{2q} \leq \|x_i - x_{i,k}\|_q^{\frac{1}{2}} \rightarrow 0, \quad k \rightarrow \infty.$$

Similarly,

$$\left\| y_i^{\frac{1}{2}} - y_{i,k}^{\frac{1}{2}} \right\|_{2r} \leq \|y_i - y_{i,k}\|_r^{\frac{1}{2}} \rightarrow 0, \quad k \rightarrow \infty.$$

Thus, $\left\| x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} \rightarrow 0$ as $k \rightarrow \infty$, and so

$$\left\| \sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - \sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} \rightarrow 0, \quad k \rightarrow \infty.$$

Therefore, $\left\| \sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right\|_{2p} \rightarrow \left\| \sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right\|_{2p}$, $k \rightarrow \infty$. Hence,

$$(3.4) \quad \left\| \left(\sum_{i=1}^n x_{i,k}^{\frac{1}{2}} y_{i,k}^{\frac{1}{2}} \right)^2 \right\|_p \rightarrow \left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p, \quad k \rightarrow \infty.$$

By an argument similar to the one presented above, we obtain

$$\|x_i y_j - x_{i,k} y_{j,k}\|_p \rightarrow 0, \quad k \rightarrow \infty.$$

It follows that

$$\left\| \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) - \left(\sum_{i=1}^n x_{i,k} \right) \left(\sum_{i=1}^n y_{i,k} \right) \right\|_p \rightarrow 0, \quad k \rightarrow \infty,$$

which tells us that

$$(3.5) \quad \left\| \left(\sum_{i=1}^n x_{i,k} \right) \left(\sum_{i=1}^n y_{i,k} \right) \right\|_p \rightarrow \left\| \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) \right\|_p, \quad k \rightarrow \infty.$$

Combining (3.3) and (3.4) with (3.5), we have

$$\left\| \left(\sum_{i=1}^n x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \right)^2 \right\|_p \leq \left\| \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right) \right\|_p. \quad \square$$

Theorem 3.3 includes a special case as follows.

COROLLARY 3.4. *Let $1 \leq p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $s, t > 0$. For $t, s > 0$, let $x \in L^{tq}(\mathcal{M})^+$, $y \in L^{sr}(\mathcal{M})^+$. Then*

$$\|x^{t+s} + y^{t+s}\|_p \leq \|(x^t + y^t)(x^s + y^s)\|_p.$$

Proof. If we replace n, x_1, y_1, x_2, y_2 by $2, x^t, x^s, y^t, y^s$, respectively, in Theorem 3.3, we deduce that

$$\|x^{t+s} + y^{t+s}\|_p \leq \|(x^t + y^t)(x^s + y^s)\|_p. \quad \square$$

LEMMA 3.5. *Let $x, y \in S(\mathcal{M})^+$ and $0 \leq q \leq 1$. If $1 \leq p < \infty$, then*

$$\|x^q y^{1-q}\|_p \leq \|qx + (1-q)y\|_p.$$

Proof. The result follows immediately from Theorem 3.3 of [8]. \square

Now, using Lemma 3.5, we get our another main result of this note.

THEOREM 3.6. *Let $1 \leq p < \infty$ and $0 \leq q \leq 1$. For all $x, y \in L^{2p}(\mathcal{M})$, we have*

$$\|xy^*\|_p^2 \leq \|qx^*x + (1-q)y^*y\|_p \|(1-q)x^*x + qy^*y\|_p.$$

Proof. Since $S(\mathcal{M})$ is dense in $L^s(\mathcal{M})$ ($0 < s \leq \infty$), then there exist

$$\{x_i\}_{k=1}^\infty, \{y_i\}_{k=1}^\infty \subseteq S(\mathcal{M})$$

such that

$$\|x_i - x\|_{2p} \rightarrow 0, \quad \|y_i - y\|_{2p} \rightarrow 0, \quad k \rightarrow \infty.$$

By inequality (2.1), Lemma 2 in [3] and Lemma 2.5 in [7], we have

$$\begin{aligned}\|x_i y_i^*\|_p^2 &= \left(\int_0^\infty \mu_t(y_i x_i^* x_i y_i^*)^{\frac{p}{2}} dt \right)^{\frac{2}{p}} = \left(\int_0^\infty \mu_t(x_i^* x_i y_i^* y_i)^{\frac{p}{2}} dt \right)^{\frac{2}{p}} \\ &= \left(\int_0^\infty \mu_t((y_i^* y_i)^q (x_i^* x_i)^{1-q} (x_i^* x_i)^q (y_i^* y_i)^{1-q})^{\frac{p}{2}} dt \right)^{\frac{2}{p}} \\ &= \|(y_i^* y_i)^q (x_i^* x_i)^{1-q} (x_i^* x_i)^q (y_i^* y_i)^{1-q}\|_{\frac{p}{2}} \\ &\leq \|(y_i^* y_i)^q (x_i^* x_i)^{1-q}\|_p \|(x_i^* x_i)^q (y_i^* y_i)^{1-q}\|_p\end{aligned}$$

for all $0 \leq q \leq 1$. It follows from Lemma 3.5 that

$$\|(y_i^* y_i)^q (x_i^* x_i)^{1-q}\|_p \leq \|q(y_i^* y_i) + (1-q)(x_i^* x_i)\|_p$$

and

$$\|(x_i^* x_i)^q (y_i^* y_i)^{1-q}\|_p \leq \|q(x_i^* x_i) + (1-q)(y_i^* y_i)\|_p.$$

Hence,

$$\|x_i y_i^*\|_p^2 \leq \|q(y_i^* y_i) + (1-q)(x_i^* x_i)\|_p \|q(x_i^* x_i) + (1-q)(y_i^* y_i)\|_p.$$

A similar argument to the proof of Theorem 3.3 shows that

$$\|xy^*\|_p^2 \leq \|q(y^* y) + (1-q)(x^* x)\|_p \|q(x^* x) + (1-q)(y^* y)\|_p. \quad \square$$

Acknowledgment. The authors would like to thank the editor and anonymous referees for their helpful comments and suggestions on the quality improvement of the manuscript.

REFERENCES

- [1] K. Audenaert. A norm inequality for pairs of commuting positive semidefinite matrices. *Electronic Journal of Linear Algebra*, 30:80–84, 2015.
- [2] K. Audenaert. Interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities. *Operators and Matrices*, 9:475–479, 2015.
- [3] A. Bikhentayev. Majorization for products of measurable operators. *International Journal of Theoretical Physics*, 37:571–576, 1998.
- [4] K.M. Chong. Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications. *Canadian Journal of Mathematics*, 26:1321–1340, 1974.
- [5] P.G. Dodds and K.T. Dodds. On a submajorization inequality of T. Ando. *Operator Theory: Advances and Applications*, Operator Theory in Function Spaces and Banach Lattices, Birkhäuser Basel, 75:113–131, 1995.
- [6] P.G. Dodds and F.A. Sukochev. Submajorisation inequalities for convex and concave functions of sums of measurable operators. *Positivity*, 13:107–124, 2009.
- [7] T. Fack and H. Kosaki. Generalized s -numbers of τ -measurable operators. *Pacific Journal of Mathematics*, 123:269–300, 1986.
- [8] D.R. Farenick and S.M. Manjegani. Young’s inequality in operator algebras. *Journal of the Ramanujan Mathematical Society*, 20:107–124, 2005.
- [9] S. Hayajneh and F. Kittaneh. Trace inequalities and a question of Bourin. *Bulletin of the Australian Mathematical Society*, 88:384–389, 2013.
- [10] H. Kosaki. An inequality of Araki-Lieb-Thirring (von Neumann algebra case). *Proceedings of the American Mathematical Society*, 114:477–481, 1992.
- [11] T. Kosem. Inequalities between $\|f(A + B)\|$ and $\|f(A) + f(B)\|$. *Linear Algebra and its Applications*, 418:153–160, 2006.
- [12] M. Lin. Remarks on two recent results of Audenaert. *Linear Algebra and its Applications*, 489:24–29, 2016.
- [13] M. Lin. Inequalities related to 2×2 block PPT matrices. *Operators and Matrices*, 9:917–924, 2015.
- [14] G. Pisier and Q. Xu. Noncommutative L^p -spaces. In: *Handbook of the Geometry of Banach Spaces*, Vol. 2, North-Holland, Amsterdam, 1459–1517, 2003.