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REVERSE JENSEN-MERCER TYPE OPERATOR INEQUALITIES*

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Abstract. Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$ with spectrum in an interval $[a,b]$ and $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a unital positive linear map, where $\mathcal{K}$ is also a Hilbert space. Let $m, M \in J$ with $m < M$ such that either $m + M \le a + b$ and $A \le m$, or $m + M \ge a + b$ and $A \ge M$. If f is convex on J, then the inequality

$$f(m1_{\mathcal{K}} + M1_{\mathcal{K}} - \phi(A)) \ge f(m)1_{\mathcal{K}} + f(M)1_{\mathcal{K}} - \phi(f(A)),$$

is proved. A variant of this inequality is established for superquadratic functions. The results obtained are used to prove some comparison inequalities between operators of power and quasi-arithmetic mean's type.

Key words. Jensen-Mercer operator inequality, Convex function, Superquadratic function, Operator power mean, Operator quasi-arithmetic mean.

AMS subject classifications. 47A63, 47A64, 15A60.

1. Introduction.

Throughout the paper, $B(\mathcal{H})$ and $B(\mathcal{K})$ denote C^*-algebras of all bounded linear operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and 1 denotes the identity operator. For an interval J, we denote by $\mathcal{C}(J)$ the set of all real valued continuous functions on J and by $\sigma(J)$ the set of all selfadjoint operators on $\mathcal{H}$ with spectra in J and by $sp(A)$ the spectrum of an operator A on a Hilbert space. The Jensen-Mercer inequality

$$f\left(a + b - \sum_{i=1}^{n} w_i x_i\right) \le f(a) + f(b) - \sum_{i=1}^{n} w_i f(x_i) \tag{1.1}$$

for convex function $f : [a,b] \to \mathbb{R}$, real numbers $x_1, \dots, x_n \in [a,b]$ and real numbers $w_i \ge 0$ with $\sum\limits_{i=1}^{n} w_i = 1$, was proved in [7]. The following operator variant of (1.1), which is called the Jensen-Mercer operator inequality, was proved in [6]:

$$f\left(m1_{\mathcal{K}} + M1_{\mathcal{K}} - \sum_{j=1}^{n} \Phi_j(A_j)\right) \le f(m)1_{\mathcal{K}} + f(M)1_{\mathcal{K}} - \sum_{j=1}^{n} \Phi_j(f(A_j)), \tag{1.2}$$

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where, $f \in C([m, M])$ is convex on $[m, M]$, $\Phi_1, \ldots, \Phi_n$ are positive linear maps from $B(\mathcal{H})$ into $B(\mathcal{K})$ with $\sum_{j=1}^{n} \Phi_j(1_{\mathcal{H}}) = 1_{\mathcal{K}}$ and $A_1, \ldots, A_n \in B(\mathcal{H})$ are selfadjoint operators with the spectra in $[m, M]$. An other version of the Jensen-Mercer operator inequality on finite-dimensional Hilbert spaces can be found in [5].

A variant of (1.2) for superquadratic functions, which is a refinement of the Jensen-Mercer operator inequality for convex functions, reads as follows [3]:

$$\begin{aligned}
& f((m+M)1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(A_j)) + \tfrac{1}{M-m}(\sum_{j=1}^{k} \phi_j(A_j) - m1_{\mathcal{K}}) f(M1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(A_j)) \\
& \quad + \tfrac{1}{M-m}(M1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(A_j)) f(\sum_{j=1}^{k} \phi_j(A_j) - m1_{\mathcal{K}}) \\
& \leq (f(M) + f(m))1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(f(A_j)) - \tfrac{1}{M-m} \sum_{j=1}^{k} \phi_j((A_j - m1_{\mathcal{H}}) f(M1_{\mathcal{H}} - A_j)) \\
& \quad - \tfrac{1}{M-m} \sum_{j=1}^{k} \phi_j((M1_{\mathcal{H}} - A_j) f(A_j - m1_{\mathcal{H}})),
\end{aligned}$$

where, $0 < m < M$ and $f \in \mathcal{C}([0, \infty))$ is superquadratic. A function $f : [0, \infty) \to \mathbb{R}$ is said to be superquadratic if for every $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) - f(x) - f(|y - x|) \geq C_x(y - x)$$

for all $y \geq 0$. Some basic properties and examples of superquadratic functions can be found in [1, 2].

In this paper, we assume that the spectra of selfadjoint operators $A_1, \ldots, A_k$ do not intersect the interval (m, M) and obtain reverse Jensen-Mercer type operator inequalities. In Section 2, we prove this type of inequalities for convex functions and superquadratic functions. In Section 3, we use the reverse Jensen-Mercer type operator inequality for convex functions to derive some comparison inequalities between operators similar to operator power and quasi-arithmetic means of Mercer's type.

2. Main results.

We start this section with our main result in which a reverse Jensen-Mercer type operator inequality for convex functions is proved.

THEOREM 2.1. *Let $A_1, \ldots, A_k \in B(\mathcal{H})$ be selfadjoint operators with spectra in $J = [a, b]$ and $\phi_1, \ldots, \phi_k$ be positive linear maps from $B(\mathcal{H})$ into $B(\mathcal{K})$ such that $\sum_{j=1}^{k} \phi_j(1_{\mathcal{H}}) = 1_{\mathcal{K}}$. Suppose that $f \in \mathcal{C}(J)$ is convex and $m, M \in J$ with $m < M$. Then*

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we have

$$f\left(m1_{\mathcal{K}}+M1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right)\geq (f(m)+f(M))1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(f(A_j)), \tag{2.1}$$

if one of the following statements is satisfied:

(1) $m+M=a+b$ and $A_j\in\sigma(J\setminus(m,M))$ for every $1\leq j\leq k$ and $\sum\limits_{j=1}^{k}\phi_j(A_j)\in\sigma(J\setminus(m,M))$;
(2) $m+M\leq a+b$ and for every $1\leq j\leq k$, $A_j\leq m$;
(3) $m+M\geq a+b$ and for every $1\leq j\leq k$, $A_j\geq M$.

Proof. If $t\in J\setminus(m,M)$, then $t\leq m$ or $t\geq M$. Suppose that $t\leq m$. Hence, $M-t\geq M-m$, so $0<\frac{M-m}{M-t}\leq 1$. Now since f is convex on J, we have

$$f(m)=f\left(\frac{M-m}{M-t}t+\frac{m-t}{M-t}M\right)\leq\frac{M-m}{M-t}f(t)+\frac{m-t}{M-t}f(M).$$

Hence, we obtain

$$f(t)\geq\frac{t-m}{M-m}f(M)+\frac{M-t}{M-m}f(m). \tag{2.2}$$

Similarly, it is proved that the inequality (2.2) holds true for every $t\geq M$.

Now suppose that the statement (1) is satisfied. Since $m+M=a+b$, if $t\in J\setminus(m,M)$, then $M+m-t\in J\setminus(m,M)$. Now it follows from (2.2) that

$$f(M+m-t)\geq\frac{M-t}{M-m}f(M)+\frac{t-m}{M-m}f(m)\qquad(t\in J\setminus(m,M)). \tag{2.3}$$

Since $\sum\limits_{j=1}^{k}\phi_j(A_j)\in\sigma(J\setminus(m,M))$, then by using functional calculus, it follows from (2.3) that

$$\begin{aligned}f\left((M+m)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right)&\geq\frac{M1_{\mathcal{K}}-\sum\limits_{j=1}^{k}\phi_j(A_j)}{M-m}f(M)\\&\quad+\frac{\sum\limits_{j=1}^{k}\phi_j(A_j)-m1_{\mathcal{K}}}{M-m}f(m).\end{aligned} \tag{2.4}$$

On the other hand, since for every $1\leq j\leq k$, $A_j\in\sigma(J\setminus(m,M))$, then by using functional calculus, it follows from (2.2) that

$$f(A_j)\geq\frac{A_j-m1_{\mathcal{H}}}{M-m}f(M)+\frac{M1_{\mathcal{H}}-A_j}{M-m}f(m)\qquad(1\leq j\leq k). \tag{2.5}$$

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Since for every $1 \leq j \leq k$, ϕ_j is a positive operator, it follows from (2.5) that

$$\phi_j(f(A_j)) \geq \frac{\phi_j(A_j) - m\phi_j(1_{\mathcal{H}})}{M-m} f(M) + \frac{M\phi_j(1_{\mathcal{H}}) - \phi_j(A_j)}{M-m} f(m),$$

and hence, we have

$$\sum_{j=1}^{k} \phi_j(f(A_j)) \geq \frac{\sum_{j=1}^{k} \phi_j(A_j) - m1_{\mathcal{K}}}{M-m} f(M) + \frac{M1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(A_j)}{M-m} f(m).$$

It follows that

$$\begin{aligned} (f(M)+f(m))1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(f(A_j)) &\leq \frac{M1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(A_j)}{M-m} f(M) \\ &+ \frac{\sum_{j=1}^{k} \phi_j(A_j) - m1_{\mathcal{K}}}{M-m} f(m). \end{aligned} \tag{2.6}$$

Now using inequalities (2.4) and (2.6), we obtain inequality (2.1).

If one of the statements (2) and (3) is satisfied, then the inequality (2.1) is proved in the same way. □

REMARK 2.2. In fact, the proof of Theorem 2.1 shows that if one of the statements (1),(2) and (3) is satisfied, then the following series of inequalities holds

$$\begin{aligned} f\left((M+m)1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(A_j)\right) &\geq \frac{M1_{\mathcal{K}} - \sum\limits_{j=1}^{k} \phi_j(A_j)}{M-m} f(M) \\ &+ \frac{\sum\limits_{j=1}^{k} \phi_j(A_j) - m1_{\mathcal{K}}}{M-m} f(m) \\ &\geq (f(M)+f(m))1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(f(A_j)). \end{aligned} \tag{2.7}$$

EXAMPLE 2.3. The function $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by $\phi(A) = DAD$, where $D = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, is a unital positive linear map. The function $f(t) = t^2$ is convex on $\mathbb{R}$ and for hermitian matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ we have $0 \leq A \leq 2I$. Now for an interval $[m, M]$, a simple calculation yields

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$$f(mI+MI-\phi(A))-[f(m)I+f(M)I-\phi(f(A))]$$

$$=\begin{pmatrix} -6m-6M+2mM+12 & 6m+6M-12 \\ -2m-2M+4 & 2mM+2m+2M-4 \end{pmatrix}. \tag{2.8}$$

Now for example, if $[m,M]=[2,3]$, then the matrix (2.8) is equal to the positive matrix $\begin{pmatrix} -6 & 18 \\ -6 & 18 \end{pmatrix}$ and if $[m,M]=[3,5]$, then it is equal to the strictly positive matrix $\begin{pmatrix} -6 & 36 \\ -12 & 42 \end{pmatrix}$.

Now, a variant of inequality (2.1) is proved for superquadratic functions.

THEOREM 2.4. *Let $f\in\mathcal{C}([0,\infty))$ be a superquadratic function and $m,M\in(0,\infty)$ with $m<M$. Suppose $A_1,\dots,A_k\in B(\mathcal{H})$ are positive operators with spectra in $[0,m]$ and $\phi_1,\dots,\phi_k$ are positive linear maps with $\sum_{j=1}^{k}\phi_j(1_{\mathcal{H}})=1_{\mathcal{K}}$. Then we have*

$$\begin{aligned} f\left(m1_{\mathcal{K}}+M1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right)-2\frac{f(M-m)}{M-m}\left(m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right)& \\ -f\left(m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right)& \\ \geq f(m)1_{\mathcal{K}}+f(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(f(A_j))+\sum_{j=1}^{k}\phi_j(f(m1_{\mathcal{H}}-A_j)).& \end{aligned}$$

Proof. Since f is a superquadratic function, by definition we have

$$f(\lambda a+(1-\lambda)b)\leq\lambda f(a)+(1-\lambda)f(b)-\lambda f((1-\lambda)|b-a|)-(1-\lambda)f(\lambda|b-a|)$$

for all $a,b\geq 0$ and all $\lambda\in[0,1]$.

Let $0\leq t\leq m$. Then $0<\frac{M-m}{M-t}\leq 1$ and so

$$\begin{aligned} f(m)&=f\left(\frac{M-m}{M-t}t+\frac{m-t}{M-t}M\right)\\ &\leq\frac{M-m}{M-t}f(t)+\frac{m-t}{M-t}f(M)-\frac{M-m}{M-t}f(m-t)-\frac{m-t}{M-t}f(M-m). \end{aligned}$$

Hence, we have

$$f(t)\geq\frac{M-t}{M-m}f(m)-\frac{m-t}{M-m}f(M)+f(m-t)+\frac{m-t}{M-m}f(M-m), \tag{2.9}$$

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for all $0 \le t \le m$.

Also, for every $t \ge M$, since $0 < \frac{M-m}{t-m} \le 1$, we have

$$\begin{aligned} f(M) &= f\left(\frac{t-M}{t-m}m + \frac{M-m}{t-m}t\right) \\ &\le \frac{t-M}{t-m}f(m) + \frac{M-m}{t-m}f(t) - \frac{t-M}{t-m}f(M-m) - \frac{M-m}{t-m}f(t-M), \end{aligned}$$

and hence, it follows that

$$f(t) \ge \frac{t-m}{M-m}f(M) - \frac{t-M}{M-m}f(m) + f(t-M) + \frac{t-M}{M-m}f(M-m), \tag{2.10}$$

for every $t \ge M$. Now for every $0 \le t \le m$, we have $m + M - t \ge M$, and hence, it follows from (2.10) that

$$f(m+M-t) \ge \frac{M-t}{M-m}f(M) - \frac{m-t}{M-m}f(m) + f(m-t) + \frac{m-t}{M-m}f(M-m). \tag{2.11}$$

Now since $0 \le \sum_{j=1}^k \phi_j(A_j) \le m1_{\mathcal{K}}$, by using functional calculus, it follows from (2.11) that

$$\begin{aligned} f\left((m+M)1_{\mathcal{K}} - \sum_{j=1}^k \phi_j(A_j)\right) \ge{} & \frac{M1_{\mathcal{K}} - \sum\limits_{j=1}^k \phi_j(A_j)}{M-m} f(M) \\ & - \frac{m1_{\mathcal{K}} - \sum\limits_{j=1}^k \phi_j(A_j)}{M-m} f(m) \\ & + \frac{m1_{\mathcal{K}} - \sum\limits_{j=1}^k \phi_j(A_j)}{M-m} f(M-m) \\ & + f\left(m1_{\mathcal{K}} - \sum_{j=1}^k \phi_j(A_j)\right). \end{aligned} \tag{2.12}$$

Also, since $0 \le A_j \le m1_{\mathcal{H}}$ for every $1 \le j \le k$, by using functional calculus, it follows from (2.9) that

$$\begin{aligned} f(A_j) \ge{} & \frac{M1_{\mathcal{H}} - A_j}{M-m}f(m) - \frac{m1_{\mathcal{H}} - A_j}{M-m}f(M) \\ & + f(m1_{\mathcal{H}} - A_j) + \frac{m1_{\mathcal{H}} - A_j}{M-m}f(M-m). \end{aligned}$$

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Applying positive operators ϕ_j and summing, we obtain

$$\sum_{j=1}^{k}\phi_j(f(A_j)) \geq \frac{M1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)}{M-m}f(m)-\frac{m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)}{M-m}f(M) +\sum_{j=1}^{k}\phi_j(f(m1_{\mathcal{H}}-A_j))+\frac{m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)}{M-m}f(M-m).$$

Hence, we have

$$\begin{aligned} f(m)1_{\mathcal{K}}+f(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(f(A_j)) \leq{}& \frac{\sum_{j=1}^{k}\phi_j(A_j)-m1_{\mathcal{K}}}{M-m}f(m) \\ &+\frac{M1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)}{M-m}f(M) \\ &-\sum_{j=1}^{k}\phi_j(f(m1_{\mathcal{H}}-A_j)) \\ &-\frac{m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)}{M-m}f(M-m). \end{aligned} \tag{2.13}$$

Using inequalities (2.12) and (2.13), we obtain

$$\begin{aligned} f\left(m1_{\mathcal{K}}+M1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right)-\frac{f(M-m)}{M-m}&\left(m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right) \\ &-f\left(m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right) \\ &\geq \frac{f(m)}{M-m}\left(\sum_{j=1}^{k}\phi_j(A_j)-m1_{\mathcal{K}}\right) \\ &\quad+\frac{f(M)}{M-m}\left(M1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right) \\ &\geq f(m)1_{\mathcal{K}}+f(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(f(A_j)) \\ &\quad+\sum_{j=1}^{k}\phi_j(f(m1_{\mathcal{H}}-A_j)) \\ &\quad+\frac{f(M-m)}{M-m}\left(m1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(A_j)\right). \quad \square \end{aligned}$$

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3. Applications. In [6], the Jensen-Mercer operator inequality (1.2) is applied to derive comparison inequalities between operator power and quasi-arithmetic means of Mercer's type. For $\mathbf{A} = (A_1, \dots, A_k)$, where $A_j \in B(\mathcal{H})$ are positive operators with the spectra in $[m, M]$ and $\mathbf{\Phi} = (\phi_1, \dots, \phi_k)$, where ϕ_j are positive linear maps with $\sum_{j=1}^k \phi_j(1_\mathcal{H}) = 1_\mathcal{K}$ and a strictly monotonic function φ, the operator power mean of Mercer's type

$$\text{(3.1)}\quad \widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) = \begin{cases} \left(m^r 1_\mathcal{K} + M^r 1_\mathcal{K} - \sum_{j=1}^k \phi_j(A_j^r)\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left((\ln m)1_\mathcal{K} + (\ln M)1_\mathcal{K} - \sum_{j=1}^k \phi_j(\ln(A_j))\right), & r = 0, \end{cases}$$

and the operator quasi-arithmetic mean of Mercer's type

$$\widetilde{M}_\varphi(\mathbf{A}, \mathbf{\Phi}) = \varphi^{-1}\left(\varphi(m)1_\mathcal{K} + \varphi(M)1_\mathcal{K} - \sum_{j=1}^k \phi_j(\varphi(A_j))\right), \tag{3.2}$$

are defined in [6]. Here, we assume that the spectra of selfadjoint operators A_j do not intersect the interval (m, M) and prove some comparison inequalities between operators which are defined as (3.1) and (3.2). To do this we need some results about the function order of positive operators based on Kantorovich type inequalities. For power functions, we have the following theorem (see [4]).

Theorem 3.1. *Let A and B be positive operators on $\mathcal{H}$ such that $A \geq B$.*

i) If $p \in [0, 1]$, then $A^p \geq B^p$ (Löwner-Heinz inequality).
ii) If $p > 1$ and $B \in \sigma([m, M])$, then

$$K(m, M, p)A^p \geq B^p,$$

where a generalized Kantorovich constant $K(m, M, p)$ is defined by

$$K(m, M, p) = \frac{mM^p - Mm^p}{(1-p)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p.$$

In [8, 9], Mićić and Pečarić and Seo proved some results about the function preserving the operator order, under a general setting (see [8, Theorem 2.1] and [9, Theorem 2.1]).

Theorem 3.2. *Let A and B be positive operators on a Hilbert space $\mathcal{H}$ satisfying $sp(B) \subseteq [m, M]$ for some scalars $0 < m < M$. Let $f \in \mathcal{C}([m, M])$ be a convex function and $g \in \mathcal{C}(J)$, where J is an interval containing $[m, M] \cup sp(A)$. Suppose that either of the following conditions holds: (a) $A \geq B > 0$ and g is increasing convex on J, or (b) $B \geq A > 0$ and g is decreasing convex on J. Then for a given $\alpha > 0$,*

$$\alpha g(A) + \beta 1_\mathcal{H} \geq f(B)$$

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holds for

$$\beta = \max_{m \le t \le M} \left\{ \frac{f(M)-f(m)}{M-m} t + \frac{Mf(m)-mf(M)}{M-m} - \alpha g(t) \right\}.$$

We apply Theorem 3.2 to obtain the following result.

Corollary 3.3. *Let A and B be positive operators on a Hilbert space $\mathcal{H}$ such that $B \in \sigma([m, M])$, where $0 < m < M$. Suppose $f > 0$ is a continuous convex function on an interval J, where $[m, M] \cup sp(A) \subseteq J$. Suppose that either of the following conditions holds: (a) $A \ge B > 0$ and f is increasing, or (b) $B \ge A > 0$ and f is decreasing. Then*

$$C(m, M, f) f(A) \ge f(B),$$

where

$$C(m, M, f) := \max_{m \le t \le M} \left\{ \frac{\frac{f(M)-f(m)}{M-m} t + \frac{Mf(m)-mf(M)}{M-m}}{f(t)} \right\}.$$

Proof. Let $\alpha = C(m, M, f)$ and $g = f$. Then by Theorem 3.2, we obtain

$$C(m, M, f) f(A) + \beta 1_{\mathcal{H}} \ge f(B),$$

where

$$\beta = \max_{m \le t \le M} \left\{ \frac{f(M)-f(m)}{M-m} t + \frac{Mf(m)-mf(M)}{M-m} - C(m, M, f) f(t) \right\}.$$

Since $f > 0$ and

$$\frac{1}{f(t)} \left(\frac{f(M)-f(m)}{M-m} t + \frac{Mf(m)-mf(M)}{M-m} \right) \le C(m, M, f) \qquad (m \le t \le M),$$

we have

$$\frac{f(M)-f(m)}{M-m} t + \frac{Mf(m)-mf(M)}{M-m} \le C(m, M, f) f(t) \qquad (m \le t \le M),$$

and hence, $\beta \le 0$. Therefore, $C(m, M, f) f(A) \ge f(B)$. □

Now, we prove a comparison theorem for quasi-arithmetic means for operators.

Theorem 3.4. *Suppose $\mathbf{A} = (A_1, \ldots, A_k)$, where $A_j \in B(\mathcal{H})$ are selfadjoint operators with spectra in $[a, b]$ and $\mathbf{\Phi} = (\phi_1, \ldots, \phi_k)$, where ϕ_j are positive linear*

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maps such that $\sum_{j=1}^{k} \phi_j(1_{\mathcal{H}}) = 1_{\mathcal{K}}$*. Let* $\varphi, \psi \in \mathcal{C}([a,b])$ *such that* φ *is strictly increasing and* ψ^{-1} *is operator increasing and* $\psi \circ \varphi^{-1}$ *is convex on* $[\varphi(a), \varphi(b)]$*. Then*

$$\widetilde{M}_{\varphi}(\mathbf{A}, \mathbf{\Phi})$$

$$\geq \psi^{-1}\left(\frac{\sum_{j=1}^{k} \phi_j(\varphi(A_j)) - \varphi(m)1_{\mathcal{K}}}{\varphi(M) - \varphi(m)}\psi(m) + \frac{\varphi(M)1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)}\psi(M)\right)$$

$$\geq \widetilde{M}_{\psi}(\mathbf{A}, \mathbf{\Phi}), \tag{3.3}$$

if one of the following statements is satisfied:

(1) $\varphi(m) + \varphi(M) \leq \varphi(a) + \varphi(b)$ *and* $a1_{\mathcal{H}} \leq A_j \leq m1_{\mathcal{H}}$*,*
(2) $\varphi(m) + \varphi(M) \geq \varphi(a) + \varphi(b)$ *and* $M1_{\mathcal{H}} \leq A_j \leq b1_{\mathcal{H}}$*.*

Proof. Suppose that the statement (2) is satisfied. Since $M1_{\mathcal{H}} \leq A_j \leq b1_{\mathcal{H}}$ and φ is strictly increasing on $[a,b]$, it follows that $\varphi(M)1_{\mathcal{H}} \leq \varphi(A_j) \leq \varphi(b)1_{\mathcal{H}}$ for every $1 \leq j \leq k$. Hence, $\varphi(M)1_{\mathcal{K}} \leq \sum_{j=1}^{k} \phi_j(\varphi(A_j)) \leq \varphi(b)1_{\mathcal{K}}$ and so

$$\begin{aligned}
\varphi(a)1_{\mathcal{K}} &\leq \varphi(m)1_{\mathcal{K}} + \varphi(M)1_{\mathcal{K}} - \varphi(b)1_{\mathcal{K}} \\
&\leq \varphi(m)1_{\mathcal{K}} + \varphi(M)1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(\varphi(A_j)) \\
&\leq \varphi(m)1_{\mathcal{K}}.
\end{aligned}$$

Now applying the inequality (2.7) to the convex function $\psi \circ \varphi^{-1}$ and replacing A_j, m and M by $\varphi(A_j), \varphi(m)$ and $\varphi(M)$, respectively, we obtain

$$\begin{aligned}
&\psi \circ \varphi^{-1}\left(\varphi(m)1_{\mathcal{K}} + \varphi(M)1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(\varphi(A_j))\right) \\
&\geq \frac{\sum_{j=1}^{k} \phi_j(\varphi(A_j)) - \varphi(m)1_{\mathcal{K}}}{\varphi(M) - \varphi(m)}\psi \circ \varphi^{-1}(\varphi(m)) \\
&\quad + \frac{\varphi(M)1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(\varphi(A_j))}{\varphi(M) - \varphi(m)}\psi \circ \varphi^{-1}(\varphi(M)) \\
&\geq \psi \circ \varphi^{-1}(\varphi(m))1_{\mathcal{K}} + \psi \circ \varphi^{-1}(\varphi(M))1_{\mathcal{K}} - \sum_{j=1}^{k} \phi_j(\psi \circ \varphi^{-1}(\varphi(A_j))).
\end{aligned}$$

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Hence, we have

$$\begin{aligned}&\psi\left(\varphi^{-1}\left(\varphi(m)1_{\mathcal{K}}+\varphi(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(\varphi(A_j))\right)\right)\\ &\geq \frac{\sum_{j=1}^{k}\phi_j(\varphi(A_j))-\varphi(m)1_{\mathcal{K}}}{\varphi(M)-\varphi(m)}\psi(m)+\frac{\varphi(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(\varphi(A_j))}{\varphi(M)-\varphi(m)}\psi(M)\\ &\geq \psi(m)1_{\mathcal{K}}+\psi(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(\psi(A_j)).\end{aligned}$$

Now since ψ^{-1} is operator increasing, we obtain the inequality (3.3).

If the statement (1) is satisfied, then the inequality (3.3) is proved in the same way. □

In the following theorem, we apply Corollary 3.3 to obtain some comparison results for the operator quasi-arithmetic means when φ is strictly increasing, ψ^{-1} is increasing convex or decreasing convex and $\psi\circ\varphi^{-1}$ is convex.

Theorem 3.5. *Suppose $\mathbf{A}=(A_1,\ldots,A_k)$, where $A_j\in B(\mathcal{H})$ are selfadjoint operators with spectra in $[a,b]$ $(0<a<b)$ and $\mathbf{\Phi}=(\phi_1,\ldots,\phi_k)$, where ϕ_j are positive linear maps such that $\sum\limits_{j=1}^{k}\phi_j(1_{\mathcal{H}})=1_{\mathcal{K}}$. Let $\varphi,\psi\in\mathcal{C}([a,b])$ such that φ is strictly increasing and $\psi\circ\varphi^{-1}$ is convex on $[\varphi(a),\varphi(b)]$. Suppose that $m,M\in[a,b]$ with $m<M$ and $\varphi(m)+\varphi(M)\leq\varphi(a)+\varphi(b)$ and $a1_{\mathcal{H}}\leq A_j\leq m1_{\mathcal{H}}$ for every $1\leq j\leq k$. Then the following statements hold:*

(1) If ψ^{-1} is increasing convex and $\psi(M)>0$ and $\psi(m)+\psi(M)\leq\psi(a)+\psi(b)$, then

$$C\left(\psi(M),\psi(m)+\psi(M)-\psi(a),\psi^{-1}\right)\widetilde{M}_{\varphi}(\mathbf{A},\mathbf{\Phi})\geq\widetilde{M}_{\psi}(\mathbf{A},\mathbf{\Phi}).$$

(2) If ψ^{-1} is decreasing convex and $\psi(b)>0$ and $\psi(a)+\psi(b)\leq\psi(m)+\psi(M)$, then

$$C\left(\psi(m),\psi(a),\psi^{-1}\right)\widetilde{M}_{\varphi}(\mathbf{A},\mathbf{\Phi})\geq\widetilde{M}_{\psi}(\mathbf{A},\mathbf{\Phi}).$$

Proof. By the same reasoning as in the proof of Theorem 3.4 we obtain

$$\psi\left(\widetilde{M}_{\varphi}(\mathbf{A},\mathbf{\Phi})\right)\geq\psi(m)1_{\mathcal{K}}+\psi(M)1_{\mathcal{K}}-\sum_{j=1}^{k}\phi_j(\psi(A_j)).$$

Let $B=\psi(m)1_{\mathcal{K}}+\psi(M)1_{\mathcal{K}}-\sum\limits_{j=1}^{k}\phi_j(\psi(A_j))$.

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We prove the statement (1). Since ψ^{-1} is an increasing function, so is ψ and hence

$$0 < \psi(M) \leq B \leq \psi(M) + \psi(m) - \psi(a).$$

It follows that

$$sp(B) \subseteq [\psi(M), \psi(M) + \psi(m) - \psi(a)] \subseteq [\psi(M), \psi(b)].$$

Now by Corollary 3.3, we obtain

$$C\left(\psi(M), \psi(m) + \psi(M) - \psi(a), \psi^{-1}\right) \widetilde{M}_\varphi(\mathbf{A}, \mathbf{\Phi}) \geq \widetilde{M}_\psi(\mathbf{A}, \mathbf{\Phi}).$$

The proof of statement (2) is similar. □

Now, the following monotonicity property of power means of Mercer's type for operators can follows directly from Theorems 3.4 and 3.5 by setting power functions.

THEOREM 3.6. *Let* $0 < m < M$ *and* $0 \leq r < s$. *Suppose* $\mathbf{A} = (A_1, \ldots, A_k)$, *where* $A_j \in B(\mathcal{H})$ *are positive invertible operators with* $A_j \leq m1_{\mathcal{H}}$ *and* $\mathbf{\Phi} = (\phi_1, \ldots, \phi_k)$, *where* ϕ_j *are positive linear maps such that* $\sum_{j=1}^{k} \phi_j(1_{\mathcal{H}}) = 1_{\mathcal{K}}$. *Let*

$$m_j = \inf_{\|x\|=1} \langle A_j x, x\rangle \qquad (1 \leq j \leq k)$$

and

$$a = \min\{m_1, \ldots, m_k\}.$$

Then the following statements hold:

(1) If $0 \leq r$ *and* $1 \leq s$, *then*

$$\widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \geq \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}). \tag{3.4}$$

(2) If $0 \leq r < s < 1$, *then*

$$C\left(M^s, m^s + M^s - a^s, t^{\frac{1}{s}}\right) \widetilde{M}_r(\mathbf{A}, \mathbf{\Phi}) \geq \widetilde{M}_s(\mathbf{A}, \mathbf{\Phi}). \tag{3.5}$$

Proof. If $r > 0$ and $1 \leq s$, then define the functions $\varphi, \psi \in \mathcal{C}((0,\infty))$ by $\varphi(t) = t^r$ and $\psi(t) = t^s$. It is clear that φ is strictly increasing and $\psi \circ \varphi^{-1}(t) = t^{\frac{s}{r}}$ is convex. Also, the function ψ^{-1} is operator increasing by Löwner-Heinz inequality. Hence, by Theorem 3.4, we obtain the inequality (3.4).

If $r = 0$ and $1 \leq s$, then by applying Theorem 3.4 for the functions $\varphi(t) = \ln t$ and $\psi(t) = t^s$, we obtain the inequality (3.4).

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If $0 < r < s < 1$, then applying Theorem 3.5 for the strictly increasing function $\varphi(t) = t^r$ and the increasing convex function $\psi^{-1}(t) = t^{\frac{1}{s}}$, we obtain desired inequality (3.5).

Finally, for $r = 0$ and $0 < s < 1$, consider the functions $\varphi(t) = \ln t$ and $\psi^{-1}(t) = t^{\frac{1}{s}}$ and apply Theorem 3.5. $\square$

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