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A NOTE ON A CONJECTURE FOR THE DISTANCE LAPLACIAN MATRIX*

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Abstract. In this note, the graphs of order n having the largest distance Laplacian eigenvalue of multiplicity $n-2$ are characterized. In particular, it is shown that if the largest eigenvalue of the distance Laplacian matrix of a connected graph G of order n has multiplicity $n-2$, then $G \cong S_n$ or $G \cong K_{p,p}$, where $n = 2p$. This resolves a conjecture proposed by M. Aouchiche and P. Hansen in [M. Aouchiche and P. Hansen. A Laplacian for the distance matrix of a graph. *Czechoslovak Mathematical Journal*, 64(3):751–761, 2014.]. Moreover, it is proved that if G has P_5 as an induced subgraph then the multiplicity of the largest eigenvalue of the distance Laplacian matrix of G is less than $n-3$.

Key words. Distance Laplacian matrix, Laplacian matrix, Largest eigenvalue, Multiplicity of eigenvalues.

AMS subject classifications. 05C12, 05C50, 15A18.

1. Introduction.

Let $G = (V, E)$ be a connected graph and the distance (the length of a shortest path) between vertices v_i and v_j of G be denoted by $d_{i,j}$. The distance matrix of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to $d_{i,j}$, $i, j = 1, 2, \ldots, n$. The transmission $\mathrm{Tr(v_i)}$ of a vertex v_i is defined as the sum of the distances from v_i to all other vertices in G. For more details about the distance matrix we suggest, for example, [5]. M. Aouchiche and P. Hansen [3] introduced the Laplacian for the distance matrix of a connected graph G as $\mathcal{D}^L(G) = \mathrm{Tr(G)} - \mathcal{D}(\mathrm{G})$, where $\mathrm{Tr(G)}$ is the diagonal matrix of vertex transmissions. We write $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L = 0)$, for the distance Laplacian spectrum of a connected graph G, the $\mathcal{D}^L$-spectrum, and assume that the eigenvalues are arranged in a nonincreasing order. The multiplicity of the eigenvalue ∂_i^L is denoted by $m(\partial_i^L)$, for $1 \le i \le n$. We often use exponents to exhibit the multiplicity of the distance Laplacian eigenvalues when we write the $\mathcal{D}^L$-spectrum. The distance Laplacian matrix has been recently

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studied ([2, 4, 6]) and, in [4], M. Aouchiche and P. Hansen proposed some conjectures about it. Among them, we consider in this work the following one:

Conjecture 1.1. [4] If G is a graph on $n \geq 3$ vertices and $G \ncong K_n$, then $m(\partial_1^L(G)) \leq n-2$ with equality if and only if G is the star S_n or $n = 2p$ for the complete bipartite graph $K_{p,p}$.

In this paper, we prove the conjecture. In order to obtain this result we analyze how the existence of P_4 as an induced subgraph influences the $\mathcal{D}^L$-spectrum of a connected graph. We conclude that, in this case, the largest distance Laplacian eigenvalue has multiplicity less than or equal to $n-3$. This fact motivated us to also investigate the influence of an induced P_5 subgraph in the $\mathcal{D}^L$-spectrum of a graph. We prove that if a graph has an induced P_5 subgraph then the largest eigenvalue of its distance Laplacian matrix has multiplicity at most $n-4$. Although we do not make a general approach by characterizing the graphs that have the largest distance Laplacian eigenvalue with multiplicity $n-3$, some considerations on this topic are made.

2. Preliminaries. In what follows, $G = (V, E)$, or just G, denotes a graph with n vertices and $\overline{G}$ denotes its complement. The diameter of a connected graph G is denoted by $diam(G)$. As usual, we write, respectively, P_n, C_n, S_n and K_n, for the path, the cycle, the star and the complete graph, all with n vertices. We denote by $K_{p,p}$ and by $K_{p,p,p}$ the balanced complete bipartite and tripartite graph, respectively. Now, we recall the definitions of some operations with graphs that will be used. For this, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint graphs:

- The *union* of G_1 and G_2 is the graph $G_1 \cup G_2$ (or $G_1 + G_2$), whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$;
- The *complete product* or *join* of graphs G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2.

A graph G is a cograph, also known as a decomposable graph, if no induced subgraph of G is isomorphic to P_4 [1]. About the cographs, we also have the following characterizations:

Theorem 2.1. [1] *Given a graph G, the foIlowing statements are equivalent:*

- *G is a cograph.*
- *The complement of any connected subgraph of G with at least two vertices is disconnected.*
- *Every connected subgraph of G has diameter less than or equal to* 2.

We denote by $(\mu_1, \mu_2, \ldots, \mu_n = 0)$ the L-spectrum of G, i.e., the spectrum of the Laplacian matrix of G, and assume that the eigenvalues are labeled such that

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$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. It is well known that the multiplicity of the Laplacian eigenvalue 0 is equal to the number of components of G and that $\mu_{n-i}(\overline{G}) = n - \mu_i(G)$, $\forall 1 \leq i \leq n-1$ (see [8] for more details).

The following results regarding the distance Laplacian matrix are already known.

THEOREM 2.2. [3] *Let G be a connected graph on n vertices with diam$(G) \leq 2$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ be the Laplacian spectrum of G. Then the distance Laplacian spectrum of G is $2n - \mu_{n-1} \geq 2n - \mu_{n-2} \geq \cdots \geq 2n - \mu_1 > \partial_n^L = 0$. Moreover, for every $i \in \{1, 2, \ldots, n-1\}$ the eigenspaces corresponding to μ_i and to $2n - \mu_i$ are the same.*

THEOREM 2.3. [3] *Let G be a connected graph on n vertices. Then $\partial_{n-1}^L = n$ if and only if $\overline{G}$ is disconnected. Moreover, the multiplicity of n as an eigenvalue of $\mathcal{D}^L$ is one less than the number of components of $\overline{G}$.*

THEOREM 2.4. [3] *If G is a connected graph on $n \geq 2$ vertices then $m(\partial_1^L) \leq n-1$ with equality if and only if G is the complete graph K_n.*

We finish this section enunciating the Cauchy interlacing theorem, that will be necessary for what follows::

THEOREM 2.5. [7] *Let A be a real symmetric matrix of order n with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and let M be a principal submatrix of M with order $m \leq n$ and eigenvalues $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_m(M)$. Then $\lambda_i(A) \geq \lambda_i(M) \geq \lambda_{i+n-m}(A)$, for all $1 \leq i \leq m$.*

3. Proof of the conjecture. The next lemmas will be useful to prove the main results of this section:

LEMMA 3.1. *If G is a connected graph on $n \geq 2$ vertices and Laplacian spectrum equal to $(n, \mu_2, \ldots, \mu_2, \mu_2, 0)$, with $\mu_2 \neq n$, then $G \cong S_n$ or $G \cong K_{p,p}$, where $n = 2p$.*

Proof. In this case, the L-spectrum of $\overline{G}$ is $(n - \mu_2, n - \mu_2, \ldots, n - \mu_2, 0, 0)$ and, then, $\overline{G}$ has exactly 2 components. As each component has no more than two distinct Laplacian eigenvalues, both are isolated vertices or complete graphs. Since the two components also have all nonzero eigenvalues equal, we have $\overline{G} \cong K_1 \cup K_{n-1}$ or $\overline{G} \cong K_p \cup K_p$, where $n = 2p$. Therefore, $G \cong S_n$ or $G \cong K_{p,p}$. On the other hand, it is already known that the L-spectrum of S_n and $K_{p,p}$ are, respectively, $(n, 1, \ldots, 1, 0)$ and $(n, p, \ldots, p, 0)$. $\square$

LEMMA 3.2. *Let A be a real symmetric matrix of order n with largest eigenvalue λ and M the $m \times m$ principal submatrix of A obtained from A by excluding the $n - m$ last rows and columns. If M also has λ as an eigenvalue, associated with the normalized eigenvector $\boldsymbol{x} = (x_1, \ldots, x_m)$, then $\boldsymbol{x}^* = (x_1, \ldots, x_m, 0, \ldots, 0)$ is a*

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corresponding eigenvector to λ in A.

Proof. As λ is an eigenvalue of M corresponding to $\mathbf{x}$, then $\lambda = \langle M\mathbf{x}, \mathbf{x}\rangle$. So, it is enough to see that $\langle M\mathbf{x}, \mathbf{x}\rangle = \langle A\mathbf{x}^*, \mathbf{x}^*\rangle$. ◻

A well known result about the Laplacian matrix ([8]) says that, if G is a graph with at least one edge then $\mu_1 \geq \Delta + 1$, where Δ denotes the maximum degree of G. It is possible to get an analogous lower bound for the largest distance Laplacian eigenvalue of a connected graph G:

Theorem 3.3. *If G is a connected graph then $\partial_1^L(G) \geq \max\limits_{i\in V} \mathrm{Tr}(\mathrm{v_i}) + 1$. Equality is attained if and only if $G \cong K_n$.*

Proof. Suppose, without loss of generality, that $\mathrm{Tr}(\mathrm{v_1}) = \max\limits_{\mathrm{i\in V}} \mathrm{Tr}(\mathrm{v_i}) = \mathrm{Tr_{max}}$ and let $\mathbf{x} = \left(1, \frac{-1}{n-1}, \frac{-1}{n-1}, \ldots, \frac{-1}{n-1}\right)$. Then

$$\partial_1^L(G) = \max_{y\perp 1} \frac{\left\langle D^L\mathbf{y}, \mathbf{y}\right\rangle}{\|\mathbf{y}\|^2} \geq \frac{\left\langle D^L\mathbf{x}, \mathbf{x}\right\rangle}{\|\mathbf{x}\|^2} = \left(1 + \frac{1}{n-1}\right)^2 \left(\frac{\sum\limits_{i=1}^{n} d_{1,i}}{\|\mathbf{x}\|^2}\right) = \frac{n^2 \mathrm{Tr_{max}}}{(n-1)^2 \|\mathbf{x}\|^2}.$$

Since, $\|\mathbf{x}\|^2 = \dfrac{n}{n-1}$, we obtain

$$\partial_1^L(G) \geq \frac{n}{n-1}\mathrm{Tr_{max}} = \mathrm{Tr_{max}} + \frac{\mathrm{Tr_{max}}}{n-1} \geq \mathrm{Tr_{max}} + 1. \tag{3.1}$$

If the equality is attained for a connected graph G then, from (3.1), we conclude that $\mathrm{Tr_{max}} = n - 1$. As $G \cong K_n$ is the unique graph with this property and $\partial_1^L(K_n) = n$, the result is proven. ◻

In order to solve Conjecture 1.1, we first investigate how the existence of P_4 as an induced subgraph influences the multiplicity of the largest eigenvalue of the distance Laplacian matrix of a graph:

Theorem 3.4. *If the connected graph G has at least 4 vertices and it is not a cograph then $m(\partial_1^L) \leq n - 3$.*

Proof. Let G be a connected graph on $n \geq 4$ vertices which is not a cograph. Then G has P_4 as an induced subgraph. Let M be the principal submatrix of $\mathcal{D}^L(G)$ associated with this induced subgraph and denote the eigenvalues of M by $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. Suppose that $m(\partial_1^L) \geq n-2$. By Cauchy interlacing (Theorem 2.5) is easy to check that $\lambda_1 = \lambda_2 = \partial_1^L$. By Lemma 3.2, if $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are eigenvectors associated to ∂_1^L in M, then $\mathbf{x}^* = (x_1, x_2, x_3, x_4, 0, \ldots, 0)$ and $\mathbf{y}^* = (y_1, y_2, y_3, y_4, 0, \ldots, 0)$ are eigenvectors associated to ∂_1^L in $\mathcal{D}^L(G)$. As $\mathbf{x}^*, \mathbf{y}^* \perp \mathbf{1}$, with a linear combination of this vectors, is possible to get $z^* = (z_1, z_2, 0, z_4, 0, \ldots, 0)$

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such that $\mathbf{z}^*\perp\mathbf{1}$ and it is still an eigenvector for $\mathcal{D}^L(G)$ associated to ∂_1^L. Thus, $\mathbf{z} = (z_1, z_2, 0, z_4)$ is an eigenvector for M such that $z_1 + z_2 + z_4 = 0$.

Now, we observe that there are just two options for the matrix M:

$$M_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 \\ -1 & t_2 & -1 & -2 \\ -2 & -1 & t_3 & -1 \\ -3 & -2 & -1 & t_4 \end{bmatrix} \quad \text{or} \quad M_2 = \begin{bmatrix} t_1 & -1 & -2 & -2 \\ -1 & t_2 & -1 & -2 \\ -2 & -1 & t_3 & -1 \\ -2 & -2 & -1 & t_4 \end{bmatrix},$$

where t_1, t_2, t_3, t_4 denote the transmissions of the vertices that induce P_4 in $\mathcal{D}^L(G)$.

From the third entry of $M_1\mathbf{z} = \lambda_1\mathbf{z}$ it follows that $-2z_1 - z_2 - z_4 = 0$. This, together with the fact that $z_1 + z_2 + z_4 = 0$, allow us to conclude that $(0, 1, 0, -1)$ is an eigenvector corresponding to ∂_1^L in M_1. From the first entry of $M_1\mathbf{z} = \lambda_1\mathbf{z}$, we have a contradiction. Similarly we have a contradiction, considering M_2 instead of M_1. ☐

The next theorem resolves the Conjecture 1.1:

THEOREM 3.5. *If G is a graph on $n \geq 3$ vertices and $G \ncong K_n$, then $m(\partial_1^L(G)) \leq n-2$ with equality if and only if G is the star S_n or the complete bipartite graph $K_{p,p}$, if $n = 2p$.*

Proof. As $G \ncong K_n$, we already know that $m(\partial_1^L(G)) \leq n - 2$ (Theorem 2.4). Therefore, it remains to check for which graphs we have $m(\partial_1^L(G)) = n - 2$. Let G be a connected graph satisfying this property. Thus, $m(\partial_{n-1}^L(G)) = 1$. We consider two cases, when $\partial_{n-1}^L(G) = n$ and when $\partial_{n-1}^L(G) \neq n$:

- If $\partial_{n-1}^L(G) = n$, the $\mathcal{D}^L$-spectrum of G is $\big(\partial_1^L, \partial_1^L, \ldots, \partial_1^L, n, 0\big)$, with $\partial_1^L(G) \neq n$. By Theorem 2.3, $\overline{G}$ is disconnected and has exactly two components. Furthermore, as G is connected and $\overline{G}$ is disconnected, $diam(G) \leq 2$. So, by Theorem 2.2, the L-spectrum of G is $\big(n, 2n - \partial_1^L, \ldots, 2n - \partial_1^L, 2n - \partial_1^L, 0\big)$ and, from Lemma 3.1, $G \cong S_n$ or $G \cong K_{p,p}$;
- If $\partial_{n-1}^L(G) \neq n$, the $\mathcal{D}^L$-spectrum of G is $\big(\partial_1^L, \partial_1^L, \ldots, \partial_1^L, \partial_{n-1}^L, 0\big)$ with $\partial_1^L \neq \partial_{n-1}^L$ and $\partial_{n-1}^L \neq n$. We claim there is no graph with this property. Indeed, by Theorem 2.3, as $\partial_{n-1}^L \neq n$, $\overline{G}$ is also connected. By Theorem 2.1, G has P_4 as an induced subgraph and, therefore, by Theorem 3.4, G cannot have a distance Laplacian eigenvalue with multiplicity $n - 2$.

It is already known [4] the $\mathcal{D}^L$-spectra of the star and the complete bipartite graph, and this complete the proof:

- $\mathcal{D}^L$-spectrum of S_n : $((2n - 1)^{(n-2)}, n, 0)$;
- $\mathcal{D}^L$-spectrum of $K_{p,p}$: $((3p)^{(n-2)}, n, 0)$. ☐

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4. Graphs with P_5 as forbidden subgraph. In the previous section, we established a relationship between the $\mathcal{D}^L$-spectrum of a connected graph and the existence of a P_4 induced subgraph. Then, it is natural to think how the existence of a P_5 induced subgraph could influence its $\mathcal{D}^L$-spectrum. In this case, we prove the following theorem, regarding the largest distance Laplacian eigenvalue:

THEOREM 4.1. *If G is a connected graph on $n \geq 5$ vertices and $m(\partial_1^L(G)) = n-3$ then G does not have a P_5 as induced subgraph.*

Proof. Suppose that G has a P_5 as an induced subgraph and let M be the principal submatrix of $\mathcal{D}^L(G)$ corresponding to the vertices in this P_5. Denote the eigenvalues of M by $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$. If $m(\partial_1^L) = n-3$, by Cauchy interlacing theorem it follows that $\lambda_1 = \lambda_2 = \partial_1^L$. By Lemma 3.2, if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5)$ are eigenvectors associated to ∂_1^L for M, then $\mathbf{x}^* = (x_1, x_2, x_3, x_4, x_5, 0, \ldots, 0)$ and $\mathbf{y}^* = (y_1, y_2, y_3, y_4, y_5, 0, \ldots, 0)$ are eigenvectors for $\mathcal{D}^L(G)$, associated to ∂_1^L. As $\mathbf{x}^*, \mathbf{y}^* \perp \mathbf{1}$, with a linear combination of this vectors, is possible to get $z^* = (z_1, z_2, z_3, z_4, 0, \ldots, 0)$ such that $\mathbf{z}^* \perp \mathbf{1}$. and it is still an eigenvector for $\mathcal{D}^L(G)$ associated to ∂_1^L. Then, $\mathbf{z} = (z_1, z_2, z_3, z_4, 0)$ is an eigenvector for M such that $z_1 + z_2 + z_3 + z_4 = 0$.

Now, we observe that the matrix M can be written as

$$M = \begin{bmatrix} t_1 & -1 & -2 & -d_{1,4} & -d_{1,5} \\ -1 & t_2 & -1 & -2 & -d_{2,5} \\ -2 & -1 & t_3 & -1 & -2 \\ -d_{1,4} & -2 & -1 & t_4 & -1 \\ -d_{5,1} & -d_{5,2} & -2 & -1 & t_5 \end{bmatrix}, \tag{4.1}$$

where t_1, t_2, t_3, t_4, t_5 denote the transmissions of the vertices that induce P_5 in $\mathcal{D}^L(G)$, $d_{1,5}$=2,3 or 4, $d_{2,5} = 2$ or 3 and $d_{1,4} = 2$ or 3. As P_5 is an induced subgraph, it is easy to check that if $d_{1,5} = 4$ then $d_{2,5} = 3$ and $d_{1,4} = 3$. Considering the following cases, we see that all possibilities lead to a contradiction:

- $d_{1,5} = 2$ and $d_{2,5} = 2$:
  As $\mathbf{z} \perp \mathbf{1}$, from the fifth entry of $M\mathbf{z} = \partial_1^L \mathbf{z}$, it follows that $z_4 = 0$. So, using also the fourth entry of this equation, we have

$$\begin{cases} -d_{1,4} z_1 - 2z_2 - z_3 = 0, \\ z_1 + z_2 + z_3 = 0. \end{cases}$$

  If $d_{1,4} = 2$, then $z_3 = 0$ and $z_1 = -z_2$. So, we can assume that $\mathbf{z} = (-1, 1, 0, 0, 0)$ is an eigenvector of M, which is a contradiction according to the third entry of the equation. If $d_{1,4} = 3$, then $z_3 = z_1$ and $z_2 = -2z_1$. So, we can assume that $\mathbf{z} = (1, -2, 1, 0, 0)$ is an eigenvector of M. From the third

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entry of the equation, we conclude that $t_3 = \partial_1^L$, which is a contradiction (Theorem 3.3).

- $d_{1,5} = 2$ and $d_{2,5} = 3$:
  As $\mathbf{z} \perp \mathbf{1}$, from the fifth entry of $M\mathbf{z} = \partial_1^L \mathbf{z}$, it follows that $z_2 = z_4 = 1$ and $z_1 + z_3 = -2$. So, we can consider $\mathbf{z} = (z_1, 1, -2 - z_1, 1, 0)$, and from the second entry of the same equation, we conclude that $t_2 = \partial_1^L$.
- If $d_{1,5} = 3$ and $d_{2,5} = 2$:
  As $\mathbf{z} \perp \mathbf{1}$, from the fifth entry of $M\mathbf{z} = \partial_1^L \mathbf{z}$, it follows that $z_1 = z_4 = 1$ and $z_2 + z_3 = -2$. So, we can consider $\mathbf{z} = (1, -2 - z_3, z_3, 1, 0)$, and we have

$$\begin{cases} t_1 + 2 - z_3 - d_{1,4} = \partial_1^L, \\ -d_{1,4} + 4 + z_3 + t_4 = \partial_1^L. \end{cases}$$

  If $d_{1,4} = 2$, by Theorem 3.3 we have

$$\begin{cases} z_3 = t_1 - \partial_1^L \le -1, \\ z_3 = \partial_1^L - t_4 \ge 1. \end{cases}$$

  If $d_{1,4} = 3$, again by Theorem 3.3, we have

$$\begin{cases} z_3 = t_1 - \partial_1^L - 1 \le -2, \\ z_3 = \partial_1^L - t_4 - 1 \ge 0. \end{cases}$$

- If $d_{1,5} = d_{2,5} = 3$:
  As $\mathbf{z} \perp \mathbf{1}$, from the fifth entry of $M\mathbf{z} = \partial_1^L \mathbf{z}$, it follows that $z_3 = -2z_4$ and $z_1 + z_2 = 1$. So, we can consider $\mathbf{z} = (z_1, 1 - z_1, -2, +1, 0)$, and we have

$$\begin{cases} -z_1 - 2t_3 - 2 = -2\partial_1^L, \\ (2 - d_{1,4})z_1 + t_4 = \partial_1^L. \end{cases}$$

  If $d_{1,4} = 2$, then $t_4 = \partial_1^L$, which is a contradiction. If $d_{1,4} = 3$, then

$$\begin{cases} z_1 = 2(\partial_1^L - t_3 - 1), \\ z_1 = t_4 - \partial_1^L, \end{cases}$$

  which is a contradiction, since Theorem 3.3 implies $z_1 < 0$ and $z_1 > 0$.
- $d_{1,5} = 4$, $d_{2,5} = 3$ and $d_{1,4} = 3$:
  As $\mathbf{z} \perp \mathbf{1}$, from the fifth entry of $M\mathbf{z} = \partial_1^L \mathbf{z}$, it follows that $-3z_1 - 2z_2 - z_3 = 0$. From this fact and the fourth entry of this equation, we obtain $t_4 z_4 = z_4 \partial_1^L$. If $z_4 \neq 0$, we get a contradiction. If $z_4 = 0$, we conclude that $-2z_1 - z_2 = 0$. So, we can consider $\mathbf{z} = (1, -2, 1, 0, 0)$, which implies in $t_1 = \partial_1^L$, a contradiction. $\square$

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Although by this theorem we cannot completely describe the graphs that have largest distance Laplacian eigenvalue with multiplicity $n-3$, it is possible to obtain a partial characterization and some remarks about this issue.

PROPOSITION 4.2. *Let G be a connected graph with order $n \geq 4$ such that $m(\partial_1^L) = n-3$. If $\partial_{n-1}^L = n$ is an eingenvalue with multiplicity 2 then $G \cong K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$, or $G \cong K_{\frac{n-1}{2},\frac{n-1}{2}} \vee K_1$, or $G \cong \overline{K}_{n-2} \vee K_2$.*

Proof. As $\partial_{n-1}^L = n$, $\overline{G}$ is disconnected and $diam(G) = 2$. Moreover, by Theorem 2.2, the L-spectrum of $\overline{G}$ is

$$(n-\partial_1^L, \ldots, n-\partial_1^L, 0, 0, 0),$$

that is, $\overline{G}$ has three components, all of them with the same nonzero eigenvalue. So, the three components are isolated vertices or complete graphs with the same order, that is, $\overline{G} \cong K_{\frac{n}{3}} \cup K_{\frac{n}{3}} \cup K_{\frac{n}{3}}$, if $3 \mid n$, $\overline{G} \cong K_{\frac{n-1}{2}} \cup K_{\frac{n-1}{2}} \cup K_1$, if $2 \mid (n-1)$, or $\overline{G} \cong K_{n-2} \cup K_1 \cup K_1$.

Finally, as the graphs we have cited above have diameter 2, by Theorem 2.2, its enough to know its L-spectrum to write the $\mathcal{D}^L$-spectrum:

- $\mathcal{D}^L$-spectrum of $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$: $\left(\left(\dfrac{4n}{3}\right)^{(n-3)}, n^{(2)}, 0\right)$;
- $\mathcal{D}^L$-spectrum of $K_{\frac{n-1}{2},\frac{n-1}{2}} \vee K_1$: $\left(\left(\dfrac{3n-1}{2}\right)^{(n-3)}, n^{(2)}, 0\right)$;
- $\mathcal{D}^L$-spectrum of $\overline{K}_{n-2} \vee K_2$: $\big((2(n-1))^{(n-3)}, n^{(2)}, 0\big)$. ☐

To finish the characterization of the graphs whose largest eigenvalue of the distance Laplacian matrix has multiplicity $n-3$ we should analyze two situations:

- If $\partial_{n-1}^L = n$ is an eigenvalue with multiplicity one;
- If $\partial_{n-1}^L \neq n$.

Although we have not characterized precisely these two cases, proceeding similarly to the last proposition, we can conclude in the first case that if the $\mathcal{D}^L$-spectrum of a connected graph G is $(\partial_1^L, \ldots, \partial_1^L, \partial_{n-2}^L, n, 0)$ then the L-spectrum of $\overline{G}$ is written as $(\partial_1^L - n, \ldots, \partial_1^L - n, \partial_{n-2}^L - n, 0, 0)$. So, $\overline{G}$ is a graph with 2 components such that the largest Laplacian eigenvalue has multiplicity $n-3$. For example, the graph $G \cong K_{2,n-2}$ has this property since the $\mathcal{D}^L$-spectrum is equal to $((2n-2)^{(n-3)}, n+2, n, 0)$.

In the last case, as $\partial_{n-1}^L \neq n$, then $\overline{G}$ is a connected graph. So, G has P_4 as an induced subgraph. On the other hand, from Theorem 4.1, G does not have P_5 as an induced subgraph. For example, C_5 satisfies this condition, since its $\mathcal{D}^L$-spectrum is $\left(\frac{15+\sqrt{5}}{2}, \frac{15+\sqrt{5}}{2}, \frac{15-\sqrt{5}}{2}, \frac{15-\sqrt{5}}{2}, 0\right)$.

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