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ODD CYCLE ZERO FORCING PARAMETERS AND
THE MINIMUM RANK OF GRAPH BLOWUPS*

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Abstract. The minimum rank problem for a simple graph G and a given field F is to determine the smallest possible rank among symmetric matrices over F whose i,j-entry, $i \neq j$, is nonzero whenever i is adjacent to j, and zero otherwise; the diagonal entries can be any element in F. In contrast, loop graphs $\mathfrak{G}$ go one step further to restrict the diagonal i,i-entries as nonzero whenever i has a loop, and zero otherwise. When char $F \neq 2$, the odd cycle zero forcing number and the enhanced odd cycle zero forcing number are introduced as bounds for loop graphs and simple graphs, respectively. A relation between loop graphs and simple graphs through graph blowups is developed, so that the minimum rank problem of some families of simple graphs can be reduced to that of much smaller loop graphs.

1. Introduction. For a given graph, the *minimum rank problem* is to determine the smallest possible rank among a family of matrices associated to the graph. Depending on the types of graphs, the definitions of the associated matrices are different. In this paper, we focus on simple graphs and loop graphs, provide new bounds for both of them, and develop their relation on the minimum rank problem through graph *blowups* (which will be defined in Section 4).

A simple graph is a graph without loops or multiedges; a *loop graph* is a graph where each vertex can have at most one loop. Given a field F, the set of associated matrices of a simple graph G is denoted by $\mathcal{S}^F(G)$ and defined as the family of symmetric matrices over F whose i,j-entry, $i \neq j$, is nonzero whenever i is adjacent to j, and zero otherwise; in contrast, the associated matrices $\mathcal{S}^F(\mathfrak{G})$ of a loop graph $\mathfrak{G}$ is the family of symmetric matrices over F whose i,j-entry ($i = j$ is possible) is nonzero whenever i is adjacent to j, and zero otherwise. Note that in a loop graph, i is adjacent to itself if and only if i has a loop. To point out the difference, the diagonal entries can be any element in F for simple graphs; however, for loop graphs, the zero-nonzero pattern on the diagonal is controlled by the loops. A graph without

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any loops can be considered as a simple graph G or a loop graph $\mathfrak{G}$ without loops, but the definitions for $\mathcal{S}^F(G)$ and $\mathcal{S}^F(\mathfrak{G})$ are different, since $\mathcal{S}^F(G)$ allows free diagonal while $\mathcal{S}^F(\mathfrak{G})$ requires zero diagonal. Therefore, a simple graph is usually denoted as G and a loop graph is denoted as $\mathfrak{G}$.

The *minimum rank* of a given graph, is defined as the smallest possible rank in $\mathcal{S}^F(G)$, or $\mathcal{S}^F(\mathfrak{G})$. For a simple graph G and a loop graph $\mathfrak{G}$, the minimum ranks are written as $\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^F(G)\}$ and $\mathrm{mr}^F(\mathfrak{G}) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^F(\mathfrak{G})\}$ respectively. Equivalently, the problem of finding the minimum rank of a graph can be viewed as finding the *maximum nullity*, which is defined as $M^F(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^F(G)\}$ and $M^F(\mathfrak{G}) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^F(\mathfrak{G})\}$. This is because $\mathrm{mr}^F(G) + M^F(G) = |V(G)|$ for any simple graph G, or similarly when G is replaced by any loop graph $\mathfrak{G}$.

The minimum rank problem is a relaxation of the *inverse eigenvalue problem*, and also essentially related to *orthogonal representations* and the *Colin de Verdiére type parameters* (see [10]). For the study of the minimum rank problem, the *zero forcing number* Z was introduced in [1], and then [13] extended to each type of graph as a "universal" upper bound for the maximum nullity. That is, $M^F(G) \le Z(G)$ for any field F and any simple graph G, or when G is replaced by a loop graph $\mathfrak{G}$. Zero forcing parameters will be discussed in Section 1.1

In the sense of the maximum nullity and the zero forcing number, the relation between simple graphs and loop graphs is bridged by the *loop configurations*. A loop configuration of a simple graph G is a loop graph $\mathfrak{G}$ obtained from G by designating each vertex as having no loop or one loop. So for a given simple graph G with n vertices, there are 2^n possible loop configurations of G. Through this definition, the maximum nullity of a simple graph can be obtained from the maximum nullities of its loop configurations. That is, $M^F(G) = \max_{\mathfrak{G}} M^F(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G. Since $M^F(\mathfrak{G}) \le Z(\mathfrak{G})$ for each of the loop configurations, the *enhanced zero forcing number* $\widehat{Z}(G)$ was introduced in [4] and is defined as $\widehat{Z}(G) = \max_{\mathfrak{G}} Z(\mathfrak{G})$, where the maximum is over all loop configurations $\mathfrak{G}$ of G. In the same paper, it is shown $M^F(G) \le \widehat{Z}(G) \le Z(G)$ for any simple graph G and any field F. This suggests that the consideration of loop graphs can improve the upper bound given by $Z(G)$.

For the field of real numbers, it is known [8] that $M^{\mathbb{R}}(G) = Z(G)$ for any simple graph with $|V(G)| \le 7$, yet this is not the case for loop graphs. For example, let C_n be the cycle on n vertices, as a simple graph. A *loopless odd cycle* $\mathfrak{C}_{2k+1}^0$ is the loop configuration of C_{2k+1} without any loop. For a loopless odd cycle $\mathfrak{C}_{2k+1}^0$, its maximum nullity $M^F(\mathfrak{C}_{2k+1}^0) = 0$ for any field F with characteristic char $F \ne 2$, but $Z(\mathfrak{C}_{2k+1}^0) = 1$ [7]. That means, even for small loop graphs like $\mathfrak{C}_3^0$, there is a gap.

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When char $F \neq 2$, loopless odd cycles play an important role, and allow us to discover new upper bounds for both loop graphs and simple graphs. In Section 2, we define a new parameter called the *odd cycle zero forcing number*, $Z_{oc}(\mathfrak{G})$, for loop graphs $\mathfrak{G}$; meanwhile, Theorem 2.8 proves that $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$, and Corollary 2.9 states that $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ whenever $F = \mathbb{R}$ and $\mathfrak{G}$ is a loop configuration of a complete graph or a cycle, which fixes the gap between $Z(\mathfrak{C}^0_{2k+1})$ and $M^{\mathbb{R}}(\mathfrak{C}^0_{2k+1})$.

Following the same track of the enhanced zero forcing number, when char $F \neq 2$, the odd cycle zero forcing number for loop graphs also leads to a new bound for simple graphs. In Section 3, the *enhanced odd cycle zero forcing number* $\widehat{Z}_{oc}(G)$ for simple graphs is introduced with the property $M^F(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G) \leq Z(G)$. Example 3.3 shows that $M^{\mathbb{R}}(K_{3,3,3}) = \widehat{Z}_{oc}(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = 7$, where $K_{3,3,3}$ is the complete tripartite (simple) graph. Corollary 4.9 and Proposition 6.1 provide examples showing that $\widehat{Z}(G) - \widehat{Z}_{oc}(G)$ and $\widehat{Z}_{oc}(G) - M^{\mathbb{R}}(G)$ can be arbitrarily large.

Graph blowups are a transformation from a loop graph to a simple graph, and were used for the characterization for minimum rank over finite fields [12]. In Section 4, graph blowups are defined, and Theorem 4.7 shows that $M^F(H) = \widehat{Z}_{oc}(H)$ if $M^F(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$, provided that H is a "large" blowup of $\mathfrak{G}$. That means the maximum nullity of a graph blowup, which is a simple graph, can be obtained by the maximum nullity of a much smaller loop graph.

In Section 5, the *graph complement conjecture* for $M(G)$ is shown to be true for most graph blowups; while the graph complement conjecture for $\widehat{Z}_{oc}(G)$ is true for any simple graph.

1.1. Different types of zero forcing numbers. There are several different types of zero forcing numbers, but they all serve as upper bounds for the maximum nullity for different types of graphs. In this section, the zero forcing number $Z(G)$ for simple graphs G and the zero forcing number $Z(\mathfrak{G})$ for loop graphs $\mathfrak{G}$ will be discussed.

The zero forcing number starts by the *zero forcing game*, where vertices are blue or white and different *color-change rules* may apply on different types of graphs. For simple graphs G, the color-change rule is

- if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue;

for loop graphs $\mathfrak{G}$, the color-change rule is

- if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ (where $x = y$ is possible), then y turns blue.

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So one of the major differences is for simple graphs, x should be blue first so x can force its neighbor y to turn blue, but for loop graphs, this need not be the case. Also, we emphasize the neighbors of x for loop graphs refers to those vertices which are adjacent to x. So it is possible that x itself is the only white neighbor of x, when there is a loop on x.

On a graph with vertex set V, a subset $B \subseteq V$ is called a *zero forcing set* if setting the vertices of B blue and the others white can make the whole set V change to blue through repeated applications of the corresponding color-change rule. The zero forcing number $Z(G)$, or $Z(\mathfrak{G})$, is defined to be the minimum cardinality of a zero forcing set on a simple graph G, or a loop graph $\mathfrak{G}$, using the appropriate color-change rule.

Suppose $\mathfrak{G}$ is a loop configuration of a simple graph G. Then any zero forcing set of G is a zero forcing set of $\mathfrak{G}$, so $Z(\mathfrak{G}) \leq Z(G)$. This establishes the stated theorem

$$M^F(G) \leq \widehat{Z}(G) \leq Z(G)$$

in [4].

As mentioned, there is a gap between $M^F(\mathfrak{G})$ and $Z(\mathfrak{G})$ when $\mathfrak{G}$ is a loopless odd cycle and char $F \neq 2$. In fact, this also happens on some loop configurations of complete graphs, when $\mathfrak{C}_3^0$ appears on it. For loop configurations of complete graphs, the maximum nullity can be found in [7] and Proposition 1.1 provides the zero forcing number .

PROPOSITION 1.1. *Let K_n be the complete (simple) graph on n vertices and $\mathfrak{K}_n^{\ell(s)}$ its loop configuration with s loops. Then*

$$M^{\mathbb{R}}(\mathfrak{K}_n^{\ell(s)}) = \begin{cases} n & \text{if } n - s = 1 = n; \\ n-1 & \text{if } n - s = 0 \text{ and } 1 \leq n; \\ n-2 & \text{if } 1 \leq n - s \leq 2 \leq n; \\ n-3 & \text{if } 3 \leq n - s, \end{cases}$$

and

$$Z(\mathfrak{K}_n^{\ell(s)}) = \begin{cases} n & \text{if } n - s = 1 = n; \\ n-1 & \text{if } n - s = 0 \text{ and } 1 \leq n; \\ n-2 & \text{if } 1 \leq n - s \text{ and } 2 \leq n. \end{cases}$$

Proof. Since $M^{\mathbb{R}}(\mathfrak{G}) + \mathrm{mr}^{\mathbb{R}}(\mathfrak{G}) = |V(\mathfrak{G})|$ for every loop graph $\mathfrak{G}$, the formula for $M^{\mathbb{R}}(\mathfrak{K}_n^{\ell(s)})$ comes from Proposition 5.5 in [7]. To determine the zero forcing number, two cases are considered. When $n - s \leq 2$, the formulas for $Z(\mathfrak{K}_n^{\ell(s)})$ and $M(\mathfrak{K}_n^{\ell(s)})$ agree with each other, so it is enough to find a zero forcing set of cardinality $M(\mathfrak{K}_n^{\ell(s)})$.

When $n - s = 1 = n$, the graph has only one vertex, which makes a zero forcing set. When $n - s = 0$ and $1 \le n$, any set of $n - 1$ vertices can be a zero forcing set. When $1 \le n - s \le 2 \le n$, any set of $n - 2$ vertices with loops forms a zero forcing set.

In the case of $3 \le n - s$, by coloring all vertices blue except two vertices without loops, it becomes a zero forcing set. However, if there are 3 white vertices initially, then every vertex will have at least two white neighbors (beside itself), so it cannot be a zero forcing set. As a consequence, $n - 2$ is the zero forcing number. $\square$

PROPOSITION 1.2. [7] *Let C_n be the (simple) cycle on n vertices and $\mathfrak{C}_n$ one of its loop configurations. Then $M^{\mathbb{R}}(\mathfrak{C}_n) = Z(\mathfrak{C}_n)$ whenever $\mathfrak{C}_n$ is not a loopless odd cycle. For loopless odd cycles $\mathfrak{C}_{2k+1}^0$, $M^{\mathbb{R}}(\mathfrak{C}_{2k+1}^0) = 0$ but $Z(\mathfrak{C}_{2k+1}^0) = 1$.*

REMARK 1.3.   The equality $M^F(\mathfrak{C}_{2k+1}^0) = 0$ holds for any field F with char $F \ne 2$. This is because a loop graph $\mathfrak{G}$ with a *unique spanning generalized cycle* always has $M(\mathfrak{G}) = 0$ if char $F \ne 2$ [11] (spanning generalized cycles are called spanning composite cycles in [7]). This states that every matrix in $\mathcal{S}^F(\mathfrak{C}_{2k+1}^0)$ is nonsingular whenever char $F \ne 2$. On the other hand, $M^F(\mathfrak{C}_{2k+1}^0) = 1$ if char $F = 2$, because $Z(\mathfrak{C}_{2k+1}^0) = 1$ and the adjacency matrix of $\mathfrak{C}_{2k+1}^0$ over F has determinant 0. Symmetry is also crucial, since the asymmetric matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ follows the zero-nonzero pattern given by $\mathfrak{C}_3^0$ but has nullity 1.

2. Odd cycle zero forcing number $Z_{oc}(\mathfrak{G})$.

This section exploits Remark 1.3 to develop a new upper bound of $M^F(\mathfrak{G})$ for loop graphs $\mathfrak{G}$ when char $F \ne 2$.

For a loop graph $\mathfrak{G}$ and a subset of vertices $W \subseteq V(\mathfrak{G})$, the *induced subgraph* of $\mathfrak{G}$ on W is the loop graph obtained from $\mathfrak{G}$ by deleting all vertices outside W, which keeps all those edges and loops with their two endpoints in W. The odd cycle zero forcing number is an extension of the conventional zero forcing number by adding one more rule.

DEFINITION 2.1.   On a given loop graph $\mathfrak{G}$, where vertices are marked blue or white, the *color-change rule for Z_{oc}* (CCR-Z_{oc}) is:

(a) if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ (where $x = y$ is possible), then y turns blue.
(b) if W is the set of white vertices and $\mathfrak{G}[W]$ contains a component $\mathfrak{C}$, as a loopless odd cycle, then all vertices of $\mathfrak{C}$ turn blue.

If starting with $B \subseteq V(\mathfrak{G})$ as initial blue vertices makes the whole set $V(\mathfrak{G})$ change to blue through repeated applications of CCR-Z_{oc}, then B is a *zero forcing set for*

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Z_{oc} (ZFS-Z_{oc}) on $\mathfrak{G}$. The *odd cycle zero forcing number* is defined as

$$Z_{oc}(\mathfrak{G}) = \min\{|B| : B \text{ is a ZFS-}Z_{oc} \text{ on } \mathfrak{G}\}.$$

REMARK 2.2.   Given an initial blue set, no matter what order the rules (a) and (b) are applied, the process always stops at some unique final coloring where neither color-change rule can be used. To see this, suppose at a certain step, W is the set of white vertices and $\mathfrak{C}$ is a loopless odd cycle as a component of $\mathfrak{G}[W]$; also suppose $y \in V(\mathfrak{C})$ is the only white neighbor of x. In this situation, we can apply rule (b), and all vertices in $V(\mathfrak{C})$ turns blue; on the other hand, if we apply rule (a) instead to make y blue, then all vertices in $V(\mathfrak{C})$ will eventually turn blue, since y is a (conventional) zero forcing set of $\mathfrak{C}$. So the order of using rule (a) and rule (b) will not affect the final set of blue vertices. For implementing an algorithm, one can consider rule (b) only when rule (a) no longer applies. (A fast implementation of rule (a) exists but no fast implementation of rule (b) currently exists. This explains our preference for rule (a).)

The following concepts are helpful for understanding this new color-change rule. A *chronological list for* Z_{oc} records how a ZFS-Z_{oc} makes all vertices blue, and is defined as $(X_i \to Y_i)_{i=1}^{s}$, where at the i-th step, if rule (a) is applied, then $X_i = \{x\}$ and $Y_i = \{y\}$, while if rule (b) is applied, then $X_i = Y_i = V(\mathfrak{C})$. Here x, y, and $\mathfrak{C}$ are as those in Definition 2.1. A *zero forcing process for* Z_{oc} (ZFP-Z_{oc}) refers to the initial blue set B and its chronological list. Note that a ZFS-Z_{oc} may have different ways of applying CCR-Z_{oc}, so the chronological list for Z_{oc} with a given ZFS-Z_{oc} is not unique. Note that we do not restrict the ZFS-Z_{oc} of a chronological list to be a minimum ZFS-Z_{oc}; when it is minimum, the chronological list and ZFP-Z_{oc} are said to be *optimal*.

For a given chronological list, we can draw a corresponding digraph on $V(\mathfrak{G})$ with arcs indicated by $X_i \to Y_i$. If $X_i = \{x\}$ and $Y_i = \{y\}$, then $x \to y$ is added; if $X_i = Y_i = V(\mathfrak{C})$ for some loopless odd cycle $\mathfrak{C}$, then an odd directed cycle is added, with some circular orientation. With these definitions, each (weakly connected) component of this digraph is called a *maximal chain*.

On a digraph, a sequence of vertices with structure $v_1 \to v_2 \to \cdots \to v_n$ is called a *directed n-path*, where v_1 is called the *tail* and v_n is called the *head* of this directed path; and a sequence of vertices with the structure $v_1 \to v_2 \to \cdots \to v_n \to v_1$ is called a *directed n-cycle*. On digraphs, a directed 1-cycle or a directed 2-cycle are possible, and they are a vertex with a loop or two vertices with two arcs of both directions.

EXAMPLE 2.3.   Let $\mathfrak{G}$ be the loop graph in Figure 2.1 and $B = \{3\}$ be the initial blue set. Then the column on the left is one possible chronological list, while the column on the right is its reversal, which reverses the order of the list and switches

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the roles of X_i's and Y_i's. The reversal is also a ZFP-Z_{oc}, with the initial blue set $B' = \{6\}$.

$$
\begin{array}{ll}
\{2\} \to \{1\} & \{10\} \to \{10\} \\
\{1\} \to \{2\} & \{7,8,9\} \to \{7,8,9\} \\
\{3\} \to \{4\} & \{5\} \to \{4\} \\
\{5\} \to \{6\} & \{6\} \to \{5\} \\
\{4\} \to \{5\} & \{4\} \to \{3\} \\
\{7,8,9\} \to \{7,8,9\} & \{2\} \to \{1\} \\
\{10\} \to \{10\} & \{1\} \to \{2\}
\end{array}
$$

Following this chronological list, its maximal chains are shown in Figure 2.1, including a vertex with a directed 1-cycle on $\{10\}$, a directed 2-cycle on $\{1,2\}$, a directed path on $\{3,4,5,6\}$, and a directed odd cycle on $\{7,8,9\}$ given by rule (b).
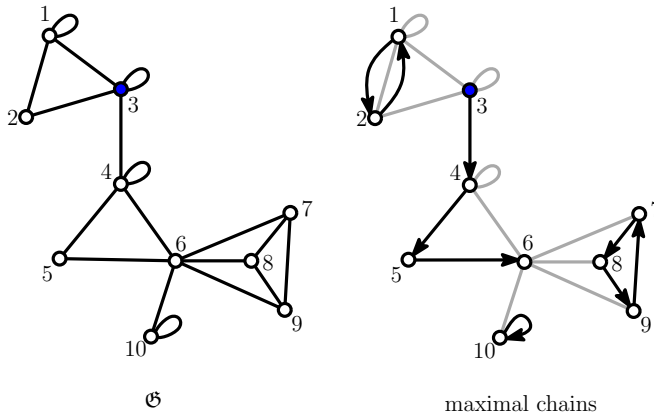

$\mathfrak{G}$ maximal chains

FIG. 2.1. *The loop graph $\mathfrak{G}$ for Example 2.3 and its maximal chains.*

Example 2.3 shows all possible types of maximal chains. Proposition 2.4 and Proposition 2.5 develop some general properties for maximal chains.

PROPOSITION 2.4. *Let $\mathfrak{G}$ be a loop graph and B a ZFS-Z_{oc} on $\mathfrak{G}$. Let ζ be a ZFP-Z_{oc} with its initial blue set B and Γ the corresponding digraph of ζ. By CCR-Z_{oc}, the following properties hold:*

(1) *for every vertex $x \in B$, the in-degree of x in Γ is 0;*
(2) *for every vertex $x \in V(\mathfrak{G}) \smallsetminus B$, the in-degree of x in Γ is 1;*
(3) *for every vertex $x \in V(\mathfrak{G})$, the out-degree of x in Γ is at most 1;*
(4) *each maximal chain is either a directed 1-cycle, a directed 2-cycle, a directed path, or a directed odd cycle given by rule (b), where an isolated vertex without any arcs on it is considered a directed 1-path;*
(5) *B is the set of tails of the directed paths in Γ.*

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Proof. Since B is a ZFS-Z_{oc}, each vertex in B is blue initially and each vertex outside B turns blue exactly once, implying (1) and (2). A directed odd cycle given by rule (b) always forms a component itself in Γ, and each vertex of it has out-degree 1. If $x \to y_1$ and $x \to y_2$ are arcs in Γ, then y_1 and y_2 are two white neighbors of x and rule (a) cannot apply. Therefore (3) holds. For (4), since every vertex of Γ has in-degree at most 1 and out-degree 1, Γ is a disjoint union of directed cycles and directed paths. Since each directed n-cycle with $n \geq 3$ cannot be obtained by rule (a), it must be a directed odd cycle given by rule (b). Finally, for (5), B corresponds to those vertices with in-degree 0, which is the set of tails of each directed path in Γ. □

In contrast to the color-change rule on simple graphs, a vertex does not need to be blue to start its force. Let $(X_i \to Y_i)_{i=1}^s$ be a chronological list for Z_{oc} on a loop graph $\mathfrak{G}$. By Proposition 2.4, each vertex $x \in V(\mathfrak{G})$ is in at most one X_i for some i. If $x \in X_i$ is blue already before $X_i \to Y_i$ applies, or $x \notin X_i$ for any i, then x is said to be *blue-first*.

PROPOSITION 2.5. *Let ζ be a ZFP-Z_{oc} on a loop graph $\mathfrak{G}$ and π a maximal chain. Then the following properties hold:*

  (1) *if π is a directed 1-cycle on the vertex x, then x is not blue-first and has a loop in $\mathfrak{G}$;*
  (2) *if π is a directed 2-cycle, then one of its two vertices is blue-first while the other is not, and the vertex which is not blue-first has no loop in $\mathfrak{G}$;*
  (3) *if π is a directed odd cycle given by rule (b), then every vertex of π is not blue-first and has no loop in $\mathfrak{G}$;*
  (4) *if x, y are in different maximal chains and x, y are not blue-first, then there is no edge between x and y in $\mathfrak{G}$.*

Proof. Directed 1-cycles and directed 2-cycles can only be given by rule (a). If π is a directed 1-cycle given by $\{x\} \to \{x\}$ in the chronological list, then x is not yet blue when this happens, and x has a loop in $\mathfrak{G}$ by rule (a). If π is a directed 2-cycle given by $\{x\} \to \{y\}$ first and $\{y\} \to \{x\}$ later in the chronological list, then x is not blue-first while y is, and x has no loop in $\mathfrak{G}$ by rule (a). Rule (b) gives (3) immediately. If x and y are as in (4) but x is adjacent to y in $\mathfrak{G}$, then neither of them can turn blue, a contradiction. □

PROPOSITION 2.6. *If $(X_i \to Y_i)_{i=1}^s$ is a chronological list for Z_{oc} on $\mathfrak{G}$, then for $i < j$ there are no edges between X_i and Y_j.*

Proof. At the i-th step, Y_j is not yet blue, since $i < j$. Suppose there is an edge between X_i and Y_j. Then Y_j provides extra white neighbors to X_i other than Y_i. Thus, X_i would not have a unique white neighbor, nor be an isolated loopless odd

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cycle. Hence, $X_i \to Y_i$ is impossible, a contradiction. $\square$

PROPOSITION 2.7. *Let ζ be a ZFP-Z_{oc} on $\mathfrak{G}$ with initial blue set B and chronological list $(X_i \to Y_i)_{i=1}^{s}$. Then $B = V(\mathfrak{G}) \smallsetminus \bigcup_{i=1}^{s} Y_i$. Also, $(Y_i \to X_i)_{i=s}^{1}$ is again a ZFP-Z_{oc}, starting with the initial blue set $B' = V(\mathfrak{G}) \smallsetminus \bigcup_{i=1}^{s} X_i$. And B' is also a ZFS-Z_{oc} on $\mathfrak{G}$ with $|B| = |B'|$. This new zero forcing process is the reversal of ζ.*

Proof. The initial blue set B of a chronological list is those vertices not being changed to blue, so $B = V(\mathfrak{G}) \smallsetminus \bigcup_{i=1}^{s} Y_i$. By Proposition 2.4, X_i's are mutually disjoint sets, and so are the Y_i's. Also, $|X_i| = |Y_i|$ for each i by definition. Hence, $|B| = |B'|$ by the choice of B'. To see the reversal works, we claim that $Y_i \to X_i$ is a legal move under CCR-Z_{oc} when $B' = \bigcup_{j=i+1}^{s} X_j$ is all blue. At this situation, $\bigcup_{j=1}^{i} X_j$ is the set of white vertices, and Proposition 2.6 states that X_i is the only white set connected to Y_i. Therefore, $Y_i \to X_i$ works consecutively from s to 1. $\square$

We note that the proof of Proposition 2.7 is analogous to that in [3] for simple graphs.

THEOREM 2.8. *For any loop graph $\mathfrak{G}$ and any field F with $\operatorname{char} F \neq 2$, $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.*

Proof. Since the color change rules for $Z(\mathfrak{G})$ are a subset of the color change rules for $Z_{oc}(\mathfrak{G})$, $Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$.

Let $n = |V(\mathfrak{G})|$, $k = Z_{oc}(\mathfrak{G})$, and B a ZFS-Z_{oc} of cardinality k. Also let $(X_i \to Y_i)_{i=1}^{s}$ be the corresponding chronological list with s steps. Let $A \in S^F(\mathfrak{G})$ be a matrix with $\operatorname{null}(A) = M^F(\mathfrak{G})$. Apply row/column permutations on A so that the columns follow the order of X_i's and the rows follow the order of Y_i's, and put all remaining columns to the right and rows to the bottom. Note that the permutations will not change the rank, but the new matrix will be of the form

$$\begin{bmatrix} A[Y_1, X_1] & & & & ? & ? \\ O & A[Y_2, X_2] & & & & ? \\ \vdots & & \ddots & & & \vdots \\ O & \cdots & O & A[Y_s, X_s] & & \\ ? & ? & \cdots & & & ? \end{bmatrix},$$

where $A[Y_j, X_i]$ is the submatrix of A induced by rows in Y_j and columns in X_i.

This contains an upper-triangular block matrix, since Proposition 2.6 ensures $A[Y_j, X_i] = O$ if $i < j$. Every diagonal block $A[Y_i, X_i]$ is either a 1×1 nonzero matrix, or a matrix described by a loopless odd cycle, which is nonsingular by Remark 1.3. This means the rank of A is at least $|\bigcup_i^s Y_i| = n - k$. Therefore $M^F(\mathfrak{G}) = \operatorname{null}(A) \leq k = Z_{oc}(\mathfrak{G})$. $\square$

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The proof of Theorem 2.8 is based on Remark 1.3, and that is why we need char $F \neq 2$.

COROLLARY 2.9. *For any loop configuration $\mathfrak{G}$ of a complete graph or a cycle, $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$.*

Proof. If there are at least 3 nonloop vertices $\{x, y, z\}$ on a loop configuration $\mathfrak{G}$ of a complete graph, then $V(\mathfrak{G}) \smallsetminus \{x, y, z\}$ is a ZFS-Z_{oc} on $\mathfrak{G}$. If $\mathfrak{G}$ is a loopless odd cycle, then the empty set is a ZFS-Z_{oc} on $\mathfrak{G}$. Together with Proposition 1.1 and Proposition 1.2, $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ for these loop graphs. $\square$

We end this section with Example 2.10, showing the gap $Z(\mathfrak{G}) - Z_{oc}(\mathfrak{G})$ can be arbitrarily large for loop graphs.
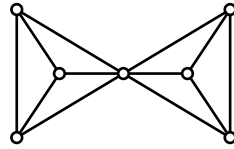

FIG. 2.2. *The graph $K_1 \bigvee (2K_3)$ for Example 2.10.*

EXAMPLE 2.10.   Let $G_n = K_1 \bigvee (nK_3)$ be the simple graph defined as the join of a vertex and n copies of K_3, the complete graph on 3 vertices. Figure 2.2 shows G_2. Let $\mathfrak{G}_n^0$ be the loop configuration of G_n without any loop and $x \in V(\mathfrak{G}_n^0)$ the vertex adjacent to all other vertices. Then $Z_{oc}(\mathfrak{G}_n^0) = 1$, since $\{x\}$ is a ZFS-Z_{oc} on $\mathfrak{G}_n^0$. However, $Z(\mathfrak{G}_n^0) = n + 1$. To see this, observe that $\mathfrak{G}_n^0 - x$ is n copies of the loopless odd cycle $\mathfrak{C}_3^0$. In $\mathfrak{G}_n^0$ each copy of $\mathfrak{C}_3^0$ needs at least one blue vertex, for otherwise there is no way to turn this copy blue; taking only these n blue vertices does not allow forcing to begin, but these n vertices along with x becomes a (conventional) zero forcing set on $\mathfrak{G}_n^0$. So $n + 1$ blue vertices is the minimum requirement. Also, $M^{\mathbb{R}}(\mathfrak{G}_n^0) = 1 = Z_{oc}(\mathfrak{G}_n^0)$, since it does not contain a unique spanning composite cycle (see [7]).

3. Enhanced odd cycle zero forcing number $\widehat{Z}_{oc}$. The enhanced zero forcing number demonstrates that an upper bound for loop graphs can lead to an upper bound for simple graphs. This also applies to the odd cycle zero forcing number.

DEFINITION 3.1.   Let G be a simple graph. Running over all loop configurations $\mathfrak{G}$ of G, the *enhanced odd cycle zero forcing number $\widehat{Z}_{oc}(G)$* for the simple graph G is

$$\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G}).$$

The proof of the next theorem follows that of Corollary 2.24 in [4].

THEOREM 3.2. *For any simple graph G and any field F with* char $F \neq 2$, $M^F(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G)$.

Proof. Let A be a matrix in $\mathcal{S}^F(G)$ such that $\mathrm{null}(A) = M^F(G)$. Following the zero-nonzero pattern on the diagonal entries of A, A must fall into $\mathcal{S}^F(\mathfrak{G})$ for some loop configuration $\mathfrak{G}$ of G. As a consequence,

$$M^F(G) = \mathrm{null}(A) \leq M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq \widehat{Z}_{oc}(G),$$

by Theorem 2.8. And again by Theorem 2.8, $Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G})$, so $\widehat{Z}_{oc}(G) \leq \widehat{Z}(G)$ by definitions. □
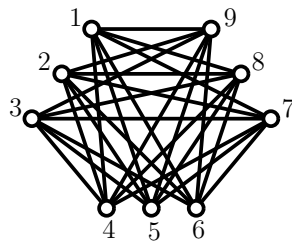

FIG. 3.1. *Labeled $K_{3,3,3}$.*

EXAMPLE 3.3. Let G be the complete tripartite graph $K_{3,3,3}$ (as a simple graph), which is shown in Figure 3.1. For this graph, we show that $Z(G) = 7 = \widehat{Z}(G)$ but $\widehat{Z}_{oc}(G) = 6 = M^{\mathbb{R}}(G)$.

We start by showing $Z(G) = 7 = \widehat{Z}(G)$. First consider the simple graph G and a zero forcing set B on G. If $|B \cap \{1, 2, 3\}| < 2$, then there is no way to turn all of $\{1, 2, 3\}$ blue. So each of the clusters $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ contains at least 2 blue vertices. But 6 vertices with 2 in each clusters is not enough to make $V(G)$ all blue. Therefore, $\{1, 2, 3, 4, 5, 7, 8\}$ is a minimum zero forcing set on G, and $Z(G) = 7$. This same argument also works for the zero forcing number of the loop graph $\mathfrak{G}^0$, where $\mathfrak{G}^0$ is the loop configuration of G without any loop. So $7 = Z(\mathfrak{G}^0) \leq \widehat{Z}(G) \leq Z(G) = 7$ and $\widehat{Z}(G) = 7$ also.

Next we show $\widehat{Z}_{oc}(G) = 6 = M^{\mathbb{R}}(G)$. Let $\mathfrak{G}$ be a loop configuration of $K_{3,3,3}$. Assume each vertex in $\{1, 2, 3\}$ has a loop. Then the initial blue set $B = \{4, 5, 6, 7, 8, 9\}$ can make all vertices blue by rule (a), so $Z_{oc}(\mathfrak{G}) \leq 6$ in this case. Similarly, $\{1, 2, 3\}$ can be replaced by the other clusters $\{4, 5, 6\}$ and $\{7, 8, 9\}$. So now assume at least one vertex in each cluster does not have a loop, say 1, 4, and 7. In this case, $\{2, 3, 5, 6, 8, 9\}$ forms a ZFS-Z_{oc}, since $\{1, 4, 7\}$ forms a loopless odd cycle and rule (b) applies. Throughout all cases, $\widehat{Z}_{oc}(G) \leq 6$. On the other hand, $M^{\mathbb{R}}(G) \geq 6$, because its adjacency matrix over $\mathbb{R}$ has nullity 6. Therefore, $\widehat{Z}_{oc}(G) = 6 = M^{\mathbb{R}}(G)$.

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Finally, if we consider the adjacency matrix of G over $\mathbb{F}_2$, the field of two elements, then its nullity is 7 instead of 6. So $7 \le M^{\mathbb{F}_2}(G) \le \widehat{Z}(G) = 7$. The discrepancy between $\widehat{Z}(G)$ and $\widehat{Z}_{oc}(G)$ is because $\widehat{Z}(G) \ge M^F(G)$ works for an arbitrary field F, but $\widehat{Z}_{oc}(G) \ge M^F(G)$ works only when char $F \ne 2$.

4. Graph blowups. The simple graph $K_{3,3,3}$ in Example 3.3 demonstrates a relation between the zero forcing number for simple graphs and that for loop graphs. Let $\mathfrak{C}_3^0$ be the loopless odd cycle on 3 vertices. The simple graph $K_{3,3,3}$ can be viewed as the simple graph obtained from $\mathfrak{C}_3^0$ by replacing each vertex with a cluster of $t = 3$ isolated vertices and replacing each edge with a complete bipartite graph joining the corresponding clusters. Example 3.3 satisfies

$$Z(K_{3,3,3}) = \widehat{Z}(K_{3,3,3}) = (t-1) \times |V(\mathfrak{C}_3^0)| + Z(\mathfrak{C}_3^0)$$

and

$$\widehat{Z}_{oc}(K_{3,3,3}) = (t-1) \times |V(\mathfrak{C}_3^0)| + Z_{oc}(\mathfrak{C}_3^0)$$

with $t = 3$.

The transformation of $\mathfrak{C}_3^0$ to $K_{3,3,3}$ is called a *blowup*. In this section, we discuss how graph blowups can bridge loop graphs and simple graphs.
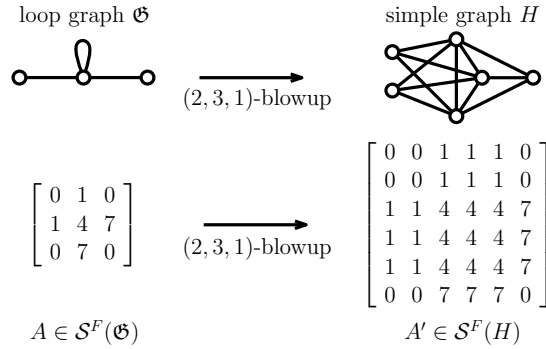

loop graph $\mathfrak{G}$ simple graph H

$(2,3,1)$-blowup

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 7 \\ 0 & 7 & 0 \end{bmatrix} \qquad \xrightarrow{(2,3,1)\text{-blowup}} \qquad A' = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 1 & 1 & 4 & 4 & 4 & 7 \\ 0 & 0 & 7 & 7 & 7 & 0 \end{bmatrix}$$

$A \in \mathcal{S}^F(\mathfrak{G})$ $A' \in \mathcal{S}^F(H)$

FIG. 4.1. *An illustration of the blowup of graphs and matrices.*

DEFINITION 4.1.   Let $\mathfrak{G}$ be a loop graph with $V(\mathfrak{G}) = \{v_i\}_{i=1}^n$, and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. The $(t_1, t_2, \ldots, t_n)$-*blowup* of $\mathfrak{G}$ is the simple graph obtained from $\mathfrak{G}$ by replacing v_i with a cluster V_i of t_i vertices and for each edge $v_i v_j$ ($i = j$ is possible), joining every vertex of V_i with every vertex of V_j.

DEFINITION 4.2.   Let $A = [a_{i,j}]$ be a symmetric matrix indexed by $\{v_i\}_{i=1}^n$. and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. Denote $n' = \sum_{i=1}^n t_i$. The $(t_1, t_2, \ldots, t_n)$-*blowup* of A is the $n' \times n'$ matrix obtained from A by replacing the i,j-entry $a_{i,j}$ of A with $a_{i,j} J_{t_i, t_j}$, where J_{t_i, t_j} is the $t_i \times t_j$ all one matrix.

These definitions are illustrated in Figure 4.1.

LEMMA 4.3. *Let $\mathfrak{G}$ be a loop graph with $V(\mathfrak{G}) = \{v_i\}_{i=1}^n$, and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. Then $\mathrm{mr}^F(H) \le \mathrm{mr}^F(\mathfrak{G})$ and $M^F(H) \ge \sum_{i=1}^n (t_i - 1) + M^F(\mathfrak{G})$ for any field F.*

Proof. Let A be a matrix in $\mathcal{S}^F(\mathfrak{G})$ and A' the $(t_1, t_2, \ldots, t_n)$-blowup of A. Then A' is a matrix in $\mathcal{S}^F(H)$. Also, since deleting repeated rows and columns does not change the rank, $\mathrm{rank}(A) = \mathrm{rank}(A')$. Therefore, $\mathrm{mr}^F(H) \le \mathrm{mr}^F(\mathfrak{G})$ and $M^F(H) \ge \sum_{i=1}^n (t_i - 1) + M^F(\mathfrak{G})$ for any field F. □

LEMMA 4.4. *Let $\mathfrak{G}$ be a loop graph on n vertices and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers with $t_i \ge 2$ for all i. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. Then*

$$Z(H) = Z(\mathfrak{H}') = \sum_{i=1}^n (t_i - 1) + Z(\mathfrak{G}),$$

where $\mathfrak{H}'$ is the loop configuration of H such that every vertex in a cluster corresponding to a clique has a loop while the others do not have a loop.

The second equality also holds if Z is replaced by Z_{oc}. That is,

$$Z_{oc}(\mathfrak{H}') = \sum_{i=1}^n (t_i - 1) + Z_{oc}(\mathfrak{G}).$$

Proof. We consider $Z(H)$ first.

Let $V(\mathfrak{G}) = \{v_i\}_{i=1}^n$ and V_i the cluster of t_i vertices. On the simple graph H, if there are two white vertices in a cluster V_i, then there is no way to make V_i all blue. So in order to be a zero forcing set on H, each V_i has at most one white vertex. This ensures $Z(H) \ge \sum_{i=1}^n (t_i - 1)$. Say the cluster V_i is *blue* if all its vertices is blue, and V_i is *one-white* if one of its vertices is white. Then $Z(H)$ will be $\sum_{i=1}^n (t_i - 1)$ plus the minimum number of blue clusters.

Assume each V_i is either blue or one-white. Denote $V_i \to V_j$ if $x \to y$ happens on H for some $x \in V_i$ and $y \in V_j$. Since $t_i \ge 2$, each one-white cluster contains at least one blue vertex. Suppose v_i has no loop in $\mathfrak{G}$, then $V_i \to V_j$ on H does not require V_i to be blue; similarly, $v_i \to v_j$ on $\mathfrak{G}$ does not require v_i to be blue. Suppose v_i has a loop in $\mathfrak{G}$, then $V_i \to V_i$ can happen when all other neighbors are blue already; this is the same case for $v_i \to v_i$. Therefore, when each cluster is either blue or one-white, $V_i \to V_j$ on H if and only if $v_i \to v_j$ on $\mathfrak{G}$. So the minimum number of blue clusters is $Z(\mathfrak{G})$ and

$$Z(H) = \sum_{i=1}^n (t_i - 1) + Z(\mathfrak{G}).$$

The same argument works when H is replaced by $\mathfrak{H}'$. It also works when rule (b) comes in. Suppose that $\{v_i\}_{i\in\alpha}$ forms a loopless odd cycle on $\mathfrak{G}$ for some index set α and rule (b) can be applied on it. Then at this step each cluster in $\{V_i\}_{i\in\alpha}$ is one-white and the only white vertices in each of them form a loopless odd cycle on $\mathfrak{H}'$. So $Z_{oc}(\mathfrak{H}') = \sum_{i=1}^{n}(t_i-1) + Z_{oc}(\mathfrak{G})$ holds. ☐

LEMMA 4.5. *Let $\mathfrak{G}$ be a loop graph on n vertices and $(t_1,t_2,\ldots,t_n)$ a sequence of n positive integers with $t_i \geq 3$ for all i. Let H be the $(t_1,t_2,\ldots,t_n)$-blowup of $\mathfrak{G}$. Then*

$$\widehat{Z}_{oc}(H) = \sum_{i=1}^{n}(t_i-1) + Z_{oc}(\mathfrak{G}).$$

Proof. Let $h = \sum_{i=1}^{n}(t_i-1) + Z_{oc}(\mathfrak{G})$. By Lemma 4.4, at least one loop configuration $\mathfrak{H}'$ of H has $Z_{oc}(\mathfrak{H}') = h$, so it is enough to show that any loop configuration $\mathfrak{H}$ of H has $Z_{oc}(\mathfrak{H}) \leq h$.

Let $\mathfrak{H}$ be a loop configuration of H and B be a minimum ZFS-Z_{oc} of $\mathfrak{G}$. We adopt the notation from Lemma 4.4. We mark V_i blue if $v_i \in B$ and one-white if $v_i \notin B$; whenever a cluster V_i is marked one-white, we pick the only white vertex to be a nonloop vertex, unless every vertex in V_i has a loop. Call this set B'. Starting with B', we can do the corresponding forces $V_i \to V_j$ whenever $v_i \to v_j$ happens in $\mathfrak{G}$. If rule (b) never applies in $\mathfrak{G}$, then B' is a ZFS-Z_{oc} of $\mathfrak{H}$ with $|B'| = h$ and we are done. So assume rule (b) first happens at some step, and it applies to a loopless odd cycle $\mathfrak{C}$ on $\mathfrak{G}$. Denote $V(\mathfrak{C}) = \{v_i\}_{i\in\alpha}$ for some index set α. If every cluster V_i in $\{V_i\}_{i\in\alpha}$ contains a nonloop vertex on the loop configuration $\mathfrak{H}$, then by the choice of B' there is a loopless odd cycle on $\mathfrak{H}$ and the process continues. Now assume at least one cluster V_a has all its vertices with loops. Say v_b and v_c are the two neighbors of v_a in $\mathfrak{C}$. We modify B' by marking V_b and V_c blue, and setting all vertices in V_a as white. This modification does not increase the number of blue vertices, since marking V_b and V_c blue add two blue vertices, but setting all V_a white loses at least two blue vertices by the fact $t_a \geq 3$. Note that V_a is an independent set since $\mathfrak{C}$ is loopless, and all its vertices has loops. By starting with the new B', the same process can go on until rule (b) applies to $\mathfrak{C}$. At this step, v_a has only two white neighbors v_b and v_c; this means at the stage where rule (b) was applied in $\mathfrak{G}$ every vertex in V_a can force itself blue, since V_b and V_c are blue initially. Now by applying rule (a) only, every cluster in $\{V_i\}_{i\in\alpha}$ turns blue eventually, and the process continues. Since all loopless odd cycles given by rule (b) are mutually isolated by Proposition 2.5, we can do the modification separately, and find a ZFS-Z_{oc} of $\mathfrak{H}$ with cardinality less than or equal to h. Therefore, $Z_{oc}(\mathfrak{H}) \leq h$ and $\widehat{Z}_{oc}(H) = h$. ☐

REMARK 4.6.   In Lemma 4.5, the assumption $t_i \geq 3$ for all i can be relaxed to $t_i \geq 3$ whenever v_i has no loop in $\mathfrak{G}$ and $t_i \geq 2$ otherwise.

THEOREM 4.7. *Let $\mathfrak{G}$ be a loop graph on n vertices and $(t_1, t_2, \ldots, t_n)$ a sequence of n positive integers. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. If $M^F(\mathfrak{G}) = Z(\mathfrak{G})$ for some field F and $t_i \geq 2$ for all i, then*

$$M^F(H) = Z(H) = \widehat{Z}(H) = \sum_{i=1}^{n}(t_i - 1) + M^F(\mathfrak{G}).$$

If $M^F(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ for some field F with $\operatorname{char} F \neq 2$ and $t_i \geq 3$ for all i, then

$$M^F(H) = \widehat{Z}_{oc}(H) = \sum_{i=1}^{n}(t_i - 1) + M^F(\mathfrak{G}).$$

Proof. This immediately comes from Lemma 4.3, Lemma 4.4, Lemma 4.5, and Theorem 2.8. ☐

COROLLARY 4.8. *Let G be a tree, a cycle, or a complete graph, and $\mathfrak{G}$ its loop configuration with $V(\mathfrak{G}) = \{v_i\}_{i=1}^{n}$. Let H be the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$ with $t_i \geq 3$ for all i. Then $M^{\mathbb{R}}(H) = \widehat{Z}_{oc}(H)$. Moreover, $M^{\mathbb{R}}(H) = Z(H)$ if G is a tree.*

Proof. If G is a tree, then $M^{\mathbb{R}}(\mathfrak{G}) = Z(\mathfrak{G})$ [5]; if G is a cycle or a complete graph, then $M^{\mathbb{R}}(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ by Corollary 2.9. By applying Theorem 4.7, the equality holds. ☐

Example 3.3 together with Lemma 4.4 and Theorem 4.7 also provide a family of simple graphs with large $\widehat{Z}(G) - \widehat{Z}_{oc}(G)$.

COROLLARY 4.9. *Let $G_n = K_1 \bigvee(nK_3)$ be the simple graph in Example 2.10 and $\mathfrak{G}_n^0$ the loop configuration of G_n without any loop. Let H_n be the $(3, 3, \ldots, 3)$-blowup of G_n. Then*

$$\widehat{Z}_{oc}(H_n) = 2 \cdot |V(\mathfrak{G}_n^0)| + 1 \ and \ \widehat{Z}(H_n) = 2 \cdot |V(\mathfrak{G}_n^0)| + n + 1.$$

5. Graph complement conjecture for $\widehat{Z}(G)$. The *graph complement conjecture* for the maximum nullity (GCC-M) for simple graphs [2] states that

$$M^F(G) + M^F(\overline{G}) \geq n - 2,$$

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where $\overline{G}$ is the complement of G. Corollary 5.1 below shows GCC-M is true for most graph blowups.

COROLLARY 5.1. *Let $\mathfrak{G}$ be a loop graph and H the $(t_1, t_2, \ldots, t_n)$-blowup of $\mathfrak{G}$. If $t_i \geq 2$ for all i, then GCC-M is true for H over any field F. That is,*

$$M^F(H) + M^F(\overline{H}) \geq |V(H)| - 2.$$

Proof. Notice that $\overline{H}$ is the $(t_1, t_2, \ldots, t_n)$-blowup of $\overline{\mathfrak{G}}$, where $\overline{\mathfrak{G}}$ is the complement of $\mathfrak{G}$ as loop graphs; that is, there is an edge (or a loop) between v_i and v_j in $\overline{\mathfrak{G}}$ if and only if there is no edge (or no loop) between v_i and v_j in $\mathfrak{G}$.

By Lemma 4.3, $M^F(H) \geq \frac{1}{2}|V(H)|$ since $t_i \geq 2$ for all i. Similarly, $M^F(\overline{H}) \geq \frac{1}{2}|V(H)|$. So

$$M^F(H) + M^F(\overline{H}) \geq |V(H)| > |V(H)| - 2,$$

and GCC-M holds for H. $\square$

If β is a graph parameter for simple graphs, the graph complement conjecture for β (GCC-β) is stated as

$$\beta(G) + \beta(\overline{G}) \geq n - 2.$$

In [9], GCC-tw, GCC-Z_+, and GCC-Z are proven to be true, where tw is the *tree-width*, Z_+ is the *positive semidefinite zero forcing number*, and Z is the zero forcing number for simple graphs. The relation between these parameters can be found in Fig. 1.1 of [4].

We claim that GCC-Z_+ implies GCC-$\widehat{Z}_{oc}$, so GCC-$\widehat{Z}_{oc}$ is also true. We need an intermediate parameter. The *loop zero forcing number* $Z_\ell(G)$ for simple graphs G is defined as $Z(\mathfrak{G})$, where $\mathfrak{G}$ is the loop configuration of G such that isolated vertices have no loop while the others have a loop [4]. Since rule (b) can never apply on $\mathfrak{G}$, $Z(\mathfrak{G}) = Z_{oc}(\mathfrak{G})$ and $Z_\ell(G) = Z_{oc}(\mathfrak{G}) \leq \widehat{Z}_{oc}(G)$. Also, it is known that $Z_+(G) \leq Z_\ell(G)$ [4]. Therefore, $Z_+(G) \leq \widehat{Z}_{oc}(G)$ for every simple graph G, which means GCC-Z_+ implies GCC-$\widehat{Z}_{oc}$.

6. Examples with $\widehat{Z}_{oc}(G) - M^{\mathbb{R}}(G)$ large. A 5-*sun*, H_5, is a simple graph obtained from C_5 by appending a leaf to each vertex. It is known that $M^{\mathbb{R}}(G) = 2 = \widehat{Z}(G)$ but $Z(G) = 3$ [1,4]. Thus, $\widehat{Z}_{oc}(G) = 2$, by Theorem 2.8. To get a discrepancy between $\widehat{Z}_{oc}(G)$ and $M^{\mathbb{R}}(G)$, we insert one more leaf to each of the leaves of H_5 and call it a *long* 5-*sun*, denoted as LH_5. A *long* 5-*sun sequence* of length n is the simple graph shown in Figure 6.1, which concatenates n copies of LH_5. Proposition 6.1 shows
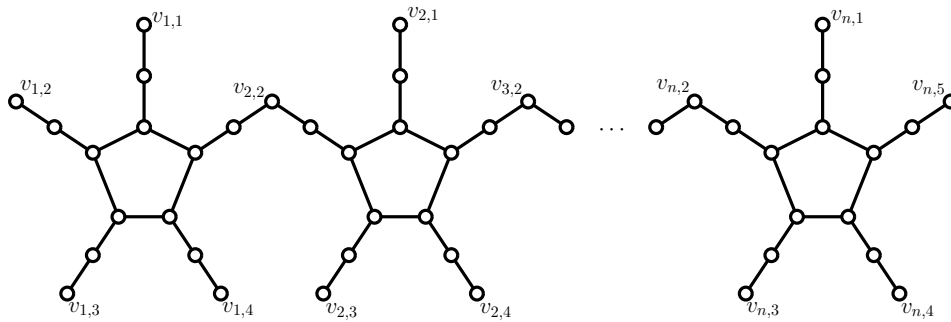
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FIG. 6.1. *A sequence of long* 5-*sun.*

that for this family of graphs and hence in general for simple graphs, $\widehat{Z}_{oc}(G) - M^{\mathbb{R}}(G)$ can be arbitrarily large.

PROPOSITION 6.1. *Let* L_n *be the long* 5-*sun sequence of length* n *described above. Then* $M^{\mathbb{R}}(L_n) = n + 1$ *and* $\widehat{Z}_{oc}(L_n) = \widehat{Z}(L_n) = Z(L_n) = 2n + 1$.

Proof. The cut-vertex reduction formula [6] states that

$$M^{\mathbb{R}}(G_1 \oplus_v G_2) = \max\{M^{\mathbb{R}}(G_1) + M^{\mathbb{R}}(G_2) - 1, M^{\mathbb{R}}(G_1 - v) + M^{\mathbb{R}}(G_2 - v) - 1\},$$

where $G_1 \oplus_v G_2$ is obtained from G_1 and G_2 by identifying the vertex v on each of them. Suppose x is a leaf on a graph G and y is the only neighbor of x. Then by applying the formula on y, one immediate observation is $M^{\mathbb{R}}(G) \geq M^{\mathbb{R}}(G - x)$; additionally, if y is of degree 2, then $M^{\mathbb{R}}(G) = M^{\mathbb{R}}(G - x)$. Therefore, $M^{\mathbb{R}}(L_1) = M^{\mathbb{R}}(H_5) = 2$. Now write L_n as $L_{n-1} \oplus_v L_1$, where v is the vertex $v_{n-1,5}$ in L_{n-1} and a leaf in L_1. Since v is a leaf on L_{n-1} and on L_1, the observation reduces the formula to be $M^{\mathbb{R}}(L_n) = M^{\mathbb{R}}(L_{n-1}) + M^{\mathbb{R}}(L_1) - 1 = M^{\mathbb{R}}(L_{n-1}) + 1$. Inductively, $M^{\mathbb{R}}(L_n) = n + 1$.

For zero forcing numbers, the set $\{v_{1,1}, v_{1,2}, v_{1,3}\} \bigcup \{v_{i,1}, v_{i,3}\}_{i=2}^n$ labeled in Figure 6.1 forms a zero forcing set on the simple graph L_n. So $Z(L_n) \leq 2n + 1$.

On the other hand, we show $Z_{oc}(\mathfrak{L}_n^\ell) = 2n + 1$, where $\mathfrak{L}_n^\ell$ is the loop configuration of L_n so that each vertex has a loop. First we make some observations. By Proposition 2.5, the maximal chains on $\mathfrak{L}_n^\ell$ can only be directed 1-cycles or directed paths, and the number of directed paths is the cardinality of the ZFS-Z_{oc}. Even more, there are no edges between any two distinct directed 1-cycles. Let $x \in V(\mathfrak{L}_n^\ell)$ be a pendent vertex, which means x has only one neighbor y other than itself. Let π_x and π_y be the maximal chain containing x and y respectively, where $\pi_x = \pi_y$ is possible. By the structure of $\mathfrak{L}_n^\ell$, if π_x is a directed path, then x is an endpoint of π_x; if π_x is a directed 1-cycle, then π_y must be a directed path and y is an endpoint of π_y. In either case, $\{x, y\}$ must contain an endpoint for some maximal chain. Now we claim $Z_{oc}(\mathfrak{L}_n^\ell) \geq 2n + 1$ by induction on n. For $n = 1$, there are 5 pendent vertices in $\mathfrak{L}_1^\ell$. Each

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directed path has only 2 endpoints, so $\lceil \frac{5}{2} \rceil = 3$ directed paths are needed. Assume $Z_{oc}(\mathfrak{L}_{n-1}^{\ell}) \geq 2n - 1$. Note that $\mathfrak{L}_n^{\ell}$ is obtained from $\mathfrak{L}_{n-1}^{\ell}$ by attaching the last copy of $\mathfrak{L}_1^{\ell}$, where $V(\mathfrak{L}_{n-1}^{\ell}) \bigcap V(\mathfrak{L}_1^{\ell}) = \{v_{n,2}\}$. There are still 4 pendent vertices on $V(\mathfrak{L}_1^{\ell})$. Only one of the 4 vertices can combine with a directed path from $\mathfrak{L}_{n-1}^{\ell}$. So at least $2n - 1 + \lceil \frac{4-1}{2} \rceil = 2n + 1$ directed paths are needed for $\mathfrak{L}_n^{\ell}$. This means

$$2n + 1 \leq Z_{oc}(\mathfrak{L}_n^{\ell}) \leq \widehat{Z}_{oc}(L_n) \leq \widehat{Z}(L_n) \leq Z(L_n) \leq 2n + 1.$$

Hence, every inequality is an equality. $\square$

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