ELA

RIGHT GUT-MAJORIZATION ON $\mathbf{M}_{N,M}$*

ASMA ILKHANIZADEH MANESH†

Abstract. Let $\mathbf{M}_{n,m}$ be the set of all n-by-m matrices with entries from $\mathbb{R}$, and suppose that $\mathbb{R}_n$ is the set of all 1-by-n real row vectors. A matrix R is called *generalized row stochastic (g-row stochastic)* if the sum of entries on every row of R is 1. For $X, Y \in \mathbf{M}_{n,m}$, it is said that X is *rgut-majorized* by Y (denoted by $X \prec_{rgut} Y$) if there exists an m-by-m upper triangular g-row stochastic matrix R such that $X = YR$. In this paper, the concept right upper triangular generalized row stochastic majorization, or rgut- majorization, is investigated and then the linear preservers and strong linear preservers of this concept are characterized on $\mathbb{R}_n$ and $\mathbf{M}_{n,m}$.

Key words. G-row stochastic matrix, (Strong) Linear preserver, Right gut-majorization.

AMS subject classifications. 15A04, 15A21.

1. Introduction. Majorization is a pre-ordering on vectors by sorting all components in non-increasing order, i.e., for each $x, y \in \mathbb{R}^n$ the vector x is said to be majorized by y and it is denoted by $x \prec y$, if $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$ for all $1 \leq k \leq n$ with eaquality for $k = n$, where $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ the non-increasing rearrangement of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The history of investigating majorization dates back to Schur [16] and Hardy et al. [9]. The reader can find that majorization has been connected with combinatorics, analytic inequalities, numerical analysis, matrix theory, probability and statistics in a book written by Marshall, Olkin, and Arnold [15]. In 1989, Ando in a vital paper [1] that was about majorization, characterized the structure of linear preservers of majorization. Dahl (1991) generalized the majorization concept to matrices. In 1994, Ando [2] gave a detailed survey of research done in the theory of majorization. In 2005, Chiang and Li [8] introduced generalized doubly stochastic matrices. In 2006, Salemi and Armandnejad used the generalized stochastic matrices and they introduced the notion of generalized majorization for matrices (see [6]). By introducing this notion many questions were raised, and some of them have been answered. We refer the interested reader to [4], [5], and [12]-[14].

A (not necessarily nonnegative) matrix R is called *g-row stochastic* if the sum of entries of every row of R is 1. Some of our notations and symbols are explained as the following.The set of all n-by-m real matrices is denoted by $\mathbf{M}_{n,m}$. The set

†Department of Mathematics, Vali-e-Asr University of Rafsanjan, PO Box 7713936417, Rafsanjan, Iran (a.ilkhani@vru.ac.ir, ailkhanizade@gmail.com).

ELA

of all n-by-1 real column vectors is denoted by $\mathbb{R}^n$. The set of all 1-by-n real row vectors is denoted by $\mathbb{R}_n$. The collection of all n-by-n upper triangular g-row stochastic matrices is denoted by $\mathcal{R}_n^{gut}$. The n-by-n matrix with all of the entries of the last column equal to one and the other entries equal to zero is denoted by E. The standard basis of $\mathbb{R}^n$ is denoted by $\{e_1, \ldots, e_n\}$. The standard basis of $\mathbb{R}_n$ is denoted by $\{\varepsilon_1, \ldots, \varepsilon_n\}$. The n-by-n matrix whose (i,j) entry is one and all other entries are zero is denoted by E_{ij}. The submatrix of A obtained from A by deleting rows $n_1, \ldots, n_l$ and columns $m_1, \ldots, m_k$ is denoted by $A(n_1, \ldots, n_l|m_1, \ldots, m_k)$. The abbreviation of $A(n_1, \ldots, n_l|n_1, \ldots, n_l)$ is denoted by $A(n_1, \ldots, n_l)$. The n-by-m matrix with columns $x_1, \ldots, x_m \in \mathbb{R}^n$ is denoted by $[x_1 \mid \cdots \mid x_m]$. The m-by-n matrix with rows $x_1, \ldots, x_m \in \mathbb{R}_n$ is denoted by $[x_1/\cdots/x_m]$. The summation of all components of a vector x in $\mathbb{R}_n$ is denoted by $tr(x)$. The set of all n-by-n permutation matrices is denoted by $\mathcal{P}_n$. The set $\{1, \ldots, k\} \subset \mathbb{N}$ is denoted by $\mathbb{N}_k$. The transpose of a given matrix A is denoted by A^t. The matrix representation of a linear function $T : \mathbb{R}_n \to \mathbb{R}_m$ with respect to the standard basis is denoted by $[T]$. The summation of all entries of ith row of $[T]$ is denoted by r_i. The ith column of the matrix representation of a linear function T is denoted by $[T]_i$. The set $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N}, \sum_{i=1}^m \lambda_i = 1, a_i \in A, \lambda_i \in \mathbb{R}, \forall i \in \mathbb{N}_m\}$, where $A \subseteq \mathbb{R}_n$, is denoted by aff(A). A linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserves an order relation $\prec$ in $\mathbf{M}_{n,m}$, if $TX \prec TY$ whenever $X \prec Y$. Also, T is said to strongly preserve if for all $X, Y \in \mathbf{M}_{n,m}$

$$X \prec Y \Leftrightarrow TX \prec TY.$$

This paper is organized as follows. In Section 2, we first introduce the relation $\prec_{rgut}$ on $\mathbb{R}_n$ and we express an equivalent condition for rgut-majorization. Finally, we obtain some results characterizing the structure of (strong) linear preservers of this relation on $\mathbb{R}_n$. One of the main results of this paper is to find the structure of linear functions $T : \mathbb{R}_n \to \mathbb{R}_m$ ($T : \mathbb{R}_n \to \mathbb{R}_n$) preserving (resp. strongly preserving) rgut-majorization. The last section of this paper studies some facts of this concept that are necessary for studying the strong linear preservers of $\prec_{rgut}$ on $\mathbf{M}_{n,m}$. Also, the strong linear preservers of $\prec_{rgut}$ on $\mathbf{M}_{n,m}$ are obtained.

2. Rgut-majorization on $\mathbb{R}_n$ and its (strong) linear preservers. When we use the doubly stochastic matrices for majorization, since the transpose of a doubly stochastic matrix is doubly stochastic too, we can obtain the concepts of left and right majorization from each other by getting transpose on the equations. But when we use the row stochastic matrices, we can not obtain the left and right majorization from each other. So, in this case, the left and right concepts are investigated in different manners. For example the concept of left matrix majorization and gw-majorization were studied in [10] and [7] respectively, but the right cases were investigated in [11]

ELA

http:/repository.uwyo.edu/ela

and [3] respectively. In this paper, we study the right case of a concept which has been investigated in [5].

In this section, we pay attention to the g-row stochastic upper triangular matrices and we introduce a new type of majorization. We obtain an equivalent condition for rgut-majorization on $\mathbb{R}_n$ and some preliminaries about $\prec_{rgut}$. Also, we characterize all linear functions $T : \mathbb{R}_n \to \mathbb{R}_m$ ($T : \mathbb{R}_n \to \mathbb{R}_n$) preserving (resp., strongly preserving) $\prec_{rgut}$.

DEFINITION 2.1. *Let $X, Y \in \mathbf{M}_{n,m}$. The matrix X is said to be rgut-majorized by Y (in symbol $X \prec_{rgut} Y$) if $X = YR$, for some $R \in \mathcal{R}_m^{gut}$.*

The following proposition gives an equivalent condition for rgut-majorization on $\mathbb{R}_n$. We state that without proof.

PROPOSITION 2.2. *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}_n$. Then $x \prec_{rgut} y$ if and only if $tr(x) = tr(y)$ and $x_i \in \mathrm{aff}(0, y_1, \ldots, y_i)$, for all $i \in \mathbb{N}_{n-1}$.*

The following lemmas are useful for finding the structure of (strong) linear preservers of rgut-majorization. Now, we may begin with the following lemma which is essential in the text.

LEMMA 2.3. *Suppose that $T : \mathbb{R}_n \to \mathbb{R}_m$ is a linear preserver of $\prec_{rgut}$, and let $[T] = [a_{ij}]$. Then the following assertions are true.*

a) $r_1 = \cdots = r_n$.

b) *If $a_{k+1i} = \cdots = a_{ni} = 0$, for each $i \in \mathbb{N}_l$, and $S : \mathbb{R}_{n-k} \to \mathbb{R}_{m-l}$ is the linear function with $[S] = [T](1, \ldots, k | 1, \ldots, l)$, then S preserves $\prec_{rgut}$.*

c) *The first column of $[T]$ is $\begin{pmatrix} a_{11} & 0 & \cdots & 0 \end{pmatrix}^t$, or $\begin{pmatrix} a_{11} & a_{11} & \cdots & a_{11} \end{pmatrix}^t$.*

d) *If there exists $i \in \mathbb{N}_m$ such that for all $l \in \mathbb{N}_{i-1}$ $[T]_l = \begin{pmatrix} a_{1l} & a_{1l} & \cdots & a_{1l} \end{pmatrix}^t$, then $[T]_i = \begin{pmatrix} a_{1i} & a_{2i} & \cdots & a_{2i} \end{pmatrix}^t$.*

e) *If there exists $i \in \mathbb{N}_m$ such that*

$$[T] = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1i-1} & * & \cdots & * \\
a_{11} & a_{22} & a_{23} & \cdots & a_{2i-1} & * & \cdots & * \\
a_{11} & a_{22} & a_{33} & \cdots & a_{3i-1} & * & \cdots & * \\
\vdots & \vdots & \vdots & & \vdots & & & \\
a_{11} & a_{22} & a_{33} & \cdots & a_{i-2i-1} & * & \cdots & * \\
a_{11} & a_{22} & a_{33} & \cdots & a_{i-1i-1} & * & \cdots & * \\
\vdots & \vdots & \vdots & & \vdots & & & \\
a_{11} & a_{22} & a_{33} & \cdots & a_{i-1i-1} & * & \cdots & *
\end{pmatrix},$$

ELA

http:/repository.uwyo.edu/ela

then $[T]_i = \begin{pmatrix} a_{1i} & a_{2i} & \cdots & a_{i-1i} & a_{ii} & \cdots & a_{ii} \end{pmatrix}^t.$

Proof. *a*) Let $i \in \mathbb{N}_n$. Since $\varepsilon_i \prec_{rgut} \varepsilon_1$, we observe that $T\varepsilon_i \prec_{rgut} T\varepsilon_1$, and hence, $r_i = r_1$.

b) Let $x' = (x_{k+1}, \ldots, x_n)$, $y' = (y_{k+1}, \ldots, y_n) \in \mathbb{R}_{n-k}$, and let $x' \prec_{rgut} y'$. Proposition 2.2 ensures that $x := (0, \ldots, 0, x_{k+1}, \ldots, x_n) \prec_{rgut} y := (0, \ldots, 0, y_{k+1}, \ldots, y_n)$, where $x, y \in \mathbb{R}_n$, and hence, $Tx \prec_{rgut} Ty$. This implies that $Sx' \prec_{rgut} Sy'$. Therefore, S preserves $\prec_{rgut}$, as desired.

c) First, we prove $a_{21} = \cdots = a_{n1}$. If there exist some j and k ($2 \leq j < k \leq n$) such that $a_{j1} \neq a_{k1}$, then by defining $x := \varepsilon_j - \varepsilon_k$ and $y := \varepsilon_1 + (-1 - \frac{a_{j1}-a_{11}}{a_{k1}-a_{j1}})\varepsilon_j + (\frac{a_{j1}-a_{11}}{a_{k1}-a_{j1}})\varepsilon_k$, we see that $x \prec_{rgut} y$, but $Tx \not\prec_{rgut} Ty$, which is a contradiction. So $a_{21} = \cdots = a_{n1}$. If $a_{21} = 0$, then $[T]_1 = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \end{pmatrix}^t$. If $a_{21} \neq 0$; since T preserves $\prec_{rgut}$ if and only if αT preserves $\prec_{rgut}$, for all $\alpha \in \mathbb{R} \setminus \{0\}$, there is no loss of generality in assuming $a_{21} = 1$. By choosing $x = (1 - a_{11})\varepsilon_2$ and $y = \varepsilon_1 - a_{11}\varepsilon_2$, we obtain $x \prec_{rgut} y$, but $Tx \not\prec_{rgut} Ty$, in contradiction to the hypothesis that T preserves $\prec_{rgut}$. Therefore, $[T]_1 = \begin{pmatrix} a_{11} & a_{11} & \cdots & a_{11} \end{pmatrix}^t$.

d) The proof is quite similar to (*c*). Suppose that there exist j and k ($2 \leq j < k \leq n$) such that $a_{ji} \neq a_{ki}$. Put $x := \varepsilon_j - \varepsilon_k$ and $y := \varepsilon_1 + (-1 - \frac{a_{ji}-a_{1i}}{a_{ki}-a_{ji}})\varepsilon_j + (\frac{a_{ji}-a_{1i}}{a_{ki}-a_{ji}})\varepsilon_k$. It easy to see that $x \prec_{rgut} y$ and $Tx \not\prec_{rgut} Ty$, which would be a contradiction. Therefore, $a_{2i} = \cdots = a_{ni}$.

e) If there exist some j and k ($i \leq j < k \leq n$) such that $a_{ji} \neq a_{ki}$, then $x := \varepsilon_j - \varepsilon_k \prec_{rgut} y := \varepsilon_{i-1} + (-1 - \frac{a_{ji}-a_{i-1i}}{a_{ki}-a_{ji}})\varepsilon_j + (\frac{a_{ji}-a_{i-1i}}{a_{ki}-a_{ji}})\varepsilon_k$, but $Tx \not\prec_{rgut} Ty$. This contradiction implies that $a_{ii} = a_{i+1i} = \cdots = a_{ni}$. $\square$

Define

$$(2.1) \qquad A_j := \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ \alpha_1^j & \alpha_2^j & \cdots & \alpha_{t_j}^j \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbf{M}_{n,t_j},$$

where $j \geq 1$, $\alpha_1^j \neq 0$, and $\begin{pmatrix} \alpha_1^j & \alpha_2^j & \cdots & \alpha_{t_j}^j \end{pmatrix}$ is the jth row of A_j.

Next, note that k_j is the number of columns of B_j. Also, define

$$(2.2) \qquad B_1 := \begin{pmatrix} \alpha_1^j & \alpha_2^j & \cdots & \alpha_{k_1}^j \\ \vdots & \vdots & & \vdots \\ \alpha_1^j & \alpha_2^j & \cdots & \alpha_{k_1}^j \end{pmatrix} \in \mathbf{M}_{n,k_1},$$

where $\alpha_1^j \neq 0$, and

$$(2.3) \qquad B_j := \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ \beta_1^j & \beta_2^j & \cdots & \beta_{k_j}^j \\ \alpha_1^j & \alpha_2^j & \cdots & \alpha_{k_j}^j \\ \vdots & \vdots & & \vdots \\ \alpha_1^j & \alpha_2^j & \cdots & \alpha_{k_j}^j \end{pmatrix} \in \mathbf{M}_{n,k_j},$$

where $j \geq 2$, $\begin{pmatrix} \beta_1^j & \beta_2^j & \cdots & \beta_{k_j}^j \end{pmatrix}$ is the $j-1^{th}$ row of B_j, and $\alpha_i^j \neq \beta_i^j$, for each $i \in \mathbb{N}_{k_j}$.

LEMMA 2.4. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function such that $r_1 = \cdots = r_n$. Suppose that one of the following conditions holds:*

a) $[T] = \begin{pmatrix} B_1 & \cdots & B_l & [T]_m \end{pmatrix}$,

b) $[T] = \begin{pmatrix} B_1 & \cdots & B_{n-1} & * & [T]_m \end{pmatrix}$,

where B_1 and B_j $(j \geq 2)$ are the same as in (2.2) and (2.3), respectively, and $\sum_{j=1}^{l} k_j = m - 1$. Then T preserves $\prec_{rgut}$.

Proof. Let us suppose that $[T] = \begin{pmatrix} B_1 & \cdots & B_l & * & [T]_m \end{pmatrix}$. We know that $B_j \in \mathbf{M}_{n,k_j}$, for each $j \in \mathbb{N}_l$. We prove that if $l = n - 1$ or $\sum_{j=1}^{l} k_j = m - 1$ (that is, $[T] = \begin{pmatrix} B_1 & \cdots & B_l & [T]_m \end{pmatrix}$), then T preserves $\prec_{rgut}$. First, suppose that $k_1 = \cdots = k_l = 1$. Then

$$[T] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{11} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{11} & a_{22} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{11} & a_{22} & a_{33} & \cdots & a_{nm} \end{pmatrix},$$

and $a_{11} \neq 0$, $a_{12} \neq a_{22}$, $a_{23} \neq a_{33}$, and so on. Let $x, y \in \mathbb{R}_n$, and let $x \prec_{rgut} y$. Then

$$Tx = \left(a_{11}tr(x), a_{12}x_1 + a_{22}\left(\sum_{i=2}^{n} x_i\right), \ldots, \sum_{i=1}^{l-1} a_{il}x_i + a_{ll}\left(\sum_{i=l}^{n} x_i\right), \ldots, \sum_{i=1}^{n} a_{im}x_i \right),$$

and

$$Ty = \left(a_{11}tr(y), a_{12}y_1 + a_{22}\left(\sum_{i=2}^{n} y_i\right), \ldots, \sum_{i=1}^{l-1} a_{il}y_i + a_{ll}\left(\sum_{i=l}^{n} y_i\right), \ldots, \sum_{i=1}^{n} a_{im}y_i \right).$$

We see that $tr(Tx) = tr(Ty)$. If $tr(y) \neq 0$, then $Tx \prec_{rgut} Ty$. Otherwise, $tr(y) = 0$, and consequently, $(Tx)_2 = (a_{12} - a_{22})x_1$ and $(Ty)_2 = (a_{12} - a_{22})y_1$. If $(Ty)_2 \neq 0$, then $Tx \prec_{rgut} Ty$. If $(Ty)_2 = 0$; this means that $y_1 = 0$, and so $(Tx)_2 = 0$, and hence, $(Tx)_3 = (a_{23} - a_{33})x_2$ and $(Ty)_3 = (a_{23} - a_{33})y_2$. By continuing this, we immediately observe that if one of $y_2, \ldots, y_l \neq 0$, then $Tx \prec_{rgut} Ty$. If not; so $y_2 = \cdots = y_l = 0$, and then $(Tx)_i = (Ty)_i = 0$, for each $i \in \mathbb{N}_l$. Thus, if $[T] = \begin{pmatrix} B_1 & \cdots & B_l & [T]_m \end{pmatrix}$, then $Tx \prec_{rgut} Ty$. If $l = n - 1$, then $[T] = \begin{pmatrix} B_1 & \cdots & B_{n-1} & * & [T]_m \end{pmatrix}$, and clearly, $Tx \prec_{rgut} Ty$.

Next, assume that there exists k_j for some $j \in \mathbb{N}_l$ such that $k_j > 1$. In a similar fashion the same as above, we can complete the proof. □

LEMMA 2.5. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function such that one of the following conditions holds.*

a) $[T] = \begin{pmatrix} A_1 & \cdots & A_n & * & [T]_m \end{pmatrix}$, *where A_j $(j \in \mathbb{N}_n)$ is the same as in (2.1).*

b) $[T] = \begin{pmatrix} A'_1 & \cdots & A'_k & * \\ 0 & \cdots & 0 & B \end{pmatrix}$, *where $A'_j \in \boldsymbol{M}_{k,t_j}$ $(j \in \mathbb{N}_k)$ is the same as in (2.1), and $B \in \boldsymbol{M}_{n-k,m-\sum_{j=1}^{k} t_j}$ can be the zero matrix, or one of the forms (a) or (b) of Lemma 2.4.*

If $r_1 = \cdots = r_n$, then T preserves $\prec_{rgut}$.

Proof. Let $x, y \in \mathbb{R}_n$, and let $x \prec_{rgut} y$. In all cases it easy to check that $tr(Tx) = tr(Ty)$. First, suppose that $[T] = \begin{pmatrix} A_1 & \cdots & A_n & * & [T]_m \end{pmatrix}$. As we explained in the proof of Lemma 2.4, we can suppose that $t_1 = \cdots = t_n = 1$. So we have

$$[T] = \begin{pmatrix} a_{11} & a_{12} & * & \cdots & a_{1m} \\ 0 & a_{22} & * & \cdots & a_{2m} \\ 0 & 0 & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & a_{nm} \end{pmatrix},$$

ELA

http://repository.uwyo.edu/ela

where $a_{11} \neq 0$, $a_{22} \neq 0$, and so on. Then

$$Tx = \left(a_{11}x_{11}, a_{12}x_1 + a_{22}x_2, \ldots, \sum_{i=1}^{n} a_{in}x_i, * \right),$$

and

$$Ty = \left(a_{11}y_{11}, a_{12}y_1 + a_{22}y_2, \ldots, \sum_{i=1}^{n} a_{in}y_i, * \right).$$

If $y_1 \neq 0$, then $Tx \prec_{rgut} Ty$. Otherwise, let $y_1 = 0$, and then $(Tx)_2 = a_{22}x_2$ and $(Ty)_2 = a_{22}y_2$. If $y_2 \neq 0$, then there is nothing to prove. If $y_2 = 0$; so $(Tx)_2 = (Ty)_2 = 0$, $(Tx)_3 = a_{33}x_3$, and $(Ty)_3 = a_{33}y_3$. By continuing this, we see that if one of $y_3, \ldots, y_n \neq 0$, then $Tx \prec_{rgut} Ty$. If not; then $y = 0$, and so $Tx \prec_{rgut} Ty$.

Next, assume that (b) holds. Without loss of generality we can suppose that $t_1 = \cdots = t_k = 1$. Then

$$Tx = \left(a_{11}x_{11}, a_{12}x_1 + a_{22}x_2, \ldots, \sum_{i=1}^{k} a_{ik}x_i, * \right),$$

and

$$Ty = \left(a_{11}y_{11}, a_{12}y_1 + a_{22}y_2, \ldots, \sum_{i=1}^{k} a_{ik}y_i, * \right).$$

In a similar fashion as in the proof of (a), if one of $y_1, \ldots, y_k \neq 0$, then $Tx \prec_{rgut} Ty$. Otherwise, $(Tx)_i = (Ty)_i = 0$, for each $i \in \mathbb{N}_k$, and Lemma 2.4 ensures then that $Tx \prec_{rgut} Ty$. $\square$

We are now ready to prove one of the main theorems of this section.

THEOREM 2.6. *Let $T : \mathbb{R}_n \to \mathbb{R}_m$ be a linear function. Then T preserves $\prec_{rgut}$ if and only if $r_1 = \cdots = r_n$ and there exists a permutation matrix $P \in \mathcal{P}_m$ such that one of the following conditions occurs.*

a) $[T] = 0$,

b) $[T] = \begin{pmatrix} B_1 & \cdots & B_{n-1} & * & [T]_m \end{pmatrix} P$,

c) $[T] = \begin{pmatrix} B_1 & \cdots & B_l & [T]_m \end{pmatrix} P$,

d) $[T] = \begin{pmatrix} A_1 & \cdots & A_n & * & [T]_m \end{pmatrix} P$,

where B_1, B_j $(j \geq 2)$, and A_j $(j \in \mathbb{N}_n)$ are the same as in (2.2), (2.3), (2.1), respectively, in (b) $\sum_{j=1}^{n-1} k_j \leq m - 1$ and in (c) $\sum_{j=1}^{l} k_j = m - 1$.

e) $[T] = \begin{pmatrix} A'_1 & \cdots & A'_k & * \\ 0 & \cdots & 0 & B \end{pmatrix} P$, where $A'_j \in \boldsymbol{M}_{k,t_j}$ $(j \in \mathbb{N}_k)$ is the same as in (2.1), and $B \in \boldsymbol{M}_{n-k,m-\sum_{j=1}^{k} t_j}$ can be the zero matrix, or one of the forms (a) or (b) of Lemma 2.4.

Proof. Let us first prove the sufficiency condition. Clearly, if $[T] = 0$, then T preserves $\prec_{rgut}$. If (b) or (c) holds from Lemma 2.4, and if (d) or (e) holds by Lemma 2.5, then T preserves $\prec_{rgut}$.

To prove the necessity of the conditions, assume that T preserves $\prec_{rgut}$ and (a) does not hold. Suppose that the first nonzero column of $[T]$ is the ith column. Lemma 2.3 ensures that there are two possibilities, the first of which is $[T]_i = (a_{1i} \ a_{1i} \ \ldots \ a_{1i})^t$. In this event, Lemma 2.3 ensures that

$$[T]_{i+1} = \begin{pmatrix} a_{1i+1} & a_{1i+1} \ldots & a_{1i+1} \end{pmatrix}^t \ \text{ or } \ [T]_{i+1} = \begin{pmatrix} a_{1i+1} & a_{2i+1} & \ldots & a_{2i+1} \end{pmatrix}^t.$$

By continuing this process and from Lemma 2.3 (e), we obtain

$$[T] = \begin{pmatrix} B_1 & \cdots & B_{n-1} & * & [T]_m \end{pmatrix} P, \ \text{ or } \ [T] = \begin{pmatrix} B_1 & \cdots & B_l & [T]_m \end{pmatrix} P,$$

for some $P \in \mathcal{P}_m$, where B_1 and B_j $(j \geq 2)$ are the same as in (2.2) and (2.3), respectively, and $\sum_{j=1}^{l} k_j = m - 1$. The second possibility is $[T]_i = (a_{1i} \ 0 \ \cdots \ 0)^t$. In this event, Lemma 2.3 ensures that $[T]_{i+1} = (a_{1i+1} \ a_{2i+1} \ \cdots \ a_{2i+1})^t$ or $[T]_{i+1} = (a_{1i+1} \ a_{2i+1} \ 0 \ \cdots \ 0)^t$. If $[T]_{i+1} = (a_{1i+1} \ a_{2i+1} \ \cdots \ a_{2i+1})^t$, then $[T] = \begin{pmatrix} A'_1 & * \\ 0 & * \end{pmatrix} P$, for some $P \in \mathcal{P}_m$, where $A'_1 \in \mathbf{M}_1$. Here let $S : \mathbb{R}_{n-1} \to \mathbb{R}_{m-1}$ be the linear function with $[S] = [T](1|1)$. Lemma 2.3 ensures then that S preserves $\prec_{rgut}$. So $[S]$ can be the zero matrix or one of forms (a) or (b) of Lemma 2.4. Otherwise, $[T]_{i+1} = (a_{1i+1} \ a_{2i+1} \ 0 \ \cdots \ 0)^t$. By continuing this, we observe that there exists some $k \in \mathbb{N}_n$ such that $[T] = \begin{pmatrix} A_1 & \cdots & A_k & * & [T]_m \end{pmatrix} P$, for some $P \in \mathcal{P}_m$, where A_j $(j \in \mathbb{N}_k)$ is the same as in (2.1). If $k = n$, then we have (d). If not, then $k < n$. Consider $A_j = \begin{pmatrix} A'_j \\ 0 \end{pmatrix}$, where $A'_j \in \mathbf{M}_{k,t_j}$ $(j \in \mathbb{N}_k)$ and consequently $[T] = \begin{pmatrix} A'_1 & \cdots & A'_k & * \\ 0 & \cdots & 0 & * \end{pmatrix} P$. Let $S : \mathbb{R}_{n-k} \to \mathbb{R}_{m-\sum_{j=1}^{k} t_j}$ be the linear function with $[S] = [T](1,\ldots,k|1,\ldots,\sum_{j=1}^{k} t_j)$. Lemma 2.3 ensures then that S preserves $\prec_{rgut}$, and so $[S]$ can be the zero matrix or one of forms (a) or (b) of Lemma 2.4. $\square$

$$\boxed{\textbf{ELA}}$$
http:/repository.uwyo.edu/ela

For better understanding of the preceding theorem we bring the following examples.

EXAMPLE 2.7. Let $[T] = \begin{pmatrix} 1 & 0 & 2 & 7 & 1 & -1 & 5 & 14 \\ 1 & 1 & 2 & 8 & -2 & 4 & 0 & 15 \\ 1 & 1 & 2 & 8 & 3 & 9 & 2 & 3 \\ 1 & 1 & 2 & 8 & 3 & 6 & 8 & 0 \end{pmatrix}$. Theorem 2.6 (b)

ensures that T preserves $\prec_{rgut}$.

EXAMPLE 2.8. Let $[T] = \begin{pmatrix} 3 & 0 & 2 & 1 \\ 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \end{pmatrix}$. Then by Theorem 2.6 (c), T preserves

$\prec_{rgut}$.

EXAMPLE 2.9. Let $[T] = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. Theorem 2.6 (e) then ensures that T

preserves $\prec_{rgut}$.

EXAMPLE 2.10. Let $[T] = \begin{pmatrix} 1 & 0 & 1 & 0 & 8 \\ 1 & 2 & 3 & 1 & 3 \\ 1 & 2 & 3 & 2 & 2 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 & 0 \end{pmatrix}$. We see that $x = (0, 0, 1, -1, 0)$

$\prec_{rgut} y = (0, 1, -\frac{3}{2}, \frac{1}{2}, 0)$, but $Tx \not\prec_{rgut} Ty$. Then T does not preserve $\prec_{rgut}$. Notice that this case is from the form $[T] = \begin{pmatrix} B_1 & \cdots & B_l & * & [T]_m \end{pmatrix} P$, where $l \neq n - 1$. We see that $\sum_{j=1}^{l} k_j < m - 1$. By Theorem 2.6, T does not preserve $\prec_{rgut}$.

EXAMPLE 2.11. Let $[T] = \begin{pmatrix} 1 & 0 & 1 & 0 & 8 \\ 1 & 5 & 3 & 1 & 3 \\ 1 & 5 & 3 & 2 & 2 \\ 1 & 5 & 3 & 4 & 0 \\ 1 & 5 & 3 & 4 & 0 \end{pmatrix}$. We see that $x = (0, 0, 1, -1, 0)$

$\prec_{rgut} y = (0, 1, -2, 1, 0)$, but $Tx \not\prec_{rgut} Ty$. Then T does not preserve $\prec_{rgut}$. Notice that $[T] = \begin{pmatrix} B_1 & B_2 & B_4 & [T]_m \end{pmatrix}$. We observe that some of the middle $B_j^{'}$s could not be void. By Theorem 2.6, T does not preserve $\prec_{rgut}$.

REMARK 2.12. Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear preserver of $\prec_{rgut}$, and let $[T] = [a_{ij}]$. If there exists $k \in \mathbb{N}_{n-1}$ such that the first column of $[T](1, \ldots, k - 1)$ has the form $(\alpha, \ldots, \alpha)^t$, then T is not invertible.

Now, we focus on finding strong linear preservers of $\prec_{rgut}$ on $\mathbb{R}_n$. We need the following lemma to prove the next theorem.

LEMMA 2.13. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function. If T strongly preserves $\prec_{rgut}$, then T is invertible.*

Proof. Let $x \in \mathbb{R}_n$, and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_{rgut}$, this implies that $x \prec_{rgut} 0$. So $x = 0$, and thus, T is invertible. ☐

In the following theorem, the structure of linear functions $T : \mathbb{R}_n \to \mathbb{R}_n$ strongly preserving rgut-majorization will be characterized.

THEOREM 2.14. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear function. Then T strongly preserves $\prec_{rgut}$ if and only if $[T] = \alpha A$, for some $\alpha \in \mathbb{R} \setminus \{0\}$ and an invertible matrix $A \in \mathcal{R}_n^{gut}$.*

Proof. Assume first T strongly preserves $\prec_{rgut}$. So T is invertible and T preserves $\prec_{rgut}$, and hence, by Remark 2.12, there exist $\alpha \in \mathbb{R} \setminus \{0\}$ and an invertible matrix $A \in \mathcal{R}_n^{gut}$ such that $[T] = \alpha A$.

Conversely, suppose that $[T] = \alpha A$, for some $\alpha \in \mathbb{R} \setminus \{0\}$ and an invertible matrix $A \in \mathcal{R}_n^{gut}$. From Theorem 2.6, it is enough to show that if $Tx \prec_{rgut} Ty$, for each $x, y \in \mathbb{R}_n$, then $x \prec_{rgut} y$. Without loss of generality assume that $\alpha = 1$. Let $[T] = [a_{ij}]$. Clearly, $tr(x) = tr(y)$. Fix $i \in \mathbb{N}_{n-1}$. If there exists some $j \in \mathbb{N}_i$ such that $y_j \neq 0$, then $x_i \in \operatorname{aff}(0, y_1, \ldots, y_i)$. Otherwise, as $y_1 = \cdots = y_i = 0$, $(Ty)_1 = \cdots = (Ty)_i = 0$. This means that $(Tx)_1 = \cdots = (Tx)_i = 0$, and then $x_1 = \cdots = x_i = 0$. Therefore, T strongly preserves $\prec_{rgut}$. ☐

As a result of preceding theorems we can express the following corollary. It will be needed in the next section.

COROLLARY 2.15. *Suppose that $T : \mathbb{R}_n \to \mathbb{R}_n$ preserves $\prec_{rgut}$. Then T strongly preserves $\prec_{rgut}$ if and only if T is invertible.*

3. Strong linear preservers of rgut-majorization on $\mathbf{M}_{n,m}$. In this section, we characterize linear functions that strongly preserves rgut-majorization on $\mathbf{M}_{n,m}$.

LEMMA 3.1. *Let $A \in \mathbf{M}_n$. Then the following conditions are equivalent.*

a) *For all $D \in \mathcal{R}_n^{gut}$, $AD = DA$.*

b) *For some $\alpha, \beta \in \mathbb{R}$, $A = \alpha I + \beta E$.*

c) *For all $D \in \mathcal{R}_n^{gut}$ and for all $x, y \in \mathbb{R}_n$, $(xD + yDA) \prec_{rgut} (x + yA)$.*

Proof. $(a \Rightarrow b)$ Consider the matrices $D := E - E_{in} + E_{ij}$, where $i, j \in \mathbb{N}_n$ and $i \leq j$.

ELA

$(b \Rightarrow c)$ Let $D \in \mathcal{R}_n^{gut}$. Then $ED = E = DE$, and so $(xD + yDA) \prec_{rgut} (x + yA)$.

$(c \Rightarrow a)$ Fix $i \in \mathbb{N}_n$. Set $x = -\varepsilon_i A$ and $y = \varepsilon_i$. So by the hypothesis, $(-\varepsilon_i AD + \varepsilon_i DA) \prec_{rgut} (-\varepsilon_i A + \varepsilon_i A)$, for all $D \in \mathcal{R}_n^{gut}$. Hence, $(-DA + AD)\varepsilon_i = 0$, for all $D \in \mathcal{R}_n^{gut}$. It implies that $AD = DA$, for all $D \in \mathcal{R}_n^{gut}$. $\square$

For each $i, j \in \mathbb{N}_n$, consider the embedding $E^j : \mathbb{R}_m \to \mathbf{M}_{n,m}$ and the projection $E_i : \mathbf{M}_{n,m} \to \mathbb{R}_m$, where $E^j(x) = e_j x$ and $E_i(A) = \varepsilon_i A$. It is easy to show that for every linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$, $TX = T[x_1/ \cdots /x_n] = [\sum_{j=1}^n T_1^j x_j / \cdots / \sum_{j=1}^n T_n^j x_j]$, where $T_i^j = E_i T E^j$.

It is easy to see that if $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is a linear preserver of $\prec_{rgut}$, then T_i^j preserves $\prec_{rgut}$ on $\mathbb{R}_m$, for all $i, j \in \mathbb{N}_n$.

LEMMA 3.2. *Let* $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ *preserve* $\prec_{rgut}$. *If for an* $i \in \mathbb{N}_n$ *there exists* $k \in \mathbb{N}_n$ *such that* T_i^k *is invertible, then* $\sum_{j=1}^n x_j A_i^j = (\sum_{j=1}^n \alpha_i^j x_j) A_i^k + (\sum_{j=1}^n \beta_i^j x_j) E$, *for some* $\alpha_i^j, \beta_i^j \in \mathbb{R}$, *where* $A_i^j = [T_i^j]$.

Proof. It can be assumed without loss of generality that $i, k = 1$ and $j = 2$. We show that $A_1^2 = \alpha_1^2 A_1^1 + \beta_1^2 E$, for some $\alpha_1^2, \beta_1^2 \in \mathbb{R}$. Let $x, y \in \mathbb{R}_m$ and $D \in \mathcal{R}_m^{gut}$. Since $[x/y/0/\cdots/0]D \prec_{rgut} [x/y/0/\cdots/0]$, then $T[xD/yD/0/\cdots/0] \prec_{rgut} T[x/y/0/\cdots/0]$, and hence, $[T_1^1 xD + T_1^2 yD/ */*] \prec_{rgut} [T_1^1 x + T_1^2 y/ */*]$. It shows that $T_1^1 xD + T_1^2 yD \prec_{rgut} T_1^1 x + T_1^2 y$. Thus, $xDA_1^1 + yDA_1^2 \prec_{rgut} xA_1^1 + yA_1^2$. We can see $xD + yDA_1^2(A_1^1)^{-1} \prec_{rgut} x + yA_1^2(A_1^1)^{-1}$, for all $x, y \in \mathbb{R}_m$ and $D \in \mathcal{R}_m^{gut}$. Lemma 3.1 ensures that $A_1^2 = \alpha_1^2 A_1^1 + \beta_1^2 E$, for some $\alpha_1^2, \beta_1^2 \in \mathbb{R}$. $\square$

LEMMA 3.3. *Let* $T : \mathbb{R}_n \to \mathbb{R}_n$ *preserve* $\prec_{rgut}$, *and suppose that* $[T] = [a_{ij}]$ *is an upper triangular matrix. If there exists some* $t \in \mathbb{N}_{n-2}$ *such that* $a_{tt} = 0$, *then* $a_{t+1t+1} = \cdots = a_{n-1n-1} = 0$.

Proof. Suppose that $t \in \mathbb{N}_{n-2}$ and $a_{tt} = 0$. Consider the linear function $S : \mathbb{R}_{n-t+1} \to \mathbb{R}_{n-t}$, where $[S] = [T](1, \ldots, t-1|1, \ldots, t)$. Lemma 2.3 ensures that S preserves $\prec_{rgut}$ on $\mathbb{R}_{n-t+1}$. Using Lemma 2.3 again, we see that $a_{t+1t+1} = 0$. Similarly, one shows that $a_{t+2t+2} = \cdots = a_{n-1n-1} = 0$. $\square$

LEMMA 3.4. *Let* $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ *strongly preserve* $\prec_{rgut}$. *Then for each* $i \in \mathbb{N}_n$ *there exists some* $j \in \mathbb{N}_n$ *such that* T_i^j *is invertible.*

Proof. Define $I = \{i \in \mathbb{N}_n \mid T_i^j \notin IN_m, \forall j \in \mathbb{N}_n\}$, where the set of all m-by-m invertible matrices is denoted by IN_m. We claim that I is empty. Assume, if possible, that I is not empty. There is no loss of generality in assuming $I = \{1, 2, \ldots, k\}$, where $k \in \mathbb{N}_n$. There are two cases to consider. First, let $I = \{1, 2, \ldots, n\}$. Since T_i^j preserves $\prec_{rgut}$ and also it is not invertible, so $T_i^j(\varepsilon_{m-1} - \varepsilon_m) = 0$, for each $i, j \in \mathbb{N}_n$. As $X = [\varepsilon_{m-1} - \varepsilon_m/ \cdots /\varepsilon_{m-1} - \varepsilon_m] \in \mathbf{M}_{n,m}$ is nonzero, and $TX = 0$, we obtain a contradiction. Thus, this case can not happen.

ELA

http:/repository.uwyo.edu/ela

Next, let $I = \{1, 2, \ldots, k\}$, where $k < n$. By Lemma 3.2, for each i $(k+1 \leq i \leq n)$ and $j \in \mathbb{N}_n$, there exist invertible matrices A_i and $\alpha_i^j, \beta_i^j \in \mathbb{R}$ such that $\sum_{j=1}^n x_j A_i^j = (\sum_{j=1}^n \alpha_i^j x_j) A_i + (\sum_{j=1}^n \beta_i^j x_j) E$. Then there exist $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, not all zero, such that $\gamma_1(\alpha_{k+1}^1, \ldots, \alpha_n^1) + \cdots + \gamma_n(\alpha_{k+1}^n, \ldots, \alpha_n^n) = 0$. Define $x_j = \gamma_j(\varepsilon_{m-1} - \varepsilon_m)$, for each $j \in \mathbb{N}_n$, and choose $X = [x_1/\cdots/x_n] \in \mathbf{M}_{n,m}$. We conclude that $TX = 0$, but $X \neq 0$, which is a contradiction. Therefore, for each $i \in \mathbb{N}_n$, there exists $j \in \mathbb{N}_n$ such that T_i^j is invertible. $\square$

LEMMA 3.5. *Let* $T : \mathbf{M}_{n,m} \rightarrow \mathbf{M}_{n,m}$ *satisfy* $TX = RX + SXE$, *for some* $R, S \in \mathbf{M}_n$. *Then* T *is invertible if and only if* $R(R+S)$ *is invertible.*

Proof. First, assume that $R(R+S)$ is invertible. Let $X \in \mathbf{M}_{n,m}$, and let $TX = 0$. Multiple this relation by E. Since $R+S$ is invertible, we see that $XE = 0$. Put $XE = 0$ in the relation $TX = 0$. Hence, $RX = 0$ and, as R is invertible, we conclude that $X = 0$. Therefore, T is invertible.

Next, suppose that T is invertible. If R is not invertible, then there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $Rx = 0$. Define $X = [x \mid -x \mid 0 \mid \cdots \mid 0] \in \mathbf{M}_{n,m}$. Hence, $TX = 0$ while $X \neq 0$, which would be a contradiction. Thus, R is invertible.

Now, assume that $R + S$ is not invertible. So there exists $y \in \mathbb{R}^n \setminus \{0\}$ such that $(R + S)y = 0$. Define $Y = [0 \mid 0 \mid \cdots \mid y] \in \mathbf{M}_{n,m}$. It follows that $TY = 0$, but $Y \neq 0$, which is a contradiction. Therefore, $R + S$ is invertible. $\square$

The last theorem of this paper, which is our main result in this section, characterizes the strong linear preservers of $\prec_{rgut}$ on $\mathbf{M}_{n,m}$.

THEOREM 3.6. *Let* $T : \mathbf{M}_{n,m} \rightarrow \mathbf{M}_{n,m}$ *be a linear function. Then* T *strongly preserves* $\prec_{rgut}$ *if and only if* $TX = RXA + SXE$, *for some* $R, S \in \mathbf{M}_n$ *and an invertible matrix* $A \in \mathcal{R}_m^{gut}$ *such that* $R(R + S)$ *is invertible.*

Proof. First, we prove the sufficiency of the conditions. First, let $X, Y \in \mathbf{M}_{n,m}$ such that $X \prec_{rgut} Y$. So there exists $D \in \mathcal{R}_m^{gut}$ such that $X = YD$. Hence, $TX = RXA + SXE = R(YD)A + S(YD)E = (RYA)(A^{-1}DA) + (SYE)(A^{-1}DA) = (RYA + SYE)(A^{-1}DA) = TY(A^{-1}DA)$, and then $TX \prec_{rgut} TY$. Now, let $X, Y \in \mathbf{M}_{n,m}$, and let $TX \prec_{rgut} TY$. Then $TX = (TY)D$, for some matrix $D \in \mathcal{R}_m^{gut}$. It means that $RXA + SXE = RYAD + SYED$, and so $RX + SXEA^{-1} = RYADA^{-1} + SYEDA^{-1}$. Multiply this relation by E and, as $R + S$ is invertible, we conclude that $XE = YE$. Substitute $XE = YE$ in the relation $RXA + SXE = RYAD + SYED$, and then $RXA = RYAD$. So $X = Y(ADA^{-1})$, and this implies that $X \prec_{rgut} Y$. Therefore, T strongly preserves $\prec_{rgut}$.

Next, assume that T strongly preserves $\prec_{rgut}$. If $n = 1$, then the result is proved by Theorem 2.14. So we may suppose that $n > 1$. Lemma 3.4 ensures that for each

ELA

$i \in \mathbb{N}_n$ there exists some $j \in \mathbb{N}_n$ such that T_i^j is invertible. By Lemma 3.2, there exist invertible matrices $A_1, \ldots, A_n \in \mathbf{M}_m$, vectors $a_1, \ldots, a_n \in \mathbb{R}_n$, and a matrix $S' \in \mathbf{M}_n$ such that $TX = [a_1 X A_1 / \cdots / a_n X A_n] + S' X E$.

We claim that $\dim(\text{span}\{a_1, \ldots, a_n\}) \geq 2$. If not; so $\{a_1, \ldots, a_n\} \subseteq \text{span}\{a\}$, for some $a \in \mathbb{R}_n$. Since $n > 1$, we can choose $0 \neq b \in (\text{span}\{a\})^{\perp}$. Define $X = [b \mid -b \mid 0 \mid \cdots \mid 0] \in \mathbf{M}_{n,m}$. Then $XE = 0$, and also $a_i X = 0$, for all $i \in \mathbb{N}_n$. It is seen that $TX = 0$, while $X \neq 0$, a contradiction. Thus, $\text{rank}\{a_1, \ldots, a_n\} \geq 2$. Without loss of generality, assume that $\{a_1, a_2\}$ is a linearly independent set. Let $X \in \mathbf{M}_{n,m}$ and $D \in \mathcal{R}_m^{gut}$. So $XD \prec_{rgut} X$, and hence, $TXD \prec_{rgut} TX$. This follows that $a_1 X D A_1 + a_2 X D A_2 \prec_{rgut} a_1 X A_1 + a_2 X A_2$, and then

$$(3.1) \qquad a_1 XD + a_2 X D A_2 A_1^{-1} \prec_{rgut} a_1 X + a_2 X A_2 A_1^{-1},$$

for all $X \in \mathbf{M}_{n,m}$, for all $D \in \mathcal{R}_m^{gut}$. Since $\{a_1, a_2\}$ is linearly independent, for every $x, y \in \mathbb{R}_m$ there exists $B_{x,y} \in \mathbf{M}_{n,m}$ such that $a_1 B_{x,y} = x$ and $a_2 B_{x,y} = y$. By substituting $X = B_{x,y}$ in (3.1), we obtain $xD + yDA_2 A_1^{-1} \prec_{rgut} x + yA_2 A_1^{-1}$, for all $D \in \mathcal{R}_m^{gut}$ and $x, y \in \mathbb{R}_m$. Lemma 3.1 ensures then that $A_2 = \alpha A_1 + \beta E$, for some $\alpha, \beta \in \mathbb{R}$. For every $i \geq 3$, if $a_i = 0$, then we can choose $A_i = A_1$. If $a_i \neq 0$, then $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is linearly independent. Then by a similar procedure, we deduce the same result. That is, $A_i = \gamma_i A_1 + \delta_i E$, for some $\gamma_i, \delta_i \in \mathbb{R}$, or $A_i = \lambda_i A_2 + \eta_i E$, for some $\lambda_i, \eta_i \in \mathbb{R}$. Define $A = A_1$. Then for every $i \geq 2$, $A_i = \alpha_i A + \beta_i E$, for some $\alpha_i, \beta_i \in \mathbb{R}$, and hence, $TX = [a_1 X A / \cdots / a_n X A] + S' X E = RXA + SXE$, where $R = [a_1 / r_2 a_2 / \cdots / r_n a_n]$, for some $r_2, \ldots, r_n \in \mathbb{R}$ and $S = S' + [0 / \beta_2 a_2 / \cdots / \beta_n a_n]$. $\square$

REFERENCES

[1] T. Ando. Majorization, doubly stochastic matrices, and comparison of eigenvalues. *Linear Algebra and its Applications*, 118:163–248, 1989.

[2] T. Ando. Majorization and inequalities in matrix theory. *Linear Algebra and its Applications*, 199:17–67, 1994.

[3] A. Armandnejad. Right gw-majorization on $\mathbf{M}_{n,m}$. *Iranian Mathematical Society. Bulletin*, 35:69–76, 2009.

[4] A. Armandnejad and Z. Gashool. Strong linear preservers of g-tridiagonal majorization on $\mathbb{R}^n$. *Electronic Journal of Linear Algebra*, 23:115–121, 2012.

[5] A. Armandnejad and A. Ilkhanizadeh Manesh. Gut-majorization and its linear preservers. *Electronic Journal of Linear Algebra*, 23:646–654, 2012.

[6] A. Armandnejad and A. Salemi, The structure of linear preservers of gs-majorization. *Iranian Mathematical Society. Bulletin*, 32:31–42, 2006.

[7] A. Armandnejad and A. Salemi. On linear preservers of lgw-majorization on $\mathbf{M}_{n,m}$. *Iranian Mathematical Society. Bulletin*, 35:755–764, 2012.

[8] H. Chiang and C.K. Li. Generalized doubly stochastic matrices and linear preservers. *Linear and Multilinear Algebra*, 53:1–11, 2005.

[9] G.H. Hardy, J.E. Littlewood, and G. Polya. Some simple inequalities satisfied by convex functions. *Messenger of Mathematics*, 58:145–152, 1929.

ELA

[10] A.M. Hasani and M. Radjabalipour. The structure of linear operators strongly preserving majorizations of matrices. *Electronic Journal of Linear Algebra*, 15:260–268, 2006.

[11] A.M. Hasani and M. Radjabalipour. On linear preservers of (right) matrix majorization. *Linear Algebra and its Applications*, 423:255–261, 2007.

[12] A. Ilkhanizadeh Manesh. On linear preservers of sgut-majorization on $\mathbf{M}_{n,m}$. *Iranian Mathematical Society. Bulletin*, to appear.

[13] A. Ilkhanizadeh Manesh. Linear functions preserving sut-majorization on $\mathbb{R}^n$. *Iranian Journal of Mathematical Sciences and Informatics*, to appear.

[14] A. Ilkhanizadeh Manesh and A. Armandnejad. Ut-Majorization on $\mathbb{R}^n$ and its linear preservers. *Operator Theory: Advances and Applications*, 242:253–259, 2014.

[15] A.W. Marshall, I. Olkin, and B.C. Arnold, *Inequalities: Theory of majorization and its applications.* Springer, New York, 2011.

[16] I. Schur. Uber enie klasse von mittelbildungen mit anwendungen auf die determinantentheorie. *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, 22:9–20, 1923.