Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

APPLICATION OF AN IDENTITY FOR SUBTREES WITH A GIVEN EIGENVALUE*

KENJI TOYONAGA† AND CHARLES R. JOHNSON‡

Abstract. For an Hermitian matrix whose graph is a tree and for a given eigenvalue having Parter vertices, the possibilities for the multiplicity are considered. If $V = \{v_1, \ldots, v_k\}$ is a fragmenting Parter set in a tree relative to the eigenvalue λ, and T_{i+1} is the component of $T - \{v_1, v_2, \ldots, v_i\}$ in which v_{i+1} lies, it is shown that $\sum_i^k N_i = m_A(\lambda) + 2k - 1$, in which N_i is the number of components of $T_i - v_i$ in which λ is an eigenvalue. This identity is applied to make several observations, including about when a set of strong Parter vertices leaves only 3 components with λ and about multiplicities in binary trees. Furthermore, it is shown that one can construct an Hermitian matrix whose graph is a tree that has a strong Parter set V such that $|V| = k$ for each k in $1 \le k \le m - 1$ for given multiplicity $m \ge 2$ of an eigenvalue λ. Finally, some examples are given, in which the notion of a fragmenting Parter set is used.

Key words. Tree, Eigenvalues, Hermitian matrices, Multiplicity, Parter vertex.

AMS subject classifications. 05C05, 15A18, 15A57, 13H15, 05C50.

1. Introduction.

For an undirected graph G on n vertices, denote by $\mathcal{H}(G)$ the set of all n-by-n Hermitian matrices with graph G. No requirement, other than reality, is placed upon the diagonal entries of $A \in \mathcal{H}(G)$. Let $\sigma(A)$ denote the eigenvalues of A, including multiplicities. For $\lambda \in \sigma(A)$, we denote the multiplicity of λ, as an eigenvalue of A, by $m_A(\lambda)$. When there is an identified $A \in \mathcal{H}(G)$, we often speak interchangeably about the graph and the matrix, for convenience.

Our interest here is in the case in which G is a tree T and $A \in \mathcal{H}(T)$ is an Hermitian matrix. In that event, when $A \in \mathcal{H}(T)$ and $m_A(\lambda) \ge 2$, there is remarkable structure present [2, 7, 10], and there may be such structure even when $m_A(\lambda) < 2$ [2]. The multiplicities of eigenvalues of an Hermitian matrix whose graph is a tree have been studied in many papers. Parter sets or P-sets for an eigenvalue have been studied in several papers [1, 2, 4, 5, 6] as well.

*Received by the editors on August 16, 2014. Accepted for publication on October 20, 2015. Handling Editor: Bryan L. Shader.

†Department of Integrated Arts and Science, Kitakyushu National College of Technology, Kokuraminami-ku, Kitakyushu, 802-0985, Japan (toyonaga@kct.ac.jp).

‡Department of Mathematics, College of William and Mary, PO Box 8795, Williamsburg, VA 23187-8795, USA (crjohnso@math.wm.edu).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

For a vertex u of T, we denote the $(n-1)$-by-$(n-1)$ principal submatrix of $A \in \mathcal{H}(T)$, resulting from deletion of the row and column corresponding to u, by $A(u)$; its graph is $T-u$. A vertex u of T is called a *Parter vertex* [2] if $m_{A(u)}(\lambda) = m_A(\lambda)+1$. If $m_A(\lambda) \geq 2$, there is always at least one Parter vertex in T, and there may be Parter vertices even when $m_A(\lambda) < 2$. Furthermore, if $m_A(\lambda) \geq 2$, then there will be a Parter vertex u such that λ occurs as an eigenvalue in at least 3 principal submatrices of A, corresponding to branches of T at the Parter vertex u of T [2, 7]. In this case, u is called a *strong Parter vertex* [2]. In general, vertex u is Parter if and only if it has a neighbor w in T such that $m_{A[T_w-w]}(\lambda) = m_{A[T_w]}(\lambda) - 1$ in which $A[S]$ denotes the principal submatrix of A corresponding to the subgraph S of T, T_w is the branch of T at u and containing w, and $T_w - w$ is the subtree of T_w induced by deletion of w. Such a vertex in a tree is called a *downer vertex* and such a branch a *downer branch* [2].

A set V of vertices $\{v_1, \ldots, v_k\}$ is called a *Parter set* for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$ if $m_{A[T-V]}(\lambda) = m_A(\lambda) + k$. By the interlacing inequalities, it is clear that each vertex of V is Parter in T. (The converse is not generally true.) If T has a Parter vertex for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, its removal from T will leave some components (subtrees) in which λ is an eigenvalue of the corresponding principal submatrix of A. Some of these components may also include Parter vertices (relative to the component). Repeated removal of such Parter vertices in each component will eventually leave components in each of which λ appears no more than once. In this event, the Parter set of removed vertices is called a *fragmenting Parter set* (f-Parter set, for short).

In the field of validated numerical analysis, it is generally considered that it is difficult to enclose a multiple eigenvalue with large multiplicity in a narrow interval. In [9], a numerical method for validating existence of an eigenvalue with large multiplicity of an Hermitian matrix whose graph is a tree is given; the fragmenting subgraph obtained by removing Parter vertices and software INTLAB [8] is used. INTLAB is the Matlab toolbox for reliable computing and self-validating algorithms. The notion of fragmenting subgraphs can be applied to numerical analysis. So, it is important to study the property of fragmenting subgraphs obtained by deleting Parter vertices in a tree.

In Figure 1.1, we give an example of an f-Parter set for a tree T with an eigenvalue λ of multiplicity 3. The numbers in Figure 1.1 represent the multiplicity of an eigenvalue λ in the subgraph under the vertex next to the number. We can easily construct an Hermitian matrix A with eigenvalue λ whose graph is the tree T in Figure 1.1. If we suppose Parter vertices in V are sequentially removed in the order v_1, v_2, then v_i is a strong Parter vertex in T_i, $i = 1, 2$. Therefore, $V = \{v_1, v_2\}$ is a fragmenting strong Parter set (strong f-Parter set, for short) for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

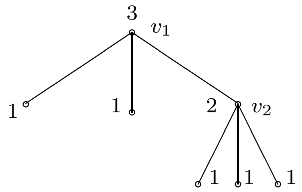

Fig. 1.1.

2. Main results. Our purpose here is to prove an identity for the system of subtrees associated with the sequential removal of vertices in an f-Parter set that is interesting by itself (Theorem 2.1) and then to apply the identity in several ways.

Let $V = \{v_1, \dots, v_k\}$ be a Parter set for $\lambda \in \sigma(A), A \in \mathcal{H}(T)$. Let T_{i+1} be the component of $T - \{v_1, \dots, v_i\}$ in which vertex v_{i+1} lies, $i = 0, \dots, k-1$. We set $T_1 = T$. If, further, v_{i+1} is a strong Parter in T_{i+1}, $i = 0, \dots, k-1$, we call V a *strong Parter set.* When the multiplicity of an eigenvalue λ is given, we consider upper bounds for the cardinality of a strong f-Parter set of the tree for eigenvalue λ. Further, in Theorem 2.2, we characterize a certain maximality of strong f-Parter sets. And we consider the number of components in which λ occurs as an eigenvalue exactly once, when we remove the strong Parter vertices in a strong f-Parter set. In Proposition 2.5, when T is a full binary tree, we give upper bounds for the cardinality of a strong f-Parter set. In Theorem 2.6, we show that given the multiplicity $m \geq 2$ of an eigenvalue, there exist Hermitian matrices, whose graph is a tree, that have a strong Parter set V for the eigenvalue with $|V| = k$ for each k in $1 \leq k \leq m-1$. Finally, we give simple examples to illustrate our results.

We also define N_i to be the number of components of $T_i - v_i$ in which λ is an eigenvalue of the corresponding principal submatrix of A. Of course, if V is a strong Parter set, then $N_i \geq 3$, for all i.

When we remove Parter vertices in V sequentially, the number of components in which λ is an eigenvalue of the corresponding principal submatrix of A satisfies the next relation.

Theorem 2.1. *Let T be a tree and $A \in \mathcal{H}(T)$. If $V = \{v_1, v_2, \dots, v_k\}$ is an f-Parter set for $\lambda \in \sigma(A)$ with $m_{A[T_i]}(\lambda) \geq 1$, $1 \leq i \leq k$, then*

$$\sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1.$$

Proof. We give a proof by induction on k. When $k = 1$, the formula holds, because of the definition of a Parter vertex. We suppose that when $|V| \leq k$, the formula

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

holds. When $|V| = k+1$, we denote the vertices in V as $V = \{v_1, v_2, \ldots, v_{k+1}\}$. We denote by $B_1, B_2, \ldots, B_l$, $l \geq 1$, the branches at v_1 in which λ is an eigenvalue of the corresponding principal submatrix. Let k_i be the number of Parter vertices in B_i that are contained in V. Then $0 \leq k_i \leq k$, and $k_1 + \cdots + k_l = k$. The Parter vertices in V that are contained in B_i must be an f-Parter set in B_i. By assumption, the next relations hold in each branch B_i respectively, in which N_{ji} denotes N_i in branch B_j :

$$\sum_{i=1}^{k_1} N_{1i} = m_{A[B_1]}(\lambda) + 2k_1 - 1\,;$$

$$\sum_{i=1}^{k_2} N_{2i} = m_{A[B_2]}(\lambda) + 2k_2 - 1\,;$$

$$\vdots$$

$$\sum_{i=1}^{k_l} N_{li} = m_{A[B_l]}(\lambda) + 2k_l - 1.$$

Here, If $k_j = 0$, then $m_{A[B_j]}(\lambda) = 1$, and we set $\sum_{i=1}^{k_j} N_{ji} = 0$. By adding the left hand sides and right hand sides of these equations, we get

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j} N_{ji} = \sum_{i=1}^{l} m_{A[B_i]}(\lambda) + 2k - l.$$

Equivalently

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j} N_{ji} + l = m_A(\lambda) + 2k + 1,$$

or

$$\sum_{i=1}^{k+1} N_i = m_A(\lambda) + 2(k+1) - 1.$$

The claim thus follows by induction. ☐

In the above theorem, we note that V is an f-Parter set, but V does not necessarily need to be a strong f-Parter set. Of course, if V is a strong f-Parter set,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

then Theorem 2.1 still holds. Furthermore, as Example 3.2 shows, we note that the condition $m_{A[T_i]}(\lambda) \geq 1$, $1 \leq i \leq k$ is necessary. Examples illustrating Theorem 2.1 occur in Example 3.2 later.

Next we consider the possible cardinalities of a strong f-Parter set for $\lambda \in \sigma(A)$ in terms of $m_A(\lambda)$.

THEOREM 2.2. *If T is a tree, $A \in \mathcal{H}(T)$ and $V = \{v_1, \ldots, v_k\}$ is a strong f-Parter set for $\lambda \in \sigma(A)$, then*

$$|V| \leq m_A(\lambda) - 1,$$

and V is maximal, that is $|V| = m_A(\lambda) - 1$, if and only if $N_i = 3$ for all i, $1 \leq i \leq k$.

Proof. Since V is a strong f-Parter set for $\lambda \in \sigma(A)$, by Theorem 2.1,

$$3|V| \leq \sum_{i=1}^{k} N_i = m_A(\lambda) + 2|V| - 1.$$

So, $|V| \leq m_A(\lambda) - 1$.

From Theorem 2.1, we have $\sum_{i=1}^{k}(N_i - 3) = m_A(\lambda) - (|V| + 1)$. Since V is a strong f-Parter set, $N_i \geq 3$. So, $|V| = m_A(\lambda) - 1$ if and only if $N_i = 3$ for all i, $1 \leq i \leq k$. ☐

The above formula gives an upper bound for V in terms of $m_A(\lambda)$ and a lower bound for $m_A(\lambda)$ in terms of $|V|$.

REMARK. If V is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, T a tree, then $T - V$ has $m_A(\lambda) + |V|$ components with λ as an eigenvalue, each of multiplicity 1.

From Theorem 2.2, given a positive integer n, we can construct an Hermitian matrix, whose graph is a tree, that has a maximal strong Parter set V with $|V| = n-1$ in which $N_i = 3$, $1 \leq i \leq n-1$.

COROLLARY 2.3. *Let T be a tree and $A \in \mathcal{H}(T)$. If $V = \{v_1, \ldots, v_k\}$ is an f-Parter set for $\lambda \in \sigma(A)$ with $m_{A[T_i]}(\lambda) \geq 1$, $1 \leq i \leq k$, then of the system of $\sum_{i=1}^{k} N_i$ subtrees of T counted in Theorem 2.1, $k-1$ subtrees have λ with multiplicity > 1.*

Proof. From Theorem 2.1,

$$\sum_{i=1}^{k} N_i - (m_A(\lambda) + k) = \sum_{i=1}^{k} N_i - (m_A(\lambda) + |V|) = k - 1.$$

From the above remark, $k-1$ subtrees at Parter vertices have λ with multiplicity > 1. ☐

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

COROLLARY 2.4. *If V is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, T a tree, then the number c of components of $T - V$ in which λ occurs as an eigenvalue exactly once satisfies*

$$m_A(\lambda) + 1 \le c \le 2\ m_A(\lambda) - 1.$$

Proof. Theorem 2.2 implies that $1 \le |V| \le m_A(\lambda) - 1$. Thus,

$$m_A(\lambda) + 1 \le |V| + m_A(\lambda) \le 2\ m_A(\lambda) - 1.$$

Since V is a strong f-Parter set, $c = m_A(\lambda) + |V|$. □

A tree T is *binary* if the degree of each vertex is at most 3. If there are no vertices of degree 2, we call it a *full binary tree.*

In a full binary tree T on v vertices, if V is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, we can get an upper bound for $|V|$ and $m_A(\lambda)$ in terms of v.

PROPOSITION 2.5. *If V is a strong f-Parter set for $\lambda \in \sigma(A)$, $A \in \mathcal{H}(T)$, T a full binary tree on v vertices then*

$$|V| \le \lfloor (v-1)/3 \rfloor$$

and

$$m_A(\lambda) \le \lfloor (v+2)/3 \rfloor.$$

Proof. Each strong Parter vertex in V has three incident edges in T_i, $1 \le i \le |V|$. Since T has $v-1$ edges, it is clear that $|V| \le \lfloor (v-1)/3 \rfloor$. Since T is a full binary tree, every strong Parter vertex in V has three branches in which λ occurs as an eigenvalue. So, $|V| = m_A(\lambda) - 1$ from Theorem 2.2. Therefore, we get an upper bound for $m_A(\lambda)$ in terms of v from the upper bound of V. □

THEOREM 2.6. *Given multiplicity $m_A(\lambda) = m \ge 2$ and $1 \le k \le m - 1$, there exists an Hermitian matrix A whose graph is a tree, that has a strong Parter set V of size k for $\lambda \in \sigma(A)$.*

Proof. In the case of $m = 2$, there exists an Hermitian matrix A whose graph is a star, whose submatrices of A corresponding to three branches at a center of the star have a simple eigenvalue λ. Then the center is a strong Parter vertex and $|V| = 1$. Next we consider the case of $m \ge 3$. Given multiplicity $m \ge 3$ of an eigenvalue λ, we can construct an Hermitian matrix A whose graph is a tree T such that $|V| = m - 1$ with $N_i = 3$, $1 \le i \le m - 1$, from Theorem 2.2. Then let

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

V be $\{p_1, p_2, \ldots, p_{m-1}\}$, and T_i be a component that contains the Parter vertex p_i when Parter vertices $\{p_1, \ldots, p_{i-1}\}$ are removed from T ($2 \leq i \leq m-1$). We set $T_1 = T$. Then there exists k such that branches at p_k in T_k have the eigenvalue λ with multiplicity at most 1. If $m_{A[T_k]}(\lambda) = l$, when p_k is removed from T_k, there exists $l+1$ branches that have the eigenvalue λ with multiplicity 1 in T_k, which we denote by $T_1, \ldots, T_{l+1}$. When we remove the Parter vertex p_k from T and connect $T_1, \ldots, T_l$ to a Parter vertex that exists in the above position of p_k in T by inserting l new edges.(cf. Example 3.3), then the graph contains $m-2$ strong Parter vertices. By repeating this procedure, we can construct a graph such that $|V| = k$ for all k, $1 \leq k \leq m-1$. □

3. Examples. We present examples for Theorem 2.2, Theorem 2.1 and Theorem 2.6.

EXAMPLE 3.1. We give an example for Theorem 2.2 in Figure 3.1. Let T be a tree, $A \in \mathcal{H}(T)$ with an eigenvalue λ of multiplicity 4. The maximal cardinality of a strong Parter set for λ relative to A is 3 from Theorem 2.2. Then there are two patterns of multiplicities as displayed.

The strong f-Parter set of T for λ relative to A is $V = \{v_1, v_2, v_3\}$, and when Parter vertices in V are sequentially removed in the order v_1, v_2, and v_3. Then v_i is a strong Parter vertex in T_i. The numbers in the figures represent the multiplicity of the eigenvalue λ relative to the submatrix of A corresponding to the subgraph under the vertex next to the number.

When Parter vertices are sequentially removed in the order v_1, v_2, and v_3, we can see that only three components in T_i, $1 \leq i \leq 3$ have the eigenvalue λ in each subgraph.

FIG. 3.1.

EXAMPLE 3.2. Here are examples for Theorem 2.1. Let T be a tree, $A \in \mathcal{H}(T)$ with an eigenvalue λ. We show the identity $\sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1$ holds in the cases of multiplicity $m_{A[T]}(\lambda) = 2, 3$ and 4, in which V is an f-Parter set, $|V| = k$, and N_i is the number of components of $T_i - v_i$ in which λ is an eigenvalue of the corresponding principal submatrix of A. When $m_{A[T]}(\lambda) = 2$, the formula in Theorem

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

2.1 holds from $|V| = k = 1$ and $N_1 = 3$. So we show the cases of $m_{A[T]}(\lambda) = 3$ and $m_{A[T]}(\lambda) = 4$ in Figure 3.2. The numbers in the figures represent the multiplicity of the eigenvalue λ relative to the submatrix of A corresponding to the subgraph under the vertex next to the number. Parter vertices in the f-Parter set are the vertices next to the number 2, 3 and 4. When $m_{A[T]}(\lambda) = 3$ there are two cases in Figure 3.2, then the formula in Theorem 2.1 holds for k and N_i, that is, $k = 1$, N_1=4 at the left graph, and $k = 2$, $N_1 + N_2 = 6$ at the right graph.

For the case of $m_{A[T]}(\lambda) = 4$, there are five cases. In the last figure, the subgraph with multiplicity 3 has the same two patterns as the case of $m_{A[T]}(\lambda) = 3$. For all the cases, we see that the formula in Theorem 2.1 holds.

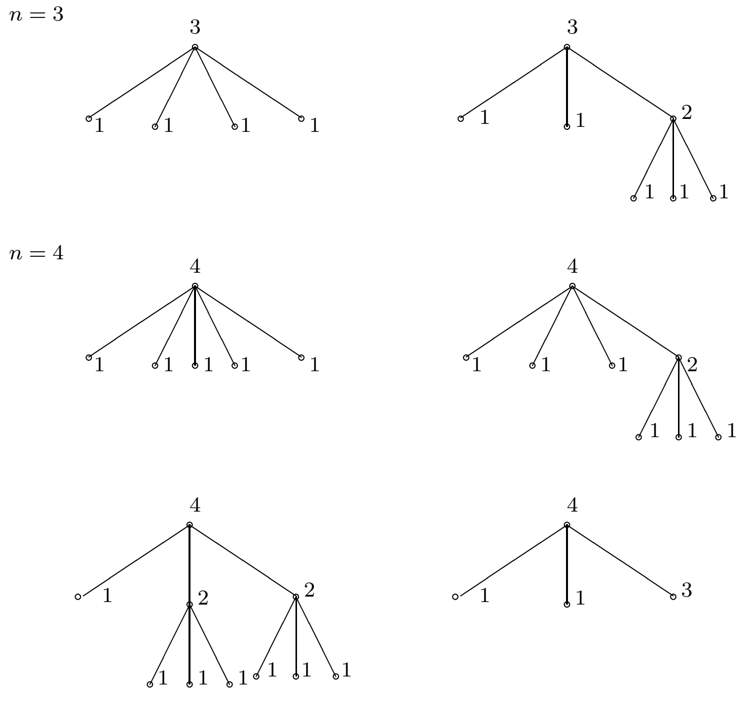

FIG. 3.2.

We note that the condition $m_{A[T_i]}(\lambda) \geq 1$, $1 \leq i \leq k$ in Theorem 2.1 is indispensable. If this condition is not satisfied, then there is a case such that the formula in Theorem 2.1 does not hold. Let A be an Hermitian matrix, and the corresponding graph be the tree T in Figure 3.3, in which vertex x_i of T corresponds to row i of A, $i = 1, \ldots, 7$. The matrix A has an eigenvalue 0 with multiplicity 2, so $m_{A[T]}(0) = 2$. Then x_1, x_6 are Parter vertices of T for 0 relative to A. Furthermore, $V = \{x_1, x_6\}$ is a Parter set of T for 0 relative to A. We denote the subgraph of T induced by vertices

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

x_5, x_6 and x_7 by T_2. Here we set $V = \{v_1, v_2\}$ by replacing $x_1 = v_1, x_6 = v_2$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

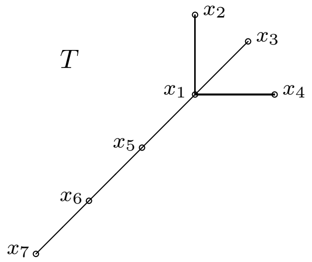

FIG. 3.3.

Let $T = T_1$, and N_i be the number of components of $T_i - v_i$, $i = 1, 2$, in which 0 is an eigenvalue of the principal submatrix of A. Since the principal submatrix of A corresponding to T_2 does not have eigenvalue 0, and $m_{A[T_2(v_2)]}(0) = 1$, then $N_1 + N_2 = 4$. Now, since $|V| = k = 2$, $m_A(\lambda) + 2k - 1 = 5$. As a result, we can say that if $m_{A[T_i]}(\lambda) \geq 1$ is not satisfied, then there is a case such that the formula in Theorem 2.1 $\sum_{i=1}^{k} N_i = m_A(\lambda) + 2k - 1$ does not hold. So, the condition $m_{A[T_i]}(\lambda) \geq 1$ in Theorem 2.1 is essential.

EXAMPLE 3.3. Finally, we give examples for Theorem 2.6 in Figure 3.4. For the case of $m_{A[T]}(\lambda) = 4$, we can construct Hermitian matrices whose graphs are trees such that $|V| = 1, 2$ and 3 respectively, with V a strong f-Parter set for λ relative to $A \in \mathcal{H}(T)$. The numbers in the figures represent the multiplicity of the eigenvalue λ relative to the submatrix of A corresponding to the subgraph under the vertex next to the number.

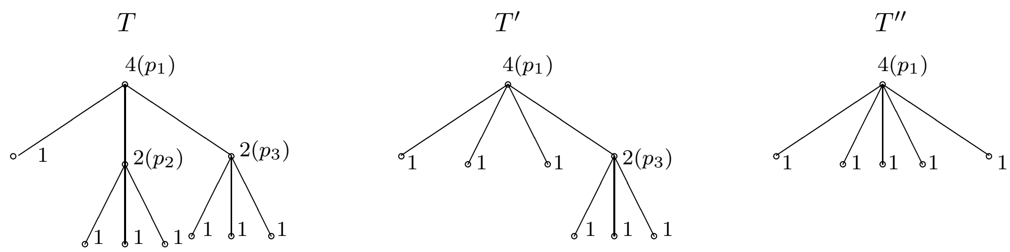

FIG. 3.4.

From Theorem 2.2, we can construct an Hermitian matrix whose graph is a tree T with $|V| = 3$ as the left graph in Figure 3.4, then strong Parter set $V = \{p_1, p_2, p_3\}$. When we remove the vertex p_2 from T, there exist three branches(vertices) at p_2 in T_2, which have the eigenvalue with multiplicity 1. Then we pick up two branches(vertices)

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 964-973, December 2015

ELA

from them, and we connect them to the vertex p_1, then we can get a tree T' with $|V| = 2$ with $V = \{p_1, p_3\}$. Furthermore, if we remove the vertex p_3 from T', and connect two branches(vertices) at p_3 to the vertex p_1, then we have a star T'', where $|V| = 1$ with $V = \{p_1\}$.

Acknowledgement. The authors would like to thank to Carlos Saiago for his helpful comments that have improved the proof of Corollary 2.4 and express sincere thanks to referees for their helpful comments that have improved the paper.

REFERENCES

[1] C.R. Johnson, A. Leal-Duarte, C.M. Saiago, B.D. Sutton, and A.J. Witt. On the relative position of multiple eigenvalues in the spectrum of an Hermitian matrix with a given graph. *Linear Algebra Appl.*, 363:147–159, 2003.

[2] C.R. Johnson, A. Leal-Duarte, and C.M. Saiago. The Parter-Wiener theorem: Refinement and generalization. *SIAM J. Matrix Anal. Appl.*, 25(2):352–361, 2003.

[3] C.R. Johnson and A. Leal-Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear Multilinear Algebra*, 46:139–144, 1999.

[4] C.R. Johnson, A. Leal-Duarte, and C.M. Saiago. The structure of matrices with a maximum multiplicity eigenvalue. *Linear Algebra Appl.*, 429:875–886, 2008.

[5] I.-J. Kim and B.L. Shader. On Fiedler- and Parter-vertices of acyclic matrices. *Linear Algebra Appl.*, 428:2601–2613, 2008.

[6] I.-J. Kim and B.L. Shader. Non-singular acyclic matrices. *Linear Multilinear Algebra*, 57(4):399–407, 2009.

[7] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *J. Soc. Indust. Appl. Math.*, 8:376–388, 1960.

[8] S.M. Rump. INTLAB-INTerval LABoratory. In: Tibor Cesndes (editor), *Developments in Reliable Computing*, Kluwer Academic Publishers, 77–104, 1999.

[9] K. Toyonaga. Numerical enclosure for multiple eigenvalues of an Hermitian matrix whose graph is a tree. *Linear Algebra Appl.*, 431:1989–1999, 2009.

[10] G. Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra Appl.*, 61:15–29, 1984.