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THE NUMERICAL RADIUS OF A WEIGHTED SHIFT OPERATOR*

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Abstract. In this paper, the point spectrum of the real Hermitian part of a weighted shift operator with weight sequence $a_1, a_2, \ldots, a_n, 1, 1, \ldots$ is investigated and the numerical radius of the weighted shift operator in terms of the weighted shift matrix with weights $a_1, a_2, \ldots, a_n$ is formulated explicitly.

Key words. Numerical radius, Weighted shift operator, Weighted shift matrix.

AMS subject classifications. 47A12, 15A60.

1. Introduction. Let A be an operator on a separable Hilbert space H. The numerical range of A is defined to be the set

$$W(A) = \{\langle Ax, x\rangle : \|x\| = 1, x \in H\}.$$

The numerical radius $w(A)$ is the supremum of the modulus of $W(A)$. It is a classical result due to Toeplitz and Hausdorff that the numerical range is a convex set. For references on the theory of numerical range, see, for instance, [1, 10, 11, 12]. We consider a weighted shift operator A with weights $(a_1, a_2, \ldots)$ on the Hilbert space

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$\ell^2(\mathbf{N})$ defined by

$$A = A(a_1, a_2, \ldots) = \begin{pmatrix} 0 & 0 & 0 & \ldots & \ldots \\ a_1 & 0 & 0 & \ddots & \ddots \\ 0 & a_2 & 0 & \ddots & \ddots \\ 0 & 0 & a_3 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\{a_n\}$ is a bounded sequence. From the operator-theoretic view point, the class of weighted shift operators contains a typical non-unitary isometry $A(1, 1, \ldots)$ (cf. [2]). The real part Hermitian operator $\Re(A) = (A + A^*)/2$ of a weighted operator A is interpreted as the adjacency matrix of weighted A_∞-graph (cf. [8]). In addition, weighted shift operators are closely related to numerical analysis and information theory (cf. [4]).

Shields [16] proved that the numerical range $W(A)$ of a weighted shift operator A is a circular disk centered at the origin. In this case, the radius of the disk equals its numerical radius $w(A)$ which is the maximal spectrum of the self-adjoint operator $\Re(A)$. Ridge [15] computed the radius for a weighted shift operator with periodic weights. Computations of the radii of weighted shift operators with typical weights such as $(r, 1, 1, \ldots)$, $(1, s, 1, 1, \ldots)$, $(r, s, 1, 1, \ldots)$ and $(r, r^2, r^3, \ldots)$ were carried out in [2, 5, 6, 18, 19].

Stout [17] provided a method to obtain the numerical radius of a weighted shift operator $A(a_1, a_2, \ldots)$ with square summable weights by introducing the analytic function

$$F_A(z) = \det(I - z\Re(A(a_1, a_2, \ldots))).$$

It is shown in [17] that the analytic function is given by

$$F_A(z) = 1 + \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k c_k z^{2k},$$

where

$$c_k = \sum a_{i_1}^2 a_{i_2}^2 \cdots a_{i_k}^2,$$

the sum being taken over

$$1 \le i_1 < i_2 < \cdots < i_k < \infty, \ \ i_2 - i_1 \ge 2, \ \ i_3 - i_2 \ge 2, \ldots, \ \ i_k - i_{k-1} \ge 2,$$

and the radius $w(A(a_1, a_2, \ldots)) = 1/\lambda$, where λ is the minimal positive root of $F_A(z) = 0$.

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In the finite-dimensional case, an $n \times n$ weighted shift matrix with weights $(a_1, a_2, \ldots, a_{n-1})$ is defined by

$$A(a_1, a_2, \ldots, a_{n-1}) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ a_1 & 0 & 0 & \ddots & \vdots \\ 0 & a_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & a_{n-1} & 0 \end{pmatrix}.$$

It is easy to see that a weighted shift matrix or operator A is unitarily equivalent to its entry-wise modulus operator $|A|$. Hence we may assume that the weights are nonnegative for the discussion of the numerical range of a weight shift matrix or operator. There have been a number of interesting papers on the properties of the numerical ranges of weighted shift matrices ([3, 6, 7, 9, 16, 17, 20]). The numerical range of a weighted matrix A is a closed disk centered at the origin. Various radii of the disk were studied, e.g.,[6, 13, 19]. In particular, $w(A(1, 1, \ldots, 1)) = \cos(\pi/(n+1))$ for the $n \times n$ shift matrix $A(1, 1, \ldots, 1)$ (cf. [13]). The numerical radii of the modified shift matrices $A(1, \ldots, 1, r, 1, \ldots, 1)$ and $A(1, \ldots, 1, r, r, 1, \ldots, 1)$ were computed respectively in [6] and [19].

In this paper, we consider weighted shift operators $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ which perturb the canonical shift operator $A(1, 1, \ldots)$, and investigate the point spectrum of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ which gives the numerical radius of the operator $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$. Furthermore, we explicitly formulate the radius $w(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ in terms of the weighted shift matrix $A(a_1, a_2, \ldots, a_n)$.

2. Weighted shift matrices. Let $A(a_1, a_2, \ldots, a_{n-1})$ be an $n \times n$ weighted shift matrix with nonnegative weights. The spectral analysis of a real symmetric tridiagonal matrix $\Re(A)$ is related to numerical analysis (cf. [4]). The Hermitian matrix $2I_n - 2\Re(A(1, \ldots, 1))$ is also discussed as the discrete Laplacian in applied mechanics ([14]).

The characteristic polynomial

$$p_n(t) = \det\Big(tI_n - \Re(A(a_1, a_2, \ldots, a_{n-1}))\Big)$$

has the recurrence

$$p_n(t) = tp_{n-1}(t) - \frac{1}{4}a_{n-1}^2 p_{n-2}(t).$$

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By the formula [17, Lemma 1],

$$p_n(t) = t^n + \sum_{1 \le k \le n/2} \left(-\frac{1}{4}\right)^k S_k(a_1, \dots, a_{n-1}) t^{n-2k}, \tag{2.1}$$

where the circularly symmetric functions

$$S_k(a_1, \dots, a_{n-1}) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n-1} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2,$$

over all $1 \le j_1 < j_2 < \cdots < j_k \le n-1$ satisfying $j_2 - j_1 \ge 2$, $j_3 - j_2 \ge 2, \dots, j_k - j_{k-1} \ge 2$. The circularly symmetric function $S_k(a_1, \dots, a_{n-1})$ is abbreviated to $S_k^{(n-1)}$ if there is no confusion. We will use Chebyshev polynomials of the second kind to find $\alpha = w(A(a_1, a_2, \dots, a_{n-1}))$. If the weights $a_1, \dots, a_{n-1}$ are positive and relatively small, e.g., less than 1, we have $0 < \alpha < 1$. Then $0 < \theta_0 = \arccos(\alpha) < \pi$ is the minimal zero of the trigonometric polynomial

$$2^n \sin\theta \, \det\Big(\cos\theta I_n - \Re(A(a_1, a_2, \dots, a_{n-1}))\Big). \tag{2.2}$$

On the other hand, if the weights $a_1, \dots, a_{n-1}$ are relatively large, say, greater than $\sec(\pi/(n+1))$, then $\alpha > 1$ and $\theta_0 = \operatorname{arccosh}(\alpha)$ is the minimal zero of the trigonometric polynomial

$$2^n \sinh\theta \, \det\Big(\cosh\theta I_n - \Re(A(a_1, a_2, \dots, a_{n-1}))\Big).$$

Let ${}_\ell C_k, 0 \le k \le \ell$, denote the binomial coefficients with boundary values ${}_\ell C_0 = {}_\ell C_\ell = 1$. For $1 \le k \le \ell - 1$, ${}_\ell C_k = \ell!/(k!(\ell-k)!)$. The following lemma is essential to expand the trigonometric polynomial (2.2).

Lemma 2.1.

$$\begin{aligned} 2^n \sin\theta \cos^n\theta &= \sum_{k=0}^{n-1} {}_{n-1}C_k \sin((n+1-2k)\theta) \\ &= \sum_{0 \le k \le n/2} ({}_{n-1}C_k - {}_{n-1}C_{k-2}) \sin((n+1-2k)\theta), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} 2^n \sinh\theta \cosh^n\theta &= \sum_{k=0}^{n-1} {}_{n-1}C_k \sinh((n+1-2k)\theta) \\ &= \sum_{0 \le k \le n/2} ({}_{n-1}C_k - {}_{n-1}C_{k-2}) \sinh((n+1-2k)\theta). \end{aligned} \tag{2.4}$$

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Proof. We compute

$$\begin{aligned}\sum_{k=0}^{n-1} {}_{n-1}C_k \sin((n+1-2k)\theta) &= \Im\Big(\sum_{k=0}^{n-1} {}_{n-1}C_k e^{i(n+1-2k)\theta}\Big)\\ &= \Im\Big(e^{i(n+1)\theta}\sum_{k=0}^{n-1} {}_{n-1}C_k e^{-2ik\theta}\Big)\\ &= \Im\Big(e^{i(n+1)\theta}\big(1+e^{-2i\theta}\big)^{n-1}\Big)\\ &= \Im\Big(e^{2i\theta}\big(e^{i\theta}+e^{-i\theta}\big)^{n-1}\Big)\\ &= 2^{n-1}\cos^{n-1}\theta\ \Im(e^{2i\theta})\\ &= 2^n \sin\theta\cos^n\theta,\end{aligned}$$

where $\Im(z)$ denotes the imaginary part of a complex number. This proves the first equality of (2.3). Next, we prove the second equality of (2.3).

Assume $n = 2m+1$ is odd. Then

$$\begin{aligned}&\sum_{0\le k\le 2m} {}_{2m}C_k \sin((2m+2-2k)\theta)\\ &\quad= {}_{2m}C_0 \sin((2m+2)\theta) + {}_{2m}C_1 \sin(2m\theta)\\ &\qquad+ \sum_{2\le k\le m} {}_{2m}C_k \sin((2m+2-2k)\theta)\\ &\qquad+ \sum_{m+2\le k\le 2m} {}_{2m}C_k \sin((2m+2-2k)\theta).\end{aligned} \tag{2.5}$$

Observe that

$$\sin((2m+2-2k)\theta) = -\sin((2m+2-2(2m+2-k))\theta) = -\sin((2m+2-2j)\theta),$$

where $j = 2m+2-k$. Since $m+2 \le k \le 2m$, we have $2 \le j \le m$. Hence,

$$\begin{aligned}\sum_{m+2\le k\le 2m} {}_{2m}C_k \sin((2m+2-2k)\theta) &= \sum_{m+2\le k\le 2m} {}_{2m}C_{2m-k} \sin((2m+2-2k)\theta)\\ &= -\sum_{2\le j\le m} {}_{2m}C_{j-2} \sin((2m+2-2j)\theta).\end{aligned} \tag{2.6}$$

Taking together (2.5) and (2.6), we have

$$\sum_{0\le k\le 2m} {}_{2m}C_k \sin((2m+2-2k)\theta) = \sum_{0\le k\le m} ({}_{2m}C_k - {}_{2m}C_{k-2}) \sin((2m+2-2k)\theta),$$

where ${}_{2m}C_{-2} = {}_{2m}C_{-1} = 0$.

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Assume $n = 2m$ is even. Similar computations yield that

$$\begin{aligned}
&\sum_{0\le k\le 2m-1} {}_{2m-1}C_k \sin((2m+1-2k)\theta)\\
&\quad = {}_{2m-1}C_0 \sin(2m+1)\theta + {}_{2m-1}C_1 \sin((2m-1)\theta)\\
&\qquad + \sum_{2\le k\le m} {}_{2m-1}C_k \sin((2m+1-2k)\theta)\\
&\qquad + \sum_{m+1\le k\le 2m-1} {}_{2m-1}C_k \sin((2m+1-2k)\theta)\\
&\quad = \sum_{0\le k\le m} ({}_{2m-1}C_k - {}_{2m-1}C_{k-2}) \sin((2m+1-2k)\theta),
\end{aligned}$$

where ${}_{2m-1}C_{-2} = {}_{2m-1}C_{-1} = 0$.

As to the formula (2.4), we compute that

$$\begin{aligned}
&\sum_{0\le k\le n-1} {}_{n-1}C_k \sinh((n+1-2k)\theta)\\
&\quad = \frac{1}{2}\Big(\sum_{0\le k\le n-1} {}_{n-1}C_k \Big(e^{(n+1-2k)\theta} - e^{(-n-1+2k)\theta}\Big)\Big)\\
&\quad = \frac{1}{2}(e^{2\theta} - e^{-2\theta})(e^{\theta} + e^{-\theta})^{n-1}\\
&\quad = \frac{1}{2}(e^{\theta} - e^{-\theta})(e^{\theta} + e^{-\theta})(e^{\theta} + e^{-\theta})^{n-1}\\
&\quad = 2^n(\frac{e^{\theta} - e^{-\theta}}{2})(\frac{e^{\theta} + e^{-\theta}}{2})^n\\
&\quad = 2^n \sinh\theta \cosh^n\theta.
\end{aligned}$$

The second equality of (2.4) can be derived in a similar way. $\square$

Applying Lemma 2.1, we expand the determinant in (2.2), which generalizes results of [6, Theorem 2.1] and [19, Theorem 2.1].

THEOREM 2.2. *Let $n \ge 2$. Then*

$$2^n \sin\theta \det\Big(\cos\theta I_n - \Re(A(a_1, a_2, \dots, a_{n-1}))\Big) = \sum_{0\le k\le [n/2]} h_k^{(n-1)} \sin((n+1-2k)\theta)$$

and

$$2^n \sinh\theta \det\Big(\cosh\theta I_n - \Re(A(a_1, a_2, \dots, a_{n-1}))\Big) = \sum_{0\le k\le [n/2]} h_k^{(n-1)} \sinh((n+1-2k)\theta),$$

where

$$\begin{aligned}
h_k^{(n-1)} &= ({}_{n-1}C_k - {}_{n-1}C_{k-2})\\
&\quad + \Big(\sum_{\ell=1}^{k-1} (-1)^{\ell} ({}_{n-1-2\ell}C_{k-\ell} - {}_{n-1-2\ell}\, C_{k-\ell-2}) S_\ell^{(n-1)}\Big) + (-1)^k S_k^{(n-1)},
\end{aligned}$$

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$0 \le k \le [n/2]$, *and* $h_0^{(n-1)} = 1$, $h_1^{(n-1)} = {}_{n-1}C_1 - S_1^{(n-1)}$.

Proof. We compute that

$$
\begin{aligned}
&2^n \sin\theta \det\Big(\cos\theta I_n - \Re(A(a_1, a_2, \ldots, a_{n-1}))\Big) \\
&\quad = 2^n \sin\theta\Big(\cos^n\theta + \sum_{1\le k\le[n/2]} \left(-\frac{1}{4}\right)^k S_k^{(n-1)} \cos^{n-2k}\theta\Big) \quad \text{(by (2.1))} \\
&\quad = 2^n \sin\theta\cos^n\theta + \sum_{1\le k\le[n/2]} (-1)^k 2^{n-2k} \sin\theta\cos^{n-2k}\theta\, S_k^{(n-1)} \\
&\quad = \sum_{0\le \ell\le[n/2]} ({}_{n-1}C_\ell - {}_{n-1}C_{\ell-2}) \sin((n+1-2\ell)\theta) \\
&\qquad + \sum_{1\le k\le[n/2]} (-1)^k S_k^{(n-1)} \times \\
&\qquad\quad \Big(\sum_{0\le\ell\le[(n-2k)/2]} ({}_{n-2k-1}C_\ell - {}_{n-2k-1}C_{\ell-2}) \sin((n-2k+1-2\ell)\theta)\Big) \\
&\qquad \text{(by Lemma2.1)} \\
&\quad = \sin((n+1)\theta) + {}_{n-1}C_1 \sin((n-1)\theta) + ({}_{n-1}C_2 - {}_{n-1}C_0)\sin((n-3)\theta) + \cdots \\
&\qquad - S_1^{(n-1)}\Big(\sin((n-1)\theta) + {}_{n-3}C_1\sin((n-3)\theta) \\
&\qquad\qquad + ({}_{n-3}C_2 - {}_{n-3}C_0)\sin((n-5)\theta) + \cdots\Big) \\
&\qquad + S_2^{(n-1)}\Big(\sin((n-3)\theta) + {}_{n-5}C_1\sin((n-5)\theta) \\
&\qquad\qquad + ({}_{n-5}C_2 - {}_{n-5}C_0)\sin((n-7)\theta) + \cdots\Big) \\
&\qquad\qquad\qquad\qquad \vdots \\
&\quad = \sum_{0\le k\le[n/2]} \Big(({}_{n-1}C_k - {}_{n-1}C_{k-2}) - S_1^{(n-1)}({}_{n-3}C_{k-1} - {}_{n-3}C_{k-3}) \\
&\qquad + S_2^{(n-1)}({}_{n-5}C_{k-2} - {}_{n-5}C_{k-4}) - \cdots\Big) \sin((n+1-2k)\theta) \\
&\quad = \sum_{0\le k\le[n/2]} h_k^{(n-1)} \sin((n+1-2k)\theta).
\end{aligned}
$$

The second assertion can be proved in a similar way. ∎

3. Weighted shift operators. The numerical radius of a weighted shift operator has attracted much attention because of its importance and complexity. Research papers on this subject include the Hilbert-Schmidt class of of weighted shift operators, i.e., with square summable weights (cf. [17]), and the modified canonical shift operator such as $(\alpha, 1, 1, \ldots)$, $(1, \alpha, 1, 1, \ldots)$ and $(\alpha, \beta, 1, 1, \ldots)$ (cf. [2, 6, 19]).

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In this section, we consider weighted shift operators $A(a_1, a_2, \ldots)$ acting on a complex Hilbert space $\ell^2(\mathbf{N})$ identified with the Hardy space H^2 satisfying

$$\lim_{\ell\to\infty} a_\ell = 1, \ \sum_{\ell=1}^{\infty} |a_\ell - 1| < \infty \quad \text{and} \quad \prod_{j=1}^{\ell} a_j \to \beta \text{ as } \ell \to \infty$$

for some $0 < \beta < \infty$. A typical weighted shift operator of this class is $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ that perturbs the canonical shift operator $A = A(1, 1, \ldots)$.

THEOREM 3.1. *Let $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ be a weighted shift operator with positive weights. Then $w(A(a_1, \ldots, a_n, 1, 1, \ldots)) > 1$ if and only if $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ has an eigenvalue greater than 1.*

Proof. Since $W(A(a_1, \ldots, a_n, 1, 1, \ldots))$ is a circular disk centered at the origin, it follows that $w(A(a_1, \ldots, a_n, 1, 1, \ldots)) = w(\Re(A(a_1, \ldots, a_n, 1, 1, \ldots)))$. The sufficiency is trivial. Assume $w(A(a_1, \ldots, a_n, 1, 1, \ldots)) > 1$. For the Hermitian operator $\Re(A(a_1, \ldots, a_n, 1, 1, \ldots))$,

$$\|\Re(A(a_1, \ldots, a_n, 1, 1, \ldots))\| = w(\Re(A(a_1, \ldots, a_n, 1, 1, \ldots))).$$

Then, by [19, Lemma 3.1], $w(\Re(A(a_1, \ldots, a_n, 1, 1, \ldots)))$ and thus $w(A(a_1, \ldots, a_n, 1, 1, \ldots))$ is an eigenvalue of $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$. $\square$

The following result characterizes the existence of an eigenvalue $\alpha \geq 0$ of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$, and it turns out the eigenvalue must be greater than 1.

THEOREM 3.2. *Let $n \geq 1$ and $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ be a weighted shift operator with positive weights. A value $\alpha \geq 0$ is an eigenvalue of $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ if and only if there is a nonzero formal power series*

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1} + \cdots$$

with $f(0) \neq 0$ belonging to the Hardy space H^2 which satisfies the following recurrence relations

$$f'(0) = \frac{2\alpha}{a_1} f(0), \ \frac{f^{(k)}(0)}{k!} = \frac{2\alpha}{a_k}\frac{f^{(k-1)}(0)}{(k-1)!} - \frac{a_{k-1}}{a_k}\frac{f^{(k-2)}(0)}{(k-2)!} \tag{3.1}$$

for $k = 2, \ldots, n$, and

$$\frac{f^{(n+1)}(0)}{(n+1)!} = 2\alpha\frac{f^{(n)}(0)}{n!} - a_n\frac{f^{(n-1)}(0)}{(n-1)!}, \tag{3.2}$$

$$\frac{f^{(m)}(0)}{m!} = 2\alpha\frac{f^{(m-1)}(0)}{(m-1)!} - \frac{f^{(m-2)}(0)}{(m-2)!} \tag{3.3}$$

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for $m = n+2, n+3, \ldots$ *In this case, the eigenvalue* $\alpha > 1$ *and*

$$\frac{f^{(n+1)}(0)}{(n+1)!} - (\alpha - \sqrt{\alpha^2 - 1})\frac{f^{(n)}(0)}{n!} = 0. \tag{3.4}$$

Proof. Assume that $\alpha \geq 0$. Clearly, α is an eigenvalue of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ if and only if there is a corresponding nonzero eigenfunction

$$f(z) = f(0) + f'(0)z + \frac{f^{(2)}(0)}{2}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \cdots \in H^2$$

satisfying $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))f = \alpha f$ which is equivalent to the recurrence relations (3.1), (3.2) and (3.3).

From the recurrence relation (3.3), the coefficients of the eigenfunction satisfy the recurrence

$$\frac{f^{(k)}(0)}{k!} = 2\alpha\frac{f^{(k-1)}(0)}{(k-1)!} - \frac{f^{(k-2)}(0)}{(k-2)!} \tag{3.5}$$

for $k = n+2, n+3, \ldots$ If $\alpha = 0$, the recurrence relation (3.5) implies that

$$\frac{f^{(n+2p)}(0)}{(n+2p)!} = (-1)^p\frac{f^{(n)}(0)}{n!} \quad \text{and} \quad \frac{f^{(n+1+2p)}(0)}{(n+1+2p)!} = (-1)^p\frac{f^{(n+1)}(0)}{(n+1)!},$$

$p = 1, 2, \ldots$ Since $f \in H^2$, it follows that

$$\frac{f^{(n+1)}(0)}{(n+1)!} = \frac{f^{(n)}(0)}{n!} = 0,$$

and this derives the function $f = 0$ from (3.1).

If $0 < \alpha < 1$ and is expressed as $\alpha = \cos\theta$ for some $0 < \theta < \pi/2$, then the difference equation (3.5) implies that

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \gamma e^{ip\theta} + \delta e^{-ip\theta},$$

$p = 0, 1, 2, \ldots$, for some constants γ and δ. Again, for $f \in H^2$, we have $\gamma = 0$ and $\delta = 0$ as well.

If $\alpha = 1$, then by (3.5),

$$\frac{f^{(n+p+2)}(0)}{(n+p+2)!} - \frac{f^{(n+p+1)}(0)}{(n+p+1)!} = \frac{f^{(n+p+1)}(0)}{(n+p+1)!} - \frac{f^{(n+p)}(0)}{(n+p)!},$$

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$p = 0, 1, 2, \ldots$, and hence

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \frac{f^{(n)}(0)}{n!} + p\eta,$$

$p = 0, 1, 2, \ldots$, for some constant η. By the same fact that $f \in H^2$, we have $\eta = 0$ and

$$\frac{f^{(n+1)}(0)}{(n+1)!} = \frac{f^{(n)}(0)}{n!} = 0,$$

and thus $f = 0$. This proves the eigenvalue $\alpha > 1$.

The characteristic equation of the difference equation (3.5) is

$$r^2 - 2\alpha r + 1 = 0.$$

The general solution of the difference equation becomes

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \mu_1(\alpha + \sqrt{\alpha^2 - 1})^p + \mu_2(\alpha - \sqrt{\alpha^2 - 1})^p,$$

$p = 0, 1, 2, \ldots$, for some constants μ_1 and μ_2. Notice that $\alpha + \sqrt{\alpha^2 - 1} > 1$. Hence, $\mu_1 = 0$, and the initial condition implies that $\mu_2 = \frac{f^{(n)}(0)}{n!}$, and this concludes (3.4). $\square$

In the following, we explicitly formulate a characteristic equation for the point spectrum of the operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ in terms of the weighted shift matrix $A(a_1, a_2, \ldots, a_n)$.

THEOREM 3.3. *Let $n \geq 1$ and $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ be a weighted shift operator with positive weights, and let $f(z)$ be a nonzero formal power series:*

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \frac{f^{(n+1)}(0)}{(n+1)!}z^{n+1} + \cdots .$$

Assume $\alpha > 1$. Then α is an eigenvalue of $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$ with eigenfunction $f(z) \in H^2$ if and only if the coefficients of $f(z)$ satisfy condition (3.1), and the value $z = \alpha - \sqrt{\alpha^2 - 1}$ is a zero of the polynomial

$$F_n(z) = Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z),$$

where

$$Q_\ell(z) = \det\Big((z^2 + 1)I_{\ell+1} - 2z\Re(A(a_1, \ldots, a_\ell))\Big),$$

$\ell = 1, 2, \ldots$ and $Q_0(z) = z^2 + 1$, $Q_{-1}(z) = 1$.

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Proof. Assume that α is an eigenvalue of the Hermitian operator $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))$, and its corresponding nonzero eigenfunction is given by

$$f(z) = f(0) + f'(0)z + \frac{f^{(2)}(0)}{2}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \cdots \in H^2.$$

The equation $\Re(A(a_1, a_2, \ldots, a_n, 1, 1, \ldots))f = \alpha f$ leads to the relation

$$\begin{aligned}&(z^2 - 2\alpha z + 1)f(z)\\ &\quad = f(0) - (a_1 - 1)z^2 f(0) - \Big(\sum_{k=1}^{n-1}((a_{k+1} - 1)z^{k+2} + (a_k - 1)z^k)\frac{f^{(k)}(0)}{k!}\Big)\\ &\qquad -(a_n - 1)\frac{f^{(n)}(0)}{n!}z^n.\end{aligned}$$

We define the above polynomial by

$$\begin{aligned}H_n(z) = f(0) &- (a_1 - 1)z^2 f(0)\\ &-\Big(\sum_{k=1}^{n-1}((a_{k+1} - 1)z^{k+2} + (a_k - 1)z^k)\frac{f^{(k)}(0)}{k!}\Big)\\ &-(a_n - 1)\frac{f^{(n)}(0)}{n!}z^n,\end{aligned} \tag{3.6}$$

and

$$F_n(z) = \frac{a_1 a_2 \cdots a_n}{f(0)} H_n(z).$$

Since $z = \alpha - \sqrt{\alpha^2 - 1}$ is a root of $z^2 - 2\alpha z + 1 = 0$, it follows that $F_n(\alpha - \sqrt{\alpha^2 - 1}) = H_n(\alpha - \sqrt{\alpha^2 - 1}) = 0$. By the relation of the sequence $f^{(k)}(0)/k!$ in (3.1), we have that

$$\begin{aligned}a_1 a_2 \cdots a_k \frac{f^{(k)}(0)}{k!}z^k &= f(0)z^k\Big((2\alpha)^k - (2\alpha)^{k-2}S_1^{(k-1)} + (2\alpha)^{k-4}S_2^{(k-1)} + \cdots\Big)\\ &= f(0)z^k\Big((2\alpha)^k + \sum_{1\le \ell\le k/2}(-1)^\ell(2\alpha)^{k-2\ell}S_\ell^{(k-1)}\Big)\\ &= f(0)\Big((z^2+1)^k + \sum_{1\le \ell\le k/2}(-1)^\ell z^{2\ell}(z^2+1)^{k-2\ell}S_\ell^{(k-1)}\Big).\end{aligned} \tag{3.7}$$

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Substituting (3.7) into (3.6), we obtain that

$$\begin{aligned}
F_n(z) &= a_1a_2\cdots a_n + (a_1a_2\cdots a_n - a_1^2a_2a_3\cdots a_n)z^2 \\
&\quad + \sum_{k=1}^{n-1}(a_{k+1}a_{k+2}\cdots a_n - a_{k+1}^2a_{k+2}\cdots a_n)z^2 \\
&\qquad \times\Big((z^2+1)^k + \sum_{1\le \ell\le k/2}(-1)^\ell z^{2\ell}(z^2+1)^{k-2\ell}S_\ell^{(k-1)}\Big) \\
&\quad + \sum_{k=1}^{n-1}(a_{k+1}a_{k+2}\cdots a_n - a_ka_{k+1}\cdots a_n) \\
&\qquad \times\Big((z^2+1)^k + \sum_{1\le \ell\le k/2}(-1)^\ell z^{2\ell}(z^2+1)^{k-2\ell}S_\ell^{(k-1)}\Big) \\
&\quad + (1-a_n)\Big((z^2+1)^n + \sum_{1\le \ell\le n/2}(-1)^\ell z^{2\ell}(z^2+1)^{n-2\ell}S_\ell^{(n-1)}\Big).
\end{aligned}$$

The polynomial $F_n(z)$ with corresponding coefficients $c_k^{(n)}$ is written as

$$F_n(z) = c_0^{(n)} + c_1^{(n)}z^2 + c_2^{(n)}z^4 + c_3^{(n)}z^6 + \cdots + c_n^{(n)}z^{2n}. \tag{3.8}$$

We compute the coefficients $c_k^{(n)}$. For $k=0$,

$$\begin{aligned}
c_0^{(n)} &= a_1a_2\cdots a_n + \sum_{k=1}^{n-1}(a_{k+1}a_{k+2}\cdots a_n - a_ka_{k+1}\cdots a_n) + (1-a_n) \\
&= 1 - a_n + (a_n - a_{n-1}a_n) + (a_{n-1}a_n - a_{n-2}a_{n-1}a_n) + \cdots \\
&\qquad + (a_2a_3\cdots a_n - a_1a_2\cdots a_n) + a_1a_2\cdots a_n \\
&= 1.
\end{aligned}$$

For $k=n$, and $n=1$, by using $a_1\, zf'(0) = (z^2+1)f(0)$, we obtain that

$$H_1(z) = f(0) - (a_1-1)z^2f(0) - (a_1-1)zf'(0), \quad F_1(z) = 1 + (1-a_1^2)z^2,$$

and hence $c_1^{(1)} = 1 - a_1^2$. For $n\ge 2$,

$$\begin{aligned}
&\frac{a_1a_2\ldots a_n}{f(0)}(1-a_n)\Big(z^n\frac{f^{(n)}(0)}{n!} + z^{n+1}\frac{f^{(n-1)}(0)}{(n-1)!}\Big) \\
&\quad = (1-a_n)\Big(z^{2n} + a_nz^2z^{2n-2} + \gamma z^{2n-2} + \cdots\Big) \\
&\quad = (1-a_n^2)z^{2n} + (1-a_n)\gamma z^{2n-2} + \cdots .
\end{aligned}$$

Hence, $c_n^{(n)} = 1 - a_n^2$.

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For $1 \leq k \leq n-1$, we deduce from the definition of $H_n(z)$ in (3.6) that

$$H_n(z) = H_{n-1}(z) - (a_n - 1)\Big(z^{n+1}\frac{f^{(n-1)}(0)}{(n-1)!} + z^n\frac{f^{(n)}(0)}{n!}\Big).$$

Then

$$\begin{aligned}
(3.9)\quad F_n(z) &= \frac{a_1a_2\cdots a_n}{f(0)}\Big(H_{n-1}(z) - (a_n-1)\Big(z^{n+1}\frac{f^{(n-1)}(0)}{(n-1)!} + z^n\frac{f^{(n)}(0)}{n!}\Big)\Big)\\
&= a_nF_{n-1}(z) + (a_n - a_n^2)\Big(z^2(z^2+1)^{n-1}\\
&\quad + \sum_{1\leq \ell\leq (n-1)/2}(-1)^\ell z^{2+2\ell}(z^2+1)^{n-1-2\ell}S_\ell^{(n-2)}\Big)\\
&\quad + (1-a_n)\Big((z^2+1)^n + \sum_{1\leq\ell\leq n/2}(-1)^\ell z^{2\ell}(z^2+1)^{n-2\ell}S_\ell^{(n-1)}\Big).
\end{aligned}$$

Comparing the coefficients of (3.8) and (3.9) , we have the following recurrence relation for $c_k^{(n)}$, $1 \leq k \leq n-1$:

$$\begin{aligned}
0 &= -c_k^{(n)} + a_nc_k^{(n-1)} + (a_n - a_n^2)_{n-1}C_{k-1} + (1-a_n)_nC_k\\
&\quad + (a_n - a_n^2)\sum_{1\leq\ell\leq k-1, 2\ell\leq n-1}(-1)^\ell{}_{n-1-2\ell}C_{k-1-\ell}S_\ell^{(n-2)}\\
&\quad + (1-a_n)\sum_{1\leq\ell\leq k, 2\ell\leq n}(-1)^\ell{}_{n-2\ell}C_{k-\ell}S_\ell^{(n-1)}\\
&= -c_k^{(n)} + B_k^{(n)} - a_n(-c_k^{(n-1)} + A_k^{(n-1)}),
\end{aligned}$$

where

$$\begin{aligned}
A_k^{(n)} &= {}_nC_k - {}_{n-1}C_{k-1}a_n^2 - {}_{n-2}C_{k-1}\sum_{j=1}^{n-1}a_j^2\\
&\quad + \sum_{\ell=2}^{k-1}(-1)^\ell[{}_{n-2\ell+1}C_{k-\ell}S_{\ell-1}^{(n-2)}a_n^2 + {}_{n-2\ell}C_{k-\ell}S_\ell^{(n-1)}] + (-1)^kS_k^{(n)}
\end{aligned}$$

and

$$\begin{aligned}
B_k^{(n)} &= {}_nC_k - {}_{n-1}C_{k-1}a_n^2 - a_n^2\sum_{1\leq\ell\leq k-1, 2\ell\leq n-2}(-1)^\ell{}_{n-1-2\ell}C_{k-1-\ell}S_\ell^{(n-2)}\\
&\quad + \sum_{1\leq\ell\leq k, 2\ell\leq n}(-1)^\ell{}_{n-2\ell}C_{k-\ell}S_\ell^{(n-1)}.
\end{aligned}$$

Direct computations show that $A_k^{(n)} = B_k^{(n)}$ and also $c_k^{(n-1)} = A_k^{(n-1)}$. Therefore, $c_k^{(n)} = A_k^{(n)} = B_k^{(n)}$ for every $n = 1, 2, \ldots$ and $1 \leq k \leq n$.

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We decompose the coefficient $c_k^{(n)} = B_k^{(n)}$ for $k \geq 2$ into two parts. By letting

$$\tilde{c}_k^{(n)} = {}_nC_k + \sum_{1 \leq \ell \leq k, \ell \leq n/2} (-1)^\ell {}_{n-2\ell}C_{k-\ell} S_\ell^{(n)}, \;\; k = 0, 1, \ldots, n$$

and

$$d_k^{(n)} = {}_{n-2}C_{k-2} + \sum_{1 \leq \ell \leq k-2, \ell \leq (n-2)/2} (-1)^\ell {}_{n-2-2\ell}C_{k-2-\ell} S_\ell^{(n-2)}, \;\; k = 2, 3, \ldots, n.$$

We compute that

$$\begin{aligned}
\tilde{c}_k^{(n)} - B_k^{(n)} &= {}_nC_k + \sum_{\ell=1}^{k} (-1)^\ell {}_{n-2\ell}C_{k-\ell} S_\ell^{(n)} - {}_nC_k - \sum_{\ell=1}^{k} {}_{n-2\ell}C_{k-\ell} S_\ell^{(n-1)} \\
&\quad + {}_{n-1}C_{k-1} a_n^2 + a_n^2 \sum_{\ell=1}^{k-1} (-1)^\ell {}_{n-1-2\ell}C_{k-1-\ell} S_\ell^{(n-2)} \\
&= a_n^2 \sum_{\ell=1}^{k} (-1)^\ell {}_{n-2\ell}C_{k-\ell} S_{\ell-1}^{(n-2)} + {}_{n-1}C_{k-1} a_n^2 \\
&\quad + a_n^2 \sum_{\ell=1}^{k-1} (-1)^\ell {}_{n-1-2\ell}C_{k-1-\ell} S_\ell^{(n-2)} \\
&= a_n^2 d_k^{(n)}.
\end{aligned}$$

This shows that

$$c_k^{(n)} = \tilde{c}_k^{(n)} - a_n^2 d_k^{(n)},$$

and thus

$$c_k^{(n)} z^{2k} = \tilde{c}_k^{(n)} z^{2k} - a_n^2 z^4 d_k z^{2k-4}$$

with $c_0^{(n)} = 1, \quad c_1^{(n)} = -S_1^{(n)} + n$. Therefore,

$$F_n(z) = G_n(z) - a_n^2 z^4 L_n(z),$$

where

$$G_n(z) = \tilde{c}_0^{(n)} + \tilde{c}_1^{(n)} z^2 + \tilde{c}_2^{(n)} z^4 + \tilde{c}_3^{(n)} z^6 + \cdots + \tilde{c}_n^{(n)} z^{2n},$$

and

$$L_n(z) = d_2^{(n)} + d_3^{(n)} z^2 + \cdots + d_n^{(n)} z^{2n-4}.$$

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Rearranging the terms of $G_n(z)$, we have that

$$\begin{aligned}G_n(z) &= (1 + {}_nC_1z^2 + {}_nC_2z^4 + \cdots + {}_nC_nz^{2n})\\ &\quad -S_1^{(n)}z^2({}_{n-2}C_0 + {}_{n-2}C_1z^2 + \cdots + {}_{n-2}C_{n-2}z^{2n-2}) + \cdots\\ &= \sum_{k=0}^{n} {}_nC_kz^{2k} + \sum_{1\le\ell\le[(n+1)/2]} (-1)^\ell S_\ell^{(n)}z^{2\ell} \sum_{j=0}^{n-2\ell} {}_{n-2\ell}C_jz^{2j}\\ &= (z^2+1)^n + \sum_{1\le\ell\le[(n+1)/2]} (-1)^\ell S_\ell^{(n)}z^{2\ell}(z^2+1)^{n-2\ell}.\end{aligned}$$

By equation (2.1), the characteristic polynomial of $2\Re(A(a_1,\dots,a_n)))$ is formulated as

$$\det\Big(xI_{n+1} - 2y\Re(A(a_1,a_2,\dots,a_n))\Big) = \sum_{0\le\ell\le(n+1)/2} (-1)^\ell S_\ell^{(n)}x^{n+1-2\ell}y^{2\ell}.$$

Then, we obtain that

$$(z^2+1)G_n(z) = \det\Big((z^2+1)I_{n+1} - 2z\Re(A(a_1,\dots,a_n))\Big) = Q_n(z). \tag{3.10}$$

Applying the Laplace expansion on the $(n+1)$-th row of the determinant (3.10), we expand

$$\begin{aligned}&\det\Big((z^2+1)I_{n+1} - 2z\Re(A(a_1,\dots,a_n))\Big)\\ &\quad= (z^2+1)\det\Big((z^2+1)I_n - 2z\Re(A(a_1,\dots,a_{n-1}))\Big)\\ &\qquad - a_n^2z^2\det\Big((z^2+1)I_{n-1} - 2z\Re(A(a_1,\dots,a_{n-2}))\Big).\end{aligned}$$

Similarly, we can prove that

$$(z^2+1)L_n(z) = \det\Big((z^2+1)I_{n-1} - 2z\Re(A(a_1,\dots,a_{n-2}))\Big).$$

Then, we have that

$$\begin{aligned}(z^2+1)F_n(z) &= (z^2+1)G_n(z) - a_n^2z^4(z^2+1)L_n(z)\\ &= (z^2+1)\det\Big((z^2+1)I_n - 2z\Re(A(a_1,\dots,a_{n-1}))\Big)\\ &\quad - a_n^2z^2(z^2+1)\det\Big((z^2+1)I_{n-1} - 2z\Re(A(a_1,\dots,a_{n-2}))\Big).\end{aligned}$$

This implies that

$$\begin{aligned}F_n(z) &= \det\Big((z^2+1)I_n - 2z\Re(A(a_1,\dots,a_{n-1}))\Big)\\ &\quad - a_n^2z^2\det\Big((z^2+1)I_{n-1} - 2z\Re(A(a_1,\dots,a_{n-2}))\Big)\\ &= Q_{n-1}(z) - a_n^2z^2Q_{n-2}(z).\end{aligned}$$

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To prove the converse part, we first compute that

$$
\begin{aligned}
F_n(z) &= Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z)\\
&= \Big((z^2+1)Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z)\Big) - z^2 Q_{n-1}(z)\\
&= Q_n(z) - z^2 Q_{n-1}(z)\\
&= \Big((z^2+1)^{n+1} + \sum_{1\le \ell\le [(n+1)/2]} (-1)^\ell z^{2\ell}(z^2+1)^{n+1-2\ell} S_\ell^{(n)}\Big)\\
&\quad -z^2\Big((z^2+1)^n + \sum_{1\le \ell\le [n/2]} (-1)^\ell z^{2\ell}(z^2+1)^{n-2\ell} S_\ell^{(n-1)}\Big) \quad \text{(by (3.10))}\\
&= z^{n+1}\Big((2\alpha)^{n+1} + \sum_{1\le \ell\le [(n+1)/2]} (-1)^\ell (2\alpha)^{n+1-2\ell} S_\ell^{(n)}\\
&\quad -z^{n+2}\Big((2\alpha)^n + \sum_{1\le \ell\le [n/2]} (-1)^\ell (2\alpha)^{n-2\ell} S_\ell^{(n-1)}\Big)\Big)\\
&= \Big(\prod_{j=1}^n a_j\Big)\frac{z^{n+1}}{f(0)}\Big(\frac{f^{(n+1)}(0)}{(n+1)!} - z\frac{f^{(n)}(0)}{n!}\Big).
\end{aligned}
$$

Hence, if $F_n(\alpha - \sqrt{\alpha^2-1}) = 0$, we have

$$\frac{f^{(n+1)}(0)}{(n+1)!} - (\alpha - \sqrt{\alpha^2-1})\frac{f^{(n)}(0)}{n!} = 0.$$

This equation guarantees that the coefficients of the eigenfunction determined by the difference equation (3.10) satisfy the relation

$$\frac{f^{(n+p)}(0)}{(n+p)!} = \mu_1(\alpha + \sqrt{\alpha^2-1})^p + \mu_2(\alpha - \sqrt{\alpha^2-1})^p,$$

$p = 0, 1, 2, \ldots$, for some constants μ_1 and μ_2, and the constant μ_1 necessarily vanishes. This asserts that the function $f(z)$ belongs to the Hardy space H^2. $\square$

We consider a special weighted shift operator $A(a_1, 1, 1, \ldots)$. If a_1 is large enough, e.g., $a_1 > \sqrt{2}$, then $\alpha = \|\Re(A(a_1,1,1,\ldots))\| > 1$ is an eigenvalue of $\Re(A(a_1,1,1,\ldots))$. By Theorem 3.3, $F_1(z) = z^2 + 1 - a_1^2 z^2$. The positive root of $F_1(z) = 0$ is $z = 1/\sqrt{a_1^2-1}$, and thus,

$$w(A(a_1,1,1,\ldots)) = \alpha = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(\sqrt{a_1^2-1} + \frac{1}{\sqrt{a_1^2-1}}\right).$$

This formula is also obtained in [2, 20]. Similarly, for the weighted shift operator $A(1, a_2, 1, 1, \ldots)$, we have

$$F_2(z) = Q_1(z) - a_2^2 z^2 Q_0(z) = (z^2+1)^2 - z^2 - a_2^2 z^2 (z^2+1).$$

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The zeros of $F_2(z) = 0$ determine the numerical radius $w(A(1, a_2, 1, \ldots))$ (cf. [6, 20]).

The polynomial $F_n(z)$ associated with the weighted shift operator $A(a_1, a_2, \ldots, a_n, 1, 1, \ldots)$ in Theorem 3.3 is also denoted by $F_n(z; a_1, \ldots, a_n)$ if it is necessary to emphasize the first n weights. In consequence of Theorem 3.3, we have the following corollaries.

Corollary 3.4. *Let $1 \le m < n$, and $F_n(z)$ be the polynomial defined in* Theorem 3.3. *Then*

$$F_n(z; a_1, \ldots, a_m, 1, \ldots, 1) = F_m(z; a_1, \ldots, a_m).$$

Proof. By mathematical induction on $n - m$, it is sufficient to prove the case $n = m + 1$. We have that

$$\begin{aligned} F_n(z; a_1, \ldots, a_m, 1) &= Q_m(z) - z^2 Q_{m-1}(z) \\ &= (z^2 + 1) Q_{m-1}(z) - a_m^2 z^2 Q_{m-2}(z) - z^2 Q_{m-1}(z) \\ &= Q_{m-1}(z) - a_m^2 Q_{m-2}(z) \\ &= F_m(z; a_1, \ldots, a_m). \quad \square \end{aligned}$$

Corollary 3.5. *Let $F_n(z)$ be the polynomial defined in* Theorem 3.3. *Then, the recurrence equation*

$$\begin{aligned} &(a_{n+2}^2 - 1)(a_{n+1}^2 - 1) F_{n+3}(z) \\ &\quad = \Big((a_{n+2}^2 - 1)(a_{n+1}^2 - 1) + (a_{n+3}^2 - 1)(a_{n+1}^2 - 1)(z^2 + 1) \Big) F_{n+2}(z) \\ &\quad\quad - \Big((a_{n+3}^2 - 1)(a_{n+1}^2 - 1)(z^2 + 1) + (a_{n+3}^2 - 1)(a_{n+2}^2 - 1) a_{n+1}^2 z^2 \Big) F_{n+1}(z) \\ &\quad\quad + \Big((a_{n+3}^2 - 1)(a_{n+2}^2 - 1) a_{n+1}^2 z^2 \Big) F_n(z) \end{aligned}$$

holds for $n \ge 0$ with $F_0(z) = 1$.

Proof. Firstly, we prove the equation

$$F_{n+1}(z) - F_n(z) = (1 - a_{n+1}^2) z^2 Q_{n-1}(z) \tag{3.11}$$

$n = 0, 1, 2, \ldots$ Equation (3.11) holds clearly for $n = 0$ with $Q_{-1}(z) = 1$. We assume that equation (3.11) is true for indices less than n. Then

$$\begin{aligned} F_{n+1}(z) &= Q_n(z) - a_{n+1}^2 z^2 Q_{n-1}(z) \\ &= (z^2 + 1) Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z) - a_{n+1}^2 z^2 Q_{n-1}(z) \\ &= (z^2 - a_{n+1}^2 z^2 + 1) Q_{n-1}(z) - a_n^2 z^2 Q_{n-2}(z) \\ &= (1 - a_{n+1}^2) z^2 Q_{n-1}(z) + F_n(z). \end{aligned}$$

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Applying equation (3.11), we compute that

$$\begin{aligned}
&(1-a_{n+2}^2)(1-a_{n+1}^2)(F_{n+3}(z)-F_{n+2}(z))\\
&\quad=(1-a_{n+2}^2)(1-a_{n+1}^2)(1-a_{n+3}^2)z^2Q_{n+1}(z)\\
&\quad=(1-a_{n+2}^2)(1-a_{n+1}^2)(1-a_{n+3}^2)z^2\Big((z^2+1)Q_n(z)-a_{n+1}^2z^2Q_{n-1}(z)\Big)\\
&\quad=(1-a_{n+3}^2)(1-a_{n+1}^2)(z^2+1)(F_{n+2}(z)-F_{n+1}(z))\\
&\qquad-(1-a_{n+3}^2)(1-a_{n+2}^2)a_{n+1}^2z^2(F_{n+1}(z)-F_n(z)),
\end{aligned}$$

which implies the desired recurrence equation. $\square$

The coefficients $c_k^{(n)}$ of the polynomial $F_n(z)$ in Theorem 3.3 and the coefficients $h_k^{(n-1)}$ in Theorem 2.2 are closely related. Comparing the coefficients $c_k^{(n)}$ and $h_k^{(n)}$, we derive the following relation.

THEOREM 3.6. *Let $c_k^{(n)}$ be the coefficients of the polynomial $F_n(z)$ in* (3.8) .

(I) Suppose $n+1=2m\geq 4$ is an even integer. Then

$$\begin{aligned}
&2^{n+1}\sin\theta\det\Big(\cos\theta I_{n+1}-\Re(A(a_1,a_2,\ldots,a_n))\Big)\\
&\quad=\sin((n+2)\theta)+c_1^{(n)}\sin(n\theta)+(c_2^{(n)}-c_n^{(n)})\sin((n-2)\theta)\\
&\qquad+(c_3^{(n)}-c_{n-1}^{(n)})\sin((n-4)\theta)+\cdots+(c_m^{(n)}-c_{m+1}^{(n)})\sin\theta.
\end{aligned}$$

(II) Suppose $n+1=2m+1\geq 3$ is an odd integer. Then

$$\begin{aligned}
&2^{n+1}\sin\theta\det\Big(\cos\theta I_{n+1}-\Re(A(a_1,a_2,\ldots,a_n))\Big)\\
&\quad=\sin((n+2)\theta)+c_1^{(n)}\sin(n\theta)+(c_2^{(n)}-c_n^{(n)})\sin((n-2)\theta)\\
&\qquad+(c_3^{(n)}-c_{n-1}^{(n)})\sin((n-4)\theta)+\cdots+(c_m^{(n)}-c_{m+2}^{(n)})\sin(2\theta).
\end{aligned}$$

In either case,

$$2^{n+1}\sin\theta\det\Big(\cos\theta I_{n+1}-\Re(A(a_1,a_2,\ldots,a_n))\Big)=\sum_{\ell=0}^{n}c_\ell^{(n)}\sin((n+2-2\ell)\theta).$$

EXAMPLE 3.7. Consider the 4×4 weighted shift matrix $A(2,1,3)$. Then, by Theorem 2.2,

$$2^4\sinh\theta\det\Big(\cosh\theta I_4-\Re(A(2,1,3))\Big)=\sinh(5\theta)-11\sinh(3\theta)+24\sinh(\theta).$$

From Theorem 3.3,

$$F_3(z)=1-11z^2+16z^4-8z^6,$$

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which yields

$$h_1^{(3)} = c_1^{(3)} = -11, \quad h_2^{(3)} = c_2^{(3)} - c_3^{(3)} = 24.$$

The eigenvalues spectra, greater than 1, of the Hermitian operator $\Re(A(2,1,3,1,1,\ldots))$ lie in the set

$$\left\{ \frac{1}{2}(z + z^{-1}) : 0 < z < 1,\ 1 - 11z^2 + 16z^4 - 8z^6 = 0 \right\}.$$

The numerical value of this set is $\{1.69504\}$ for $z = 0.3264004$.

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