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ON THE ROBUST STABILITY OF POLYNOMIAL MATRIX FAMILIES*

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Abstract. In this study, the problem of robust asymptotic stability of $n \times n$ polynomial matrix family, in both continuous-time and discrete-time cases, is considered. It is shown that in the continuous case the problem can be reduced to positivity of two specially constructed multivariable polynomials, whereas in the discrete-time case it is required three polynomials. Number of examples are given, where the Bernstein expansion method and sufficient conditions from [L.H. Keel and S.P. Bhattacharya. Robust stability via sign-definite decomposition. *IEEE Transactions on Automatic Control*, 56(1):140–145, 2011.] are applied to test positivity of the obtained multivariable polynomials. Sufficient conditions for matrix polytopes and one interesting negative result for companion matrices are also considered.

Key words. Stability, Multivariate polynomial, Bernstein expansion, Companion matrix.

AMS subject classifications. 15A18, 93D09, 93D20.

1. Introduction. It is well-known that establishing whether an uncertain system is robustly $\mathcal{D}$-stable is a key problem in automatic control, where $\mathcal{D}$ is a symmetric region of the complex plane.

If D is

- open left half plane, then robust asymptotic stability of continuous systems (Hurwitz stability),
- open unit disc, then robust asymptotic stability of discrete systems (Schur stability)

are under consideration.

On the other hand, the dependence of systems on the uncertainty is typically polynomial.

In this paper, we consider the problem of robust $\mathcal{D}$-stability of polynomial matrices, i.e., $\mathcal{D}$-stability of the real matrix family

$$A(q) = [a_{ij}(q)] \quad (i, j = 1, 2, \ldots, n), \tag{1.1}$$

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where $a_{ij}(q)$ are multivariable polynomials on $q = (q_1, q_2, \ldots, q_l) \in Q$ and $Q \subset \mathbb{R}^l$ is a box.

The stability region $\mathcal{D}$ is assumed to be

$$\mathcal{D} = \{s \in \mathbb{C} : \; a + b(s + \bar{s}) + cs\bar{s} < 0\}, \tag{1.2}$$

where a, b and c are real numbers, $b \geq 0$, $c \geq 0$, $\mathbb{C}$ is the complex plane.

In particular, if $c = 0$ and $b > 0$ the region $\mathcal{D}$ is a shifted left half plane $x < -\dfrac{a}{2b}$. If $c > 0$ the region $\mathcal{D}$ is a shifted disc

$$\left(x - \left(-\frac{b}{c}\right)\right)^2 + y^2 < \frac{b^2 - ac}{c^2},$$

where $s = x + jy$, $b^2 - ac > 0$.

Related stability problems have been studied in a lot of works (see [2, 3] and references therein). In [11], the general problem of root-clustering of a single matrix in the complex plane is considered and algebraic criterion which is necessary and sufficient for $\mathcal{D}$-stability is obtained.

In [4, 12], necessary and sufficient conditions are formulated for the zeros of an arbitrary polynomial matrix to belong to a given region $\mathcal{D}$ of the complex plane. They are expressed as an LMI feasibility problem that can be tackled with powerful interior-point methods.

The paper [5] discusses analysis and synthesis techniques for robust pole placement in LMI regions, a class of convex regions of the complex plane that embraces most practically useful stability regions. The notion of quadratic stability is generalized to these regions and results involving LMI problems are obtained.

In this report, we show that robust $\mathcal{D}$-stability of a polynomial matrix family can be reduced to positivity of multivariable polynomials. It is shown that in the continuous case the problem can be reduced to positivity of two specially constructed multivariable polynomials, whereas in the discrete-time case it is required three polynomials. Number of examples are considered, where the Bernstein expansion method [6, 8] and sufficient conditions from [14] are applied to test positivity of the obtained polynomials.

2. Continuous time case ($c = 0$). Let the region $\mathcal{D}$ (1.2) be given and $c = 0$. Recall that a matrix A is called $\mathcal{D}$-stable if all eigenvalues of A lie in $\mathcal{D}$. Since A is real, the region must be symmetric with respect to the real axis.

If $s = x + jy$ and $b > 0$, then

$$\mathcal{D} = \left\{s : \; x < -\frac{a}{2b}\right\}. \tag{2.1}$$

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We establish that in the case of (2.1) robust $\mathcal{D}$-stability of the family

$$\{A(q): \ q \in Q\} \tag{2.2}$$

can be reduced to the positivity of two specially constructed multivariable polynomials. These polynomials are defined by using the bialternate product of matrices. Here $A(q)$ is defined by (1.1).

DEFINITION 2.1 ([7, 15]). Let A and B be $n \times n$ matrices. The bialternate product $F = A \cdot B$ of A and B is defined by

$$F = (f_{ij,kl}), \quad f_{ij,kl} = \frac{1}{2}\left(\det\begin{bmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{bmatrix} + \det\begin{bmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{bmatrix}\right)$$

where $i, j, k, l \in \{1, 2, \ldots, n\}$ and $i < j$, $k < l$.

The matrix F is $r \times r$ dimensional, where $r = \frac{n(n-1)}{2}$. For a comparison the Kronecker product has dimension $n^2 \times n^2$.

THEOREM 2.2 (Eigenvalue property, [7, 9, 15]). *The eigenvalues of $2A \cdot I$ are $\lambda_i + \lambda_j$ and the eigenvalues of $A \cdot A$ are $\lambda_i.\lambda_j$ $(i < j)$, where λ_i $(i = 1, 2, \ldots, n)$ are the eigenvalues of A.*

Consider the multivariable polynomials

$$\begin{aligned} f_1(q) &:= \det\left[-\tfrac{a}{2b}I - A(q)\right], \\ f_2(q) &:= \det\left[-2(\tfrac{a}{2b}I + A(q)) \cdot I\right]. \end{aligned}$$

THEOREM 2.3. *Let the family (2.2) be given, $\mathcal{D}$ is defined by (2.1) and the family (2.2) contains at least one $\mathcal{D}$-stable member. Then the family (2.2) is robust $\mathcal{D}$-stable if and only if $f_1(q) > 0$ and $f_2(q) > 0$ for all $q \in Q$.*

Proof. For the sake of simplicity, let us carry out the proof for the Hurwitz case $a = 0$ so $f_1(q) = \det[-A(q)]$, $f_2(q) = \det[-2A(q) \cdot I]$.

($\Leftarrow$). By contrary, assume that the family (2.2) is not robust stable. Then by continuity, there exists $q_* \in Q$ such that $A(q_*)$ has a pure imaginary eigenvalue $j\omega$. If $\omega = 0$ then $\det[A(q_*)] = 0$ and $f_1(q_*) = 0$ which is a contradiction. If $\omega > 0$ then $f_2(q_*) = \det[-2A(q_*) \cdot I] = \mu_1.\mu_2. \ldots .\mu_r$ where μ_i are the eigenvalues of $-2A(q_*) \cdot I$. By Theorem 2.2 one of eigenvalues μ_i is $(-j\omega) + j\omega = 0$ and $f_2(q_*) = 0$ which is a contradiction.

($\Rightarrow$). Assume that the family (2.2) is robust Hurwitz stable. Consider the characteristic polynomial of $A(q)$:

$$p(s) = \det[sI - A(q)] = s^n + a_{n-1}(q)s^{n-1} + \cdots + a_1(q)s + a_0(q).$$

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A necessary conditions for Hurwitz stability of a monic polynomial is positivity of all coefficients, therefore $a_0(q) = p(0) = \det(-A(q)) = f_1(q) > 0$ for all $q \in Q$. On the other hand, $f_2(q) = \mu_1.\mu_2.\ \dots\ .\mu_r$ where μ_i are the eigenvalues of $-2A(q) \cdot I$. By Theorem 2.2 and stability of $A(q)$, each μ_i is real positive (is the sum of two real eigenvalues or two conjugate eigenvalues) or has a conjugate partner μ_j. Therefore, the product is positive and $f_2(q) > 0$ for all $q \in Q$. ∎

3. Discrete-time case ($c > 0$). Let the region $\mathcal{D}$ (1.2) be given and $c > 0$. Then $\mathcal{D}$ is reduced to

$$\mathcal{D} = \{(x, y) :\ (x - \delta)^2 + y^2 < r\}, \tag{3.1}$$

where

$$\delta = -\frac{b}{c}, \quad r = \left(\frac{b^2 - ac}{c^2}\right).$$

Define

$$\begin{aligned} g_1(q) &:= \det\left[(1 + \tfrac{\delta}{r})I - \tfrac{1}{r}A(q)\right], \\ g_2(q) &:= \det\left[(1 - \tfrac{\delta}{r})I + \tfrac{1}{r}A(q)\right], \\ g_3(q) &:= \det\left[I - \left(-\tfrac{\delta}{r}I + \tfrac{1}{r}A(q)\right) \cdot \left(-\tfrac{\delta}{r}I + \tfrac{1}{r}A(q)\right)\right]. \end{aligned}$$

In the case of (3.1), robust $\mathcal{D}$-stability of the family (2.2) can be reduced to positivity of the multivariable polynomials $g_1(q)$, $g_2(q)$ and $g_3(q)$.

Theorem 3.1. *Let the family (2.2) be given, $\mathcal{D}$ is defined by (3.1) and the family (2.2) contains at least one $\mathcal{D}$-stable member. Then the family (2.2) is robust $\mathcal{D}$-stable if and only if $g_1(q) > 0$, $g_2(q) > 0$ and $g_3(q) > 0$ for all $q \in Q$.*

Proof. As in the case of the proof of Theorem 2.3, for the sake of simplicity take $\delta = 0$, $r = 1$ that is $g_1(q) = \det(I - A(q))$, $g_2(q) = \det(I + A(q))$, $g_3(q) = \det(I - A(q) \cdot A(q))$.

($\Leftarrow$). By contrary, if the family (2.2) is not robust stable by continuity there exists $q_* \in Q$ such that $A(q_*)$ has an eigenvalue $\lambda = e^{j\theta}$. If $\theta = 0$ then $g_1(q_*) = 0$, if $\theta = \pi$ then $g_2(q_*) = 0$, if $0 < \theta < \pi$ then by Theorem 2.2 the matrix $A(q_*) \cdot A(q_*)$ has eigenvalue $e^{j\theta} \cdot e^{-j\theta} = 1$ so $g_3(q_*) = 0$. These contradictions proof the sufficiency.

($\Rightarrow$). Assume that $A(q)$ is robust stable and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $A(q)$. Then $|\lambda_i| < 1$ $(i = 1, 2, \dots, n)$ and

$$g_1(q) = (1 - \lambda_1)(1 - \lambda_2)\cdots(1 - \lambda_n) > 0, \quad g_2(q) = (1 + \lambda_1)(1 + \lambda_2)\cdots(1 + \lambda_n) > 0.$$

If $\nu_1, \nu_2, \dots, \nu_r$ are the eigenvalues of $A(q) \cdot A(q)$, by Theorem 2.2, ν_k has the form $\lambda_i.\lambda_j$, and therefore, $|\nu_k| < 1$ and

$$g_3(q) = (1 - \nu_1)(1 - \nu_2)\cdots(1 - \nu_r) > 0. \quad ∎$$

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EXAMPLE 3.2 ([1]). Consider the following interval matrix family

$$\mathcal{A} = \left\{ A(q) = \begin{bmatrix} 0.6 & q_1 \\ q_2 & q_3 \end{bmatrix} : \ q_1 \in [0, 0.2], \ q_2 \in [-0.78, 0], \ q_3 \in [-0.6, 0.6] \right\}.$$

The matrix $A(0,0,0)$ is Schur stable and

$$\begin{aligned} g_1(q) &= \det[I - A(q)] & g_2(q) &= \det[I + A(q)] \\ &= 0.4 - 0.4q_3 - q_1 q_2, & &= 1.6 + 1.6q_3 - q_1 q_2, \\ g_3(q) &= \det[I - A(q) \cdot A(q)] & & \\ &= 1 + q_1 q_2 - 0.6q_3. & & \end{aligned}$$

These polynomials are multilinear and by the known property of multilinear polynomials [2, p. 247] the minimum values on Q are obtained at vertices and are 0.16, 0.64 and 0.484 respectively. Therefore, $g_1(q) > 0$, $g_2(q) > 0$ and $g_3(q) > 0$ for all $q \in Q$, and by Theorem 3.1 this family is robust Schur stable.

For the comparison this example to be solved by Theorem 5.1 [1] gives the following multivariate polynomial:

$$g(q) = 2q_1 q_2 + q_1^2 q_2^2 + \cdots + 2.4t^2 q_3 - 4t^2 q_1 q_2 - 1.2t q_3^2,$$

which requires an additional investigation on positivity.

4. Bernstein expansion. One of the well-known methods to test positivity of a multivariate polynomial over a box is the Bernstein expansion. We don't give the detailed description and refer to [6]. This expansion gives bounds for the range of a multivariable polynomial. If the lower bound is not positive then the initial box should be divided into two subboxes and so on. If for a subbox the lower bound is positive this subbox should be eliminated since the polynomial is positive on this box.

EXAMPLE 4.1. Consider the following family from [4]

$$A(q) = \begin{bmatrix} 0 & 1 & 0 & 2-q \\ -1-q^2 & -2 & 7q-1 & 0 \\ -q^3 & 1-q & -1 & 0 \\ q & 0 & q^4 & -1 \end{bmatrix}$$

and $q \in [0, 1]$. For the robust Hurwitz stability of this family, by Theorem 2.3, we consider the following functions:

$$\begin{aligned} f_1(q) &= \det[-A(q)] \\ &= -q^8 + q^7 + 3q^6 - 3q^5 + 16q^4 - 23q^3 + 20q^2 - 6q + 1, \\ f_2(q) &= \det[-2A(q) \cdot I] \\ &= -q^{16} + 4q^{15} - 4q^{14} + 14q^{12} - 30q^{11} - 8q^{10} + 36q^9 - 75q^8 + 34q^7 \\ &\quad + 35q^6 - 48q^5 + 170q^4 - 298q^3 + 440q^2 - 356q + 99. \end{aligned}$$

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TABLE 4.1
Division and elimination procedure.

	f_1		f_2		
	min	max	min	max	Eliminated intervals
$[0, 1]$	0.21428	8	-4.46828	99	Divide the interval
$[0, \frac{1}{2}]$	0.35937	1.08203	3.44566	99	Eliminate
$[\frac{1}{2}, 1]$	1.08203	8	-1.92831	12.375	Divide the interval
$[\frac{1}{2}, \frac{3}{4}]$	1.08203	2.96476	-1.39587	3.44566	Divide the interval
$[\frac{3}{4}, 1]$	2.96476	8	0.840033	12.1875	Eliminate
$[\frac{1}{2}, \frac{5}{8}]$	1.08203	1.79535	-1.08237	3.44566	Divide the interval
$[\frac{5}{8}, \frac{3}{4}]$	1.79535	2.96476	-1.22773	0.84003	Divide the interval
$[\frac{1}{2}, \frac{9}{16}]$	1.08203	1.39530	0.34755	3.44566	Eliminate
$[\frac{9}{16}, \frac{5}{8}]$	1.39530	1.79535	-1.08237	0.34755	Divide the interval
$[\frac{5}{8}, \frac{11}{16}]$	1.79535	2.30740	-1.20751	-0.88253	
$[\frac{11}{16}, \frac{3}{4}]$	2.30740	2.96476	-0.88253	0.840033	

The application of the Bernstein expansion gives Table 4.1.

From the table it follows that $f_2(q) < 0$ for all $q \in [5/8, 11/16] = [0.625, 0.6875]$. That is, the family $\{A(q) : \ q \in [0, 1]\}$ is not robust stable.

5. Sign-definite decomposition. Here we describe a method from [14] to test positivity of a multivariable polynomial on a box.

Let

$$q = (q_1, q_2, \ldots, q_l)$$

be real vector, $f(q)$ be a real multivariable polynomial of $q \in Q$ where Q is a box:

$$Q = \{q : \ q_i^- \leq q_i \leq q_i^+, \ i = 1, 2, \ldots, l\}.$$

The box Q in an arbitrary location of the parameter space can always be translated to the first orthant. Therefore, one can assume that $q_i^- \geq 0$ without loss of generality. Then $f(q)$ can be written as

$$f(q) = f^+(q) - f^-(q),$$

where $f^+(q), \ f^-(q) \geq 0$ for all $q \in Q$ and this representation is called the sign-definite decomposition of $f(q)$. The functions $f^+(q)$ and $f^-(q)$ refer to the terms with positive and negative coefficients, respectively.

Define two extreme vertices of the box Q:

$$q^- := (q_1^-, q_2^-, \ldots, q_l^-), \quad q^+ := (q_1^+, q_2^+, \ldots, q_l^+).$$

PROPOSITION 5.1 ([14]). *If $f^+(q^-) - f^-(q^+) > 0$ then $f(q) > 0$, if $f^+(q^+) - f^-(q^-) < 0$ then $f(q) < 0$ for all $q \in Q$.*

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Using the sufficient condition from Proposition 5.1 we can test positivity of $f(q)$. If the condition $f^+(q^-) - f^-(q^+) > 0$ is not satisfied, as in the case of Bernstein expansion the box Q should be divided into small subboxes.

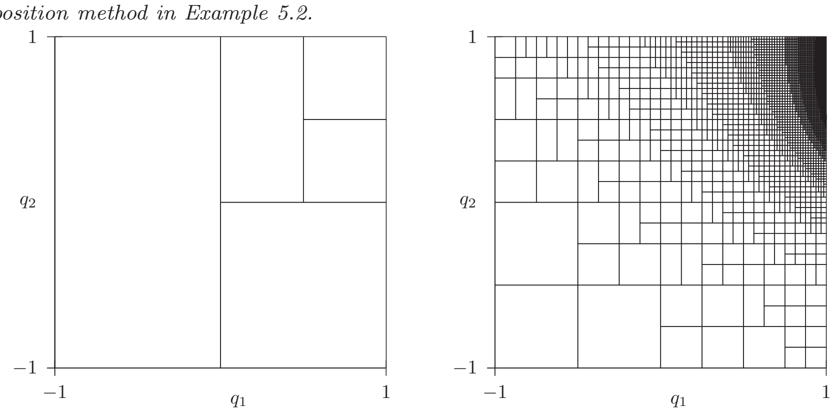
Example 5.2. Consider robust Schur stability of the family

$$A(q_1, q_2) = \begin{bmatrix} -0.14 & 0.235 & 0.29 \\ -0.94 & -0.811 & 1.246 \\ -0.22 & -0.35 & 0.95 \end{bmatrix} + q_1 \begin{bmatrix} -0.3 & 0.15 & 0.275 \\ -0.275 & -0.3 & 0.55 \\ -0.35 & -0.25 & 0.625 \end{bmatrix} + q_2 \begin{bmatrix} 0.4 & -0.1 & -0.4 \\ -0.6 & -0.325 & 0.225 \\ 0.725 & 0.225 & -0.45 \end{bmatrix},$$

where $-1 \leq q_1 \leq 1$, $-1 \leq q_2 \leq 1$. $A(0,0)$ is Schur stable.

The Bernstein expansion applied to the polynomials $g_1(q)$, $g_2(q)$ and $g_3(q)$ from Theorem 3.1 establishes their positivity after 4 steps and the family is robust Schur stable. It should be noted that sign-definite decomposition method described above establishes the positivity only after 4178 steps. The necessary and sufficient condition from [1] establishes robust Schur stability after 109 steps of the Bernstein expansion.

Fig. 5.1. *Eliminated subboxes of* $[-1,1] \times [-1,1]$ *according to the Bernstein expansion and the decomposition method in Example 5.2.*



Example 5.3. Consider robust Hurwitz stability of the family

$$A(\mathrm{q}) = \begin{bmatrix} -2 & 0 & 0 \\ 3 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} + q_1 \begin{bmatrix} -2 & -1 & -3 \\ 1 & -1 & 0 \\ 0 & 3 & -2 \end{bmatrix} + q_2 \begin{bmatrix} -2 & 1 & 0 \\ -3 & 1 & -1 \\ 2 & 0 & -3 \end{bmatrix} + q_3 \begin{bmatrix} -3 & 0 & -2 \\ -2 & -3 & -1 \\ -1 & 0 & -3 \end{bmatrix} + q_4 \begin{bmatrix} -2 & 2 & 0 \\ -1 & -2 & -3 \\ 1 & 3 & 2 \end{bmatrix},$$

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$(q_1, q_2, q_3, q_4) \in [0,1] \times [-0.1, 0.7] \times [-0.11, 0.5] \times [0,1]$. $A(0,0,0,0)$ is stable. The Bernstein expansion applied to the polynomials

$$f_1(q) = \det\left[-A(q)\right], \ f_2(q) = \det\left[-2A(q) \cdot I\right]$$

from Theorem 2.3 establishes their positivity after 16 steps and the family is Hurwitz stable.

Example 5.4. Consider the family

$$A(q) = \begin{bmatrix} 2.403 & 0.807 & -0.863 \\ 0.194 & 2.713 & 0.177 \\ 2.261 & 2.063 & 1.391 \end{bmatrix} + q_1 \begin{bmatrix} -0.54 & -0.30 & -0.50 \\ 0.32 & -0.56 & 0.6 \\ -0.2 & -0.4 & 0.30 \end{bmatrix} + q_2 \begin{bmatrix} -0.28 & -0.10 & 0.14 \\ -0.40 & 0.30 & -0.38 \\ -0.70 & 0.50 & -0.50 \end{bmatrix},$$

where $(q_1, q_2) \in [-1,1] \times [-0.6, 0.2]$ and $\mathcal{D} = \{z \in \mathbb{C} : \ (x-1)^2 + y^2 < 2\}$. Here $A(0,0)$ is $\mathcal{D}$-stable and

$$\begin{aligned} g_1(q) &:= \det\left[\tfrac{3}{2}I - \tfrac{1}{2}A(q)\right], \\ g_2(q) &:= \det\left[\tfrac{1}{2}I + \tfrac{1}{2}A(q)\right], \\ g_3(q) &:= \det\left[I - \left(-\tfrac{1}{2}I + \tfrac{1}{2}A(q)\right) \cdot \left(-\tfrac{1}{2}I + \tfrac{1}{2}A(q)\right)\right]. \end{aligned}$$

The Bernstein expansion gives answer after 3 steps and the family is robust $\mathcal{D}$-stable.

6. Simple sufficient conditions for stability and a negative result for companion matrices. Consider family (1.1) where each function $a_{ij}(q)$ is multilinear on Q, i.e., affine-linear with respect to each component q_k. Define multilinear map $T : Q \to \mathbb{R}^{n \times n}$ by $T(q) = A(q)$. By the well-known extremal property of multilinear maps ([2, p. 247])

$$\operatorname{conv} T(Q) = \operatorname{conv}\left\{T(q^1), T(q^2), \ldots, T(q^m)\right\}, \tag{6.1}$$

where "conv" stands for the convex hull, $q^1, q^2, \ldots, q^m$ are the extremal points of Q. From (6.1) it follows that the family $A(q)$ is robustly $\mathcal{D}$-stable if the extended polytope $\operatorname{conv}\left\{T(q^1), T(q^2), \ldots, T(q^m)\right\}$ is robustly $\mathcal{D}$-stable.

Now we give simple sufficient conditions for stability of matrix polytopes. Let a polytope

$$\mathcal{P} = \operatorname{conv}\{A_1, A_2, \ldots, A_m\} \tag{6.2}$$

be given. For $A \in \mathbb{R}^{n \times n}$ define $A^s = \frac{1}{2}\left(A + A^T\right)$, where "$T$" denotes the transpose.

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THEOREM 6.1. *Assume that the polytope (6.2) is given. Then*

a) *If A_i^s $(i = 1, 2, \ldots, m)$ are Hurwitz stable then $\mathcal{P}$ is robust Hurwitz stable.*
b) *If $A_i^T = A_i$ and A_i are Schur stable $(i = 1, 2, \ldots, m)$ then $\mathcal{P}$ is robust Schur stable.*

Proof.

a) Hurwitz stability of a symmetric matrix is equivalent to negative definiteness, therefore $A_i + A_i^T < 0$ or $IA_i + A_i^T I < 0$ $(i = 1, 2, \ldots, m)$, where I is the identity matrix. As a result the matrix I is a common solution to the Lyapunov inequalities for the family (6.2).
b) Let $A \in \mathcal{P}$, $A = \alpha_1 A_1 + \cdots + \alpha_m A_m$ where $\alpha_i \geq 0$, $\alpha_1 + \cdots + \alpha_m = 1$ $(i = 1, 2, \ldots, m)$. Since A_i are symmetric and Schur stable matrices $(i = 1, 2, \ldots, m)$, we get

$$\begin{aligned} -1 &< \alpha_1 \lambda_{\min}(A_1) + \cdots + \alpha_m \lambda_{\min}(A_m) \leq \lambda_{\min}(\alpha_1 A_1 + \cdots + \alpha_m A_m) \\ &\leq \lambda_{\max}(\alpha_1 A_1 + \cdots + \alpha_m A_m) \leq \alpha_1 \lambda_{\max}(A_1) + \cdots + \alpha_m \lambda_{\max}(A_m) < 1 \end{aligned}$$

by Weyl Theorem [13, p. 181]. That is A is Schur stable. ∎

EXAMPLE 6.2. Consider the family

$$A(q) = \begin{bmatrix} q_1 q_2 - 3 & 3 & 1 \\ 1 & -5 & q_1 + q_2 \\ 2 & 2q_1 q_3 & -8 \end{bmatrix} \subset \operatorname{conv}\{A_1, A_2, \ldots, A_8\},$$

where $(q_1, q_2, q_3) \in [-5, -4] \times [5, 6] \times [0, 1]$ and A_i $(i = 1, 2, \ldots, 8)$ are extremal matrices. Since A_i^s are Hurwitz stable, $A(q)$ is Hurwitz stable for all $q \in [-5, -4] \times [5, 6] \times [0, 1]$.

EXAMPLE 6.3. Consider the symmetric matrix family

$$A(q) = \begin{bmatrix} q_1 - q_2 + 1 & q_1 q_2 + q_1 - q_2 \\ q_1 q_2 + q_1 - q_2 & q_1 q_2 \end{bmatrix},$$

where $(q_1, q_2) \in [-1.2, -0.4] \times [-0.3, -0.1]$. The extremal matrices of the family are Schur stable. Therefore, $A(q)$ is Schur stable for all $q \in [-1.2, -0.4] \times [-0.3, -0.1]$.

Theorem 6.1 requires the Hurwitz stability of the symmetric parts of extreme matrices. The following proposition shows that for companion matrices this is not a case and consequently Theorem 6.1, part a) can not be applied to polynomial polytopes.

PROPOSITION 6.4. *For any real companion matrix A, the matrix $A^s = \frac{A + A^T}{2}$ is not Hurwitz stable.*

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Proof. Let A be a companion matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

with the characteristic polynomial $a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Then

$$A^s = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & -\frac{a_n}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & -\frac{a_{n-1}}{2} \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & -\frac{a_{n-2}}{2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\frac{a_n}{2} & -\frac{a_{n-1}}{2} & -\frac{a_{n-2}}{2} & \cdots & \frac{1-a_2}{2} & -a_1 \end{bmatrix}.$$

Let $b(s) = s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n$ be the characteristic polynomial of A^s. Recall that necessary condition for Hurwitz stability of A^s is positivity of all b_i $(i = 1, 2, \ldots, n)$. The coefficient b_k is the sum of the k-by-k principal minors of A^s. Consider the coefficient b_2 which is the sum of the 2-by-2 principal minors of A^s. Each such minor has the form $\begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$, and therefore, the sum of the 2-by-2 principal minors is negative that is $b_2 < 0$. Hence, A^s is not Hurwitz stable. □

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