Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

HIGHER NUMERICAL RANGES OF QUATERNION MATRICES*

NARGES HAJ ABOUTALEBI†, GHOLAMREZA AGHAMOLLAEI‡, AND HOSSEIN MOMENAEE KERMANI§

Abstract. Let n and k be two positive integers and $k \leq n$. In this paper, the notion of $k-$numerical range of $n-$square quaternion matrices is introduced. Some algebraic and geometrical properties are investigated. In particular, a necessary and sufficient condition for the convexity of $k-$numerical range of a quaternion matrix is given. Moreover, a new description of $1-$numerical range of normal quaternion matrices is also stated.

Key words. Quaternion matrices, $k-$Numerical range, Numerical range.

AMS subject classifications. 15B33, 15A60, 15A18.

1. Introduction and preliminaries. Division rings [4] are of interest because they resemble fields in every way except for commutativity of multiplication. The division ring $\mathbb{H}$ of quaternions was first discovered by W.R. Hamilton and is narrated in numerous resources; e.g., see [13, 19]. Nowadays quaternions are not only part of contemporary mathematics such as algebra, analysis, geometry, and computation; but they are also widely and heavily used in computer graphics, control theory, signal processing, altitude control, physics, and mechanics; e.g., see [1, 5, 6, 9, 11, 14, 20].

Formally, $\mathbb{H}$ which is denoted by this notation because of Hamilton, is the four-dimensional algebra over the field of real numbers $\mathbb{R}$ with the standard basis $\{1, i, j, k\}$ and multiplication rules:

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = k = -ji, \;\; jk = i = -kj, \;\; ki = j = -ik, \;\text{ and} \\ &1q = q1 = q \;\text{ for all }\; q \in \{i, j, k\}. \end{aligned}$$

If $q \in \mathbb{H}$, then there are $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$q = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k. \tag{1.1}$$

*Received by the editors on October 15, 2014. Accepted for publication on October 9, 2015. Handling Editor: Panayiotis Psarrakos.

†Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran (aboutalebi@iau-shahrood.ac.ir, nargesaboutalebi@yahoo.ca).

‡Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran (aghamollaei@uk.ac.ir, gh_aghamollaei@yahoo.com).

§Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran (momenaee@uk.ac.ir).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

Equation (1.1) is known as the canonical representation of a quaternion $q \in \mathbb{H}$, and the scalar α_0 is called the real part of q, denoted by $Re\ q$; the quantity $\alpha_0+\alpha_1 i$ is said to be the complex part of q, denoted by $Co\ q$; the part $\alpha_1 i+\alpha_2 j+\alpha_3 k$ is called the imaginary part of q, symbolized as $Im\ q$; the set of all $q \in \mathbb{H}$ such that $Re\ q = 0$ is denoted by $\mathbb{P}$; the conjugate of q is $\bar{q} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$; and the norm or length, or modulus of q is defined and denoted by $|q| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2} = (q\bar{q})^{\frac{1}{2}} = (\bar{q}q)^{\frac{1}{2}}$, which is nothing more than the Euclidean distance from the origin to the point $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ in the space $\mathbb{R}^4$. It is known that if $q, q_1, q_2 \in \mathbb{H}$, then

(a) $|q| = 0$ if and only if $q = 0$;

(b) $|q_1 + q_2| \leq |q_1| + |q_2|$, and $|q_1 q_2| = |q_1||q_2|$;

(c) $|\bar{q}| = |q|$, and $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$;

(d) if $q \neq 0$, then $q^{-1}q = qq^{-1} = 1$, where $q^{-1} = \frac{\bar{q}}{|q|^2}$.

Two quaternions x and y are said to be similar, denoted by $x \sim y$, if there exists a nonzero quaternion $q \in \mathbb{H}$ such that $x = q^{-1}yq$. It is clear that if $x \sim y$, then there exists a $q \in \mathbb{H}$ such that $|q| = 1$ and $x = \bar{q}yq$. Also, it is known, see [21, Theorem 2.2], that $x \in \mathbb{H}$ is similar to $y \in \mathbb{H}$ if and only if $Re\ x = Re\ y$ and $|Im\ x| = |Im\ y|$. For instance, i and j are similar. It is clear that $\sim$ is an equivalence relation on the quaternions. The equivalence class containing x is denoted by $[x]$.

The division ring $\mathbb{H}$ is an algebra over the field $\mathbb{R}$, and the set $\mathbb{C}$ of complex numbers appears as a real subspace of $\mathbb{H}$; that is, $\mathbb{C} = Span_{\mathbb{R}}\{1, i\} = \{q \in \mathbb{H} : qi = iq\}$. Moreover, $Span_{\mathbb{R}}\{j, k\} = \{q \in \mathbb{H} : qi = -iq\}$, and $\lambda q = q\bar{\lambda}$ for every $\lambda \in Span_{\mathbb{R}}\{1, i\}$ and $q \in Span_{\mathbb{R}}\{j, k\}$. Also, using this fact that $ij = k$, each quaternion $q \in \mathbb{H}$, as in (1.1), is uniquely represented by:

$$q = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k = (\alpha_0 + \alpha_1 i) + (\alpha_2 + \alpha_3 i)j = \gamma_1 + \gamma_2 j,$$

where $\gamma_1 = \alpha_0 + \alpha_1 i \in \mathbb{C}$ and $\gamma_2 = \alpha_2 + \alpha_3 i \in \mathbb{C}$.

Let $\mathbb{H}^n$ be the collection of all n-column vectors with entries in $\mathbb{H}$, and $M_{m\times n}(\mathbb{H})$ be the set of all $m \times n$ quaternion matrices. For the case $m = n$, $M_{n\times n}(\mathbb{H})$ is denoted by $M_n(\mathbb{H})$, i.e., the algebra of all $n \times n$ quaternion matrices. For any $1 \leq i \leq n$, $e_i \in \mathbb{H}^n$ has a 1 as its ith component and 0's elsewhere. For every $m \times n$ quaternion matrix $A = (a_{ij}) \in M_{m\times n}(\mathbb{H})$, $\bar{A} := (\bar{a}_{ij})$ is called the conjugate of A; the matrix $A^T := (a_{ji}) \in M_{n\times m}(\mathbb{H})$ is called the transpose of A; and $A^* := (\bar{A})^T \in M_{n\times m}(\mathbb{H})$ is said to be the conjugate transpose of A.

Let $A \in M_n(\mathbb{H})$. As in the complex case, A is called normal if $AA^* = A^*A$; Hermitian if $A^* = A$; and skew-Hermitian if $A^* = -A$. A quaternion λ is called a (right) eigenvalue of A if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$. If λ is an eigenvalue of A, then any element in $[\lambda]$ is also an eigenvalue of A. It is known, see [21, Theorem 5.4],

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

that A has exactly n (right) eigenvalues which are complex numbers with nonnegative imaginary parts. These eigenvalues are called the standard right eigenvalues of A. The right spectrum of A is defined and denoted by

$$\sigma_r(A) = \{\lambda \in \mathbb{H} : \ Ax = x\lambda \ \text{ for some nonzero } \ x \in \mathbb{H}^n\}.$$

For $A \in M_n(\mathbb{H})$, as in the complex case (see [7, Chapter 1]), the quaternionic numerical range of A is defined and denoted by

$$W(A) = \{x^*Ax : \ x \in \mathbb{H}^n, \ x^*x = 1\}.$$

The notion of quaternionic numerical range of matrices was first studied in 1951 by Kippenhahn [10]. In the last two decades, the study of quaternionic numerical range is revived in a series of papers; for example, see [3, 16, 18, 21, 22].

In this paper, we are going to introduce and study the higher numerical ranges of quaternionic matrices. To this end, in Section 2, we introduce, as in the complex case, the $k-$numerical range and the right $k-$spectrum of quaternion matrices. We also study the projection of the $k-$numerical range of quaternion matrices on $\mathbb{R}$ and $\mathbb{C}$. The emphasis is on the study of algebraic properties of them and their relations. Moreover, we characterize the $k-$numerical range of Hermitian quaternion matrices and we give a necessary and sufficient condition for the convexity of the $k-$numerical range of quaternion matrices. In Section 3, we show that the $1-$numerical range of a normal quaternion matrix can be characterized by its standard right eigenvalues.

2. $k-$Numerical range of quaternion matrices. Throughout this section, we assume that k and n are positive integers, and $k \leq n$. Also, I_k denotes the $k \times k$ identity matrix. A matrix $X \in M_{n\times k}(\mathbb{H})$ is called an isometry if $X^*X = I_k$, and the set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n\times k}$. For the case $k = n$, $\mathcal{X}_{n\times n}$ is denoted by $\mathcal{U}_n$, which is the set of all $n \times n$ quaternionic unitary matrices. In this section, we are going to introduce and study the notion of higher numerical ranges of quaternion matrices. To access more information about some known results in the complex case, see [12] and its references.

DEFINITION 2.1. Let $A \in M_n(\mathbb{H})$. The $k-$numerical range of A is defined and denoted by

$$W^k(A) = \left\{\frac{1}{k}\, tr(X^*AX) : \ X \in \mathcal{X}_{n\times k}\right\}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

The sets $W^k(A)$, where $k \in \{1, 2, \dots, n\}$, are generally called higher numerical ranges of A.

REMARK 2.2. Let $A \in M_n(\mathbb{H})$. Since for every $X = [x_1, x_2, \dots, x_k] \in \mathcal{X}_{n\times k}$, $\{x_1, x_2, \dots, x_k\}$ is an othonormal set in $\mathbb{H}^n$, by Definition 2.1, we have

$$W^k(A) = \left\{ \frac{1}{k} \sum_{i=1}^{k} x_i^* A x_i : \{x_1, \dots, x_k\} \text{ is an othonormal set in } \mathbb{H}^n \right\}.$$

Also, it is clear that $W^1(A) = W(A)$. So, the notion of $k-$numerical range is a generalization of the classical numerical range of quaternion matrices.

DEFINITION 2.3. Let $A \in M_n(\mathbb{H})$ have the standard right eigenvalues $\lambda_1, \dots, \lambda_n$, counting multiplicities. The right $k-$spectrum of A is defined and denoted by

$$\sigma_r^k(A) = \left\{ \frac{1}{k} \sum_{j=1}^{k} \alpha_{i_j} : 1 \le i_1 < i_2 < \cdots < i_k \le n,\ \alpha_{i_j} \in [\lambda_{i_j}] \right\}.$$

It is clear that $\sigma_r^1(A) = \sigma_r(A)$. In the following theorem, we state some basic properties of $\sigma_r^k(A)$. Recall that the convex hull of a set $S \subseteq \mathbb{H}$, which is denoted by $conv(S)$, is defined as the set of all convex linear combinations of elements of S.

THEOREM 2.4. *Let $A \in M_n(\mathbb{H})$ have the standard right eigenvalues $\lambda_1, \dots, \lambda_n$, counting multiplicities. Then the following assertions are true:*

(a) *If $\alpha \in \sigma_r^k(A)$, then $[\alpha] \subseteq \sigma_r^k(A)$;*

(b) *$conv(\mathbb{C} \bigcap \sigma_r^k(A)) = \mathbb{C} \bigcap conv(\sigma_r^k(A))$;*

(c) *If $k < n$, then $\sigma_r^{k+1}(A) \subseteq conv(\sigma_r^k(A))$. Consequently,*

$$conv(\sigma_r^n(A)) \subseteq conv(\sigma_r^{n-1}(A)) \subseteq \cdots \subseteq conv(\sigma_r(A)).$$

Proof. The part (a) follows easily from Definition 2.3.

For (b), it is clear that $conv(\mathbb{C} \bigcap \sigma_r^k(A)) \subseteq \mathbb{C} \bigcap conv(\sigma_r^k(A))$. Conversely, let $\lambda = \sum_{l=1}^m \theta_l(a_l + b_l i + c_l j + d_l k) \in \mathbb{C} \bigcap conv(\sigma_r^k(A))$, where $\theta_l \ge 0$, $\sum_{l=1}^m \theta_l = 1$, and $a_l + b_l i + c_l j + d_l k \in \sigma_r^k(A)$ for all $l = 1, \dots, m$. Then we have

$$\lambda = \sum_{l=1}^{m} \theta_l(a_l + b_l i),\ \textit{and}\ \sum_{l=1}^{m} \theta_l(c_l j + d_l k) = 0.$$

Since $a_l \pm i\sqrt{b_l^2 + c_l^2 + d_l^2} \in [a_l + b_l i + c_l j + d_l k]$, by (a), we have $a_l \pm i\sqrt{b_l^2 + c_l^2 + d_l^2} \in \mathbb{C} \bigcap \sigma_r^k(A)$ for all $l = 1, \dots, m$. So, for every $l \in \{1, \dots, m\}$, we have $a_l + b_l i =$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

$t(a_l + i\sqrt{b_l^2 + c_l^2 + d_l^2}) + (1-t)(a_l - i\sqrt{b_l^2 + c_l^2 + d_l^2}) \in conv(\mathbb{C} \bigcap \sigma_r^k(A))$, where $t = \frac{b_l + \sqrt{b_l^2+c_l^2+d_l^2}}{2\sqrt{b_l^2+c_l^2+d_l^2}}$ for the case $\sqrt{b_l^2 + c_l^2 + d_l^2} \neq 0$, and for the case $b_l = c_l = d_l = 0$, $t \in [0,1]$ is arbitrary. Therefore, $\lambda \in conv(\mathbb{C} \bigcap \sigma_r^k(A))$. Hence,

$$\mathbb{C} \bigcap conv(\sigma_r^k(A)) \subseteq conv(\mathbb{C} \bigcap \sigma_r^k(A)).$$

To prove (c), let $\alpha \in \sigma_r^{k+1}(A)$ be given. Then there exist $1 \leq i_1 < i_2 < \cdots < i_{k+1} \leq n$ and $\alpha_{i_j} \in [\lambda_{i_j}]$ such that $\alpha = \frac{1}{k+1}\sum_{j=1}^{k+1} \alpha_{i_j}$. Therefore, we have

$$\alpha = \frac{1}{k+1}\sum_{j=1}^{k+1} \alpha_{i_j} = \frac{1}{k+1}\sum_{j=1}^{k+1}(\frac{1}{k}\sum_{t=1, t\neq j}^{k+1} \alpha_{i_t}).$$

So, $\alpha \in conv(\sigma_r^k(A))$. Hence, the proof is complete. □

In the following theorem, we state some basic properties of the $k-$numerical range of quaternion matrices.

Theorem 2.5. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) $W^k(\alpha I + \beta A) = \alpha + \beta W^k(A)$ and $W^k(A + B) \subseteq W^k(A) + W^k(B)$, where $\alpha, \beta \in \mathbb{R}$ and $B \in M_n(\mathbb{H})$;

(b) $W^k(U^*AU) = W^k(A)$, where $U \in \mathcal{U}_n$;

(c) If $\lambda \in W^k(A)$, then $[\lambda] \subseteq W^k(A)$;

(d) If $B \in M_m(\mathbb{H})$ is a principal submatrix of A, and $k \leq m$, then $W^k(B) \subseteq W^k(A)$. Consequently, if $V \in \mathcal{X}_{n\times s}$, where $1 \leq s \leq n$, then $W^k(V^*AV) \subseteq W^k(A)$, and the equality holds if $s = n$;

(e) $\sigma_r^k(A) \subseteq W^k(A)$;

(f) $\bar{\alpha}W^k(A)\alpha = W^k(A)$, where $\alpha \in \mathbb{H}$ is such that $\bar{\alpha}\alpha = 1$;

(g) $W^k(A^*) = W^k(A)$;

(h) Let $A = H + K$, where H is Hermitian and K is skew-Hermitian. Moreover, let $\tilde{A} = aH + bK$, where a and b are nonzero real numbers. If $x \in \mathbb{R}$ and $y \in \mathbb{P}$, then

$$x + y \in W^k(A) \ if\ and\ only\ if\ ax + by \in W^k(\tilde{A});$$

(i) If $B \in M_{n'}$ and $k \leq \min\{n, n'\}$, then $W^k(A) \cup W^k(B) \subseteq W^k(A \oplus B)$. For the case $k = 1$,

$$conv(W^1(A \oplus B)) = conv(W^1(A) \cup W^1(B)).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

Proof. The assertions in (a) and (b) follow easily from Definition 2.1.

To prove (c), since $\lambda \in W^k(A)$, there exists a $X \in \mathcal{X}_{n\times k}$ such that

$$\lambda = \frac{1}{k}\, tr(X^*AX).$$

Now, let $\mu \in [\lambda]$ be given. Then there exists a $u \in \mathbb{H}$ such that $|u| = 1$ and $\mu = \bar{u}\lambda u$. By setting $Y := Xu$, we have $Y \in \mathcal{X}_{n\times k}$ and $\mu = \frac{1}{k}tr(Y^*AY)$. Therefore, $\mu \in W^k(A)$.

To prove (d), we assume that $B \in M_m(\mathbb{H})$ is formed by considering rows $i_1, \dots, i_m$ and the corresponding columns from A, and let $\mu \in W^k(B)$ be given. Then by Remark 2.2, there exists an othonormal set $\{x_1, \dots, x_k\}$ in $\mathbb{H}^m$ such that $\mu = \frac{1}{k}\sum_{i=1}^k x_i^* B x_i$. Now for any $1 \le i \le k$, define $y_i \in \mathbb{H}^n$ such that $(y_i)_j = 0$ for $j \ne 1, \dots, m$, and other its entries are formed by x_i. Then $\{y_1, \dots, y_k\}$ is an othonormal set in $\mathbb{H}^n$ and $\mu = \frac{1}{k}\sum_{i=1}^k y_i^* A y_i$. Therefore, $\mu \in W^k(A)$. So, $W^k(B) \subseteq W^k(A)$. If $V = [v_1, \dots, v_s] \in \mathcal{X}_{n\times s}$, then by [21, Lemma 5.2], there exist unit vectors $v_{s+1}, \dots, v_n$ in $\mathbb{H}^n$ such that $U := [v_1, \dots, v_s, v_{s+1}, \dots, v_n] \in \mathcal{U}_n$. Since V^*AV is a principal submatrix of U^*AU, the result follows from the first case and part (b). Moreover, if $s = n$, then $V \in \mathcal{U}_n$, and hence, by (b), the equality holds.

For (e), let $\mu \in \sigma_r^k(A)$ be given. Then there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\mu = \frac{1}{k}\sum_{j=1}^k \alpha_{i_j}$, where $\alpha_{i_j} \in [\lambda_{i_j}]$. Now, let $\{x_{i_1}, \dots, x_{i_k}\}$ be a set of orthonormal eigenvectors of A such that $Ax_{i_j} = x_{i_j}\alpha_{i_j}$. So, $\mu = \frac{1}{k}\sum_{j=1}^k x_{i_j} A x_{i_j}$, and hence, $\mu \in W^k(A)$.

The assertion in (f) follows from (c).

To prove (g), for every $\mu \in \mathbb{H}$, since $\mu \sim \bar{\mu}$, there exists a $\alpha \in \mathbb{H}$ such that $\bar{\alpha}\alpha = 1$ and $\bar{\alpha}\mu\alpha = \bar{\mu}$. We know that $W^k(A^*) = \overline{W^k(A)} := \{\mu \in \mathbb{H} \ : \ \bar{\mu} \in W^k(A)\}$. So, by (c), we have that $\mu \in W^k(A)$ if and only if $\bar{\mu} \in W^k(A)$ or equivalently, $\mu \in W^k(A^*)$.

For (h), let $x + y \in W^k(A)$. Then there exists a $X \in \mathcal{X}_{n\times k}$ such that

$$x + y = \frac{1}{k}tr(X^*AX) = \frac{1}{k}tr(X^*HX) + \frac{1}{k}tr(X^*KX).$$

Therefore, $x = \frac{1}{k}tr(X^*HX)$ and $y = \frac{1}{k}tr(X^*KX)$. So, we have

$$ax + by = \frac{1}{k}tr(X^*(aH)X) + \frac{1}{k}tr(X^*(bK)X) = \frac{1}{k}tr(X^*\tilde{A}X) \in W^k(\tilde{A}).$$

By a similar argument, one can prove that if $ax + by \in W^k(\tilde{A})$, then $x + y \in W^k(A)$.

Finally, the first assertion in (i) follows from (d). Now, for the second assertion in (i), let $k = 1$. By the first case, we have $conv(W(A) \cup W(B)) \subseteq conv(W(A \oplus B))$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

Conversely, let $\mu \in W(A \oplus B)$ be given. Then there exists a $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{H}^{n+n'}$ such that $x \in \mathbb{H}^n,\ y \in \mathbb{H}^{n'},\ x^*x + y^*y = 1$ and $\mu = z^*(A \oplus B)z = x^*Ax + y^*By$. If $x = 0$, then $y^*y = 1$ and $\mu = y^*By \in W(B) \subseteq conv(W(A) \cup W(B))$. The argument is analogous if $y = 0$. Now, suppose that both x and y are nonzero. Then, we have:

$$\mu = x^*x(\frac{x^*Ax}{x^*x}) + y^*y(\frac{y^*By}{y^*y}) \in conv(W(A) \cup W(B)).$$

Therefore, $conv(W(A \oplus B)) \subseteq conv(W(A) \cup W(B))$. So, the set equality holds.

Hence, the proof is complete. □

By Theorem 2.5 (c), one can prove the following corollary.

Corollary 2.6. *Let $A \in M_n(\mathbb{H})$ and $x_0+x_1i+x_2j+x_3k \in W^k(A)$. If $t \in [-1, 1]$ and $y_2, y_3 \in \mathbb{R}$ such that $(y_2^2+y_3^2)-(x_2^2+x_3^2) = (1-t^2)x_1^2$, then $x_0+tx_1i+y_2j+y_3k \in W^k(A)$.*

Using Remark 2.2 and by the same manner as in the proof of Theorem 2.4 (c), we have the following result.

Proposition 2.7. *Let $k < n$ and $A \in M_n(\mathbb{H})$. Then*

$$W^{k+1}(A) \subseteq conv(W^k(A)).$$

Consequently, $conv(W^n(A)) \subseteq conv(W^{n-1}(A)) \subseteq \cdots \subseteq conv(W(A))$.

Now, we are going to study some properties of the $k-$numerical range of Hermitian matrices. For this, we need the following lemma. Recall that if $A, B \in M_n(\mathbb{H})$, then the relation $tr(AB) = tr(BA)$ does not hold in general; for example, consider $A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}$.

Lemma 2.8. *Let $A, B \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) *$Re(tr(AB)) = Re(tr(BA))$;*

(b) *If at least one of A and B is a real matrix, then $tr(AB) = tr(BA)$;*

(c) *If A is a real diagonal matrix and $U \in \mathcal{U}_n$, then*

$$tr(U^*AU) = tr(A);$$

(d) *If $\lambda_1, \ldots, \lambda_n$ are the standard eigenvalues, counting multiplicities, of A, then*

$$Re(tr(A)) = Re\left(\sum_{i=1}^{n} \lambda_i\right).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

Proof. For (a), see [15, Lemma 2.11]. The assertions in (b) and (c) can be easily verified. The part (d) also follows from (a). So, the proof is complete. □

The following example shows that the result in Lemma 2.8 (c) does not hold in general.

Example 2.9. Let $D = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{H})$ *and* $U = \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{U}_2$. Then

$$tr(U^*DU) = j \neq i = tr(D).$$

Proposition 2.10. *Let $A \in M_n(\mathbb{H})$ be a Hermitian matrix. If $k < n$, then*

$$(n-k)W^{n-k}(A) = tr(A) - kW^k(A).$$

Proof. Since A is Hermitian, there exists a $U \in \mathcal{U}_n$ such that $U^*AU = D$, where $D \in M_n(\mathbb{H})$ is a real diagonal matrix. Using Lemma 2.8 ((c) and (d)) and Theorem 2.5 (b), we have

$$\begin{aligned}(n-k)W^{n-k}(A) &= (n-k)W^{n-k}(D) \\ &= \left\{ \sum_{i=1}^{n-k} x_i^* D x_i : \{x_1, \ldots, x_{n-k}\} \text{ is an othonormal set in } \mathbb{H}^n \right\} \\ &= \left\{ tr(U^*DU) - \sum_{i=n-k+1}^{n} x_i^* D x_i : U = [x_1 \cdots x_n] \in \mathcal{U}_n \right\} \\ &= \left\{ tr(D) - \sum_{i=n-k+1}^{n} x_i^* D x_i : \{x_{n-k+1}, \ldots, x_n\} \text{ is othonormal} \right\} \\ &= tr(A) - kW^k(A).\end{aligned}$$

Hence, the proof is complete. □

Theorem 2.11. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) $\left\{\frac{1}{n}tr(A)\right\} \subseteq W^n(A)$, *and the equality holds if A is Hermitian;*

(b) *If* $W^n(A) = \left\{\frac{1}{n}tr(A)\right\}$, *then* $\sigma_r(A) \subseteq \mathbb{R}$.

Proof. (a) At first, by setting $X := I_n$, we have $X \in \mathcal{X}_{n\times n}$ and $\frac{1}{n}tr(A) = \frac{1}{n}tr(X^*AX) \in W^n(A)$. Now, let $A \in M_n(\mathbb{H})$ be a Hermitian matrix with eigenvalues

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

$h_1, \ldots, h_n$, counting multiplicities. Then by [21, Corollary 6.2], there exists a unitary matrix $U \in \mathcal{U}_n$ such that $U^*AU = D := diag(h_1, \ldots, h_n)$. Now by Theorem 2.5 (b) and Lemma 2.8 ((c) and (d)), we have

$$W^n(A) = W^n(D) = \left\{\frac{1}{n}tr(D)\right\} = \left\{\frac{1}{n}\sum_{i=1}^{n} h_i\right\} = \left\{\frac{1}{n}tr(A)\right\}.$$

To prove (b), by Theorem 2.5 (e), $\sigma_r^n(A) = \{\frac{1}{n}tr(A)\}$. Now, let $\lambda_1, \ldots, \lambda_n$ be the standard eigenvalues, counting multiplicities, of A. So, by Theorem 2.4 (a), $\frac{1}{n}(\lambda_1 + \cdots + \lambda_i + \cdots + \lambda_n) = \frac{1}{n}(\lambda_1 + \cdots + \overline{\alpha}\lambda_i\alpha + \cdots + \lambda_n)$ for all $1 \leq i \leq n$ and for every $\alpha \in \mathbb{H}$ with $|\alpha| = 1$. Hence, for every $1 \leq i \leq n$, $\lambda_i \in \mathbb{R}$. So, the proof is complete. ∎

The following example shows that in Theorem 2.11 (a), the set equality does not hold in general.

EXAMPLE 2.12. Let $A = \begin{pmatrix} i & 0 \\ 1 & j \end{pmatrix}$. Now, by setting $X = \begin{pmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}j \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}k \end{pmatrix} \in \mathcal{X}_{2\times 2}$, we have $\frac{1}{2}i = \frac{1}{2}tr(X^*AX) \in W^2(A)$, and hence, $W^2(A) \neq \left\{\frac{1}{2}tr(A)\right\} = \left\{\frac{i+j}{2}\right\}$.

For any $A \in M_n(\mathbb{H})$, let $h_1 \leq \cdots \leq h_n$ be the eigenvalues of the Hermitian part of A, counting multiplicities. Now, we introduce the following notions:

$$a_m^{(k)} = \frac{1}{k}\sum_{i=1}^{k} h_i \quad \text{and} \quad a_M^{(k)} = \frac{1}{k}\sum_{i=n-k+1}^{n} h_i. \tag{2.1}$$

In the following theorem, we characterize the $k-$numerical range of Hermitian quaternionic matrices.

THEOREM 2.13. *Let $A \in M_n(\mathbb{H})$ be a Hermitian matrix with eigenvalues $h_1 \leq \cdots \leq h_n$. Then*

$$W^k(A) = [a_m^{(k)}, a_M^{(k)}].$$

Proof. For the cases $k = 1$ and $k = n$, the result follows from [17, Lemma IV.1.2 (i)] and Theorem 2.11 (a), respectively. Now, we assume that $1 < k < n$. Since A is Hermitian, by [21, Corollary 6.2], there exists a unitary matrix $U \in \mathcal{U}_n$ such that $U^*AU = D := diag(h_1, \ldots, h_n)$. Now, let $\mu \in W^k(A)$ be given. Thus, by Theorem 2.5 (b), $\mu \in W^k(D)$, and hence, there exists a $X = (x_{ij}) \in \mathcal{X}_{n\times k}$ such that $\mu = \frac{1}{k}tr(X^*DX)$. So,

$$\mu = \frac{1}{k}[h_1(|x_{11}|^2 + \cdots + |x_{1k}|^2) + \cdots + h_n(|x_{n1}|^2 + \cdots + |x_{nk}|^2)].$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

Since $h_i \leq h_{n-k}$ for all $i = 1, \ldots, n-k$ and $X^*X = I_k$,

$$\begin{aligned}
\mu &\leq \frac{1}{k}[h_{n-k}((|x_{11}|^2 + \cdots + |x_{1k}|^2) + \cdots + (|x_{n-k,1}|^2 + \cdots + |x_{n-k,k}|^2)) \\
&\quad + h_{n-k+1}(|x_{n-k+1,1}|^2 + \cdots + |x_{n-k+1,k}|^2) + \cdots + h_n(|x_{n1}|^2 + \cdots + |x_{nk}|^2)] \\
&= \frac{1}{k}[h_{n-k}((1 - |x_{n-k+1,1}|^2 - \cdots - |x_{n1}|^2) + \cdots + (1 - |x_{n-k+1,k}|^2 - \cdots - |x_{nk}|^2)) \\
&\quad + h_{n-k+1}(|x_{n-k+1,1}|^2 + \cdots + |x_{n-k+1,k}|^2) + \cdots + h_n(|x_{n1}|^2 + \cdots + |x_{nk}|^2)] \\
&= \frac{1}{k}[kh_{n-k} + (h_{n-k+1} - h_{n-k})(|x_{n-k+1,1}|^2 + \cdots + |x_{n-k+1,k}|^2) \\
&\quad + \cdots + (h_n - h_{n-k})(|x_{n1}|^2 + \cdots + |x_{nk}|^2)] \\
&\leq \frac{1}{k}[kh_{n-k} + h_{n-k+1} - h_{n-k} + \cdots + h_n - h_{n-k}] \\
&= \frac{1}{k}[h_{n-k+1} + \cdots + h_n] \\
&= a_M^{(k)}.
\end{aligned}$$

Therefore, $\mu \leq a_M^{(k)}$. By the same way, one can prove that $\mu \geq a_m^{(k)}$. Thus, $a_m^{(k)} \leq \mu \leq a_M^{(k)}$, and hence,

$$W^k(A) \subseteq [a_m^{(k)}, a_M^{(k)}].$$

Conversely, let $\mu \in [a_m^{(k)}, a_M^{(k)}]$ be given. Since

$$h_1 + \cdots + h_k \leq h_2 + \cdots + h_{k+1} \leq \cdots \leq h_{n-k+1} + \cdots + h_n,$$

there exists $1 \leq i \leq n-k$ such that $h_i + h_{i+1} + \cdots + h_{i+k-1} \leq k\mu \leq h_{i+1} + h_{i+2} + \cdots + h_{i+k}$.

If $h_{i+k} = h_i$, then by setting $X = (x_{ij}) \in M_{n\times k}$, where $x_{i+1,1} = \cdots = x_{i+k,k} = 1$, and $x_{rs} = 0$ elsewhere, we have $X \in \mathcal{X}_{n\times k}$ and $\mu = \frac{1}{k}tr(X^*DX) \in W^k(D) = W^k(A)$. Now, let $h_{i+k} \neq h_i$. Then by setting $X = (x_{ij}) \in M_{n\times k}$, where $x_{i+1,1} = \cdots = x_{i+k-1,k-1} = 1$, $|x_{i+k,k}|^2 = \frac{k\mu - h_i - \cdots - h_{i+k-1}}{h_{i+k} - h_i}$, $|x_{ik}|^2 = \frac{h_{i+1} + \cdots + h_{i+k} - k\mu}{h_{i+k} - h_i}$, and $x_{rs} = 0$ elsewhere, we have $X \in \mathcal{X}_{n\times k}$ and $\mu = \frac{1}{k}tr(X^*DX)$. Hence, $\mu \in W^k(D) = W^k(A)$. So, the proof is complete. ∎

Using Theorem 2.13, in the following example, we show that in Theorem 2.5 (i), the set equality

$$conv(W^k(A \oplus B)) = conv(W^k(A) \cup W^k(B))$$

does not, in general, hold for the case $k > 1$.

Example 2.14. Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in M_2(\mathbb{H})$ *and* $B = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \in M_2(\mathbb{H})$. Then by Theorem 2.13, we have

$$conv(W^2(A) \bigcup W^2(B)) = \{3\} \neq [2, 4] = conv(W^2(A \oplus B)).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

The following theorem is a generalization of [8, p. 3].

THEOREM 2.15. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) $W^k(A) \subseteq \mathbb{R}$ *if and only if A is Hermitian;*

(b) $W^k(A) \subseteq \mathbb{P}$ *if and only if A is skew-Hermitian.*

Proof. We will prove the part (a). The proof of (b) is similar.
At first, we assume that $W^k(A) \subseteq \mathbb{R}$. We will show that A is Hermitian. Let $A = H + S$, where $H = \frac{1}{2}(A + A^*)$ is the Hermitian part and $S = \frac{1}{2}(A - A^*)$ is the skew-Hermitian part of A. Since $W^k(A) \subseteq \mathbb{R}$, for any othonormal set $\{x_1, \dots, x_k\}$ in $\mathbb{H}^n$, $\sum_{i=1}^k x_i^* A x_i \in \mathbb{R}$. Therefore, $\sum_{i=1}^k x_i^* H x_i + \sum_{i=1}^k x_i^* S x_i \in \mathbb{R}$, and hence, $\sum_{i=1}^k x_i^* S x_i = 0$, because S is a skew-Hermitian matrix. So, $W^k(S) = \{0\}$, and hence, by Theorem 2.5 (e), $\sigma_r^k(S) = \{0\}$. Therefore, as the same manner in the proof of Theorem 2.11 (b), we have $S = 0$. Hence, A is Hermitian.

The converse follows from Theorem 2.13, and so, the proof is complete. ◻

For convenience, for any $A \in M_n(\mathbb{H})$, we denote the projections of $W^k(A)$ on $\mathbb{R}$ and $\mathbb{C}$ by

$$W_{\mathbb{R}}^k(A) = \{Re\ q : q \in W^k(A)\} \tag{2.2}$$

and

$$W_{\mathbb{C}}^k(A) = \{Co\ q : q \in W^k(A)\}, \tag{2.3}$$

respectively. Let $A = A_1 + A_2 j \in M_n(\mathbb{H})$, where $A_1, A_2 \in M_n(\mathbb{C})$. The complex adjoint matrix or adjoint of A is defined and denoted by

$$\chi_{_A} = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in M_{2n}(\mathbb{C}). \tag{2.4}$$

Now, we list some basic relations among $W_{\mathbb{R}}^k(A), W_{\mathbb{C}}^k(A)$ and $W^k(A)$.

PROPOSITION 2.16. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) $\mathbb{R} \cap W^k(A) \subseteq W_{\mathbb{R}}^k(A) = W^k(H) = [a_m^{(k)}, a_M^{(k)}] \subseteq \mathbb{R} \cap conv(W^k(A))$, *where H is the Hermitian part of A, and $a_m^{(k)}, a_M^{(k)}$ are the notions as in (2.1);*

(b) $\mathbb{C} \cap W^k(A) \subseteq W_{\mathbb{C}}^k(A) \subseteq W_k(\chi_{_A}) \cap conv(W^k(A))$, *where $\chi_{_A}$ is the matrix as in (2.4) and $W_k(\chi_{_A}) = \{\frac{1}{k} tr(X^* \chi_{_A} X) : X \in M_{2n \times k}(\mathbb{C}),\ X^*X = I_k\}$. The equality holds if A is a real scalar matrix;*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

(c) If $a_0 + a_1 i \in W^k_{\mathbb{C}}(A)$, then $\{a_0 + ta_1 i : \ -1 \le t \le 1\} \subseteq W^k_{\mathbb{C}}(A)$. Consequently, $\mathbb{R} \cap W^k_{\mathbb{C}}(A) = W^k_{\mathbb{R}}(A)$.

Proof. To prove (a), the first inclusion follows from (2.2). The set equality follows from Theorem 2.13. Now, let $a_0 \in W^k_{\mathbb{R}}(A)$ be given. Then there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_0 + a_1 i + a_2 j + a_3 k \in W^k(A)$. Therefore, by Theorem 2.5 (c), $a_0 - a_1 i - a_2 j - a_3 k \in W^k(A)$. Hence, $a_0 = \frac{1}{2}(a_0 + a_1 i + a_2 j + a_3 k) + \frac{1}{2}(a_0 - a_1 i - a_2 j - a_3 k) \in \mathbb{R} \cap conv(W^k(A))$.
The left inclusion in (b) is trivial. Now, let $a_0 + a_1 i \in W^k_{\mathbb{C}}(A)$ be given. Then there are $a_2, a_3 \in \mathbb{R}$ such that $q := a_0 + a_1 i + a_2 j + a_3 k \in W^k(A)$. Therefore, there exists a $X \in \mathcal{X}_{n\times k}$ such that $q = \frac{1}{k} tr(X^*AX)$. By setting $A = A_1 + A_2 j, \ X = X_1 + X_2 j$, where $A_1, A_2 \in M_n(\mathbb{C}),$ *and* $X_1, X_2 \in M_{n\times k}(\mathbb{C})$, and using this fact that $jX_2^* = X_2^T j$, we have

$$q = \frac{1}{k} tr((X_1^* - X_2^T j)(A_1 + A_2 j)(X_1 + X_2 j)).$$

Hence,

$$\frac{1}{k} tr(X_1^* A_1 X_1 - X_1^* A_2 \bar{X}_2 + X_2^T \bar{A}_1 \ \bar{X}_2 + X_2^T \bar{A}_2 X_1) = a_0 + a_1 i \in W^k_{\mathbb{C}}(A).$$

Now, by setting $Y = \begin{pmatrix} X_1 \\ -\bar{X}_2 \end{pmatrix} \in M_{2n\times k}(\mathbb{C})$ and using this fact that $X^*X = I_k$, we have $Y^*Y = I_k$ and

$$\begin{aligned} a_0 + a_1 i &= \frac{1}{k} tr\left(\begin{pmatrix} X_1^* & -X_2^T \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \begin{pmatrix} X_1 \\ -\bar{X}_2 \end{pmatrix} \right) \\ &= \frac{1}{k} tr(Y^* \chi_A Y) \in W_k(\chi_A). \end{aligned}$$

Hence, $W^k_{\mathbb{C}}(A) \subseteq W_k(\chi_A)$. By the same manner as in the proof of (a), we have $W^k_{\mathbb{C}}(A) \subseteq conv(W^k(A))$. So, the second inclusion also holds. If $A = \alpha I_n$, where $\alpha \in \mathbb{R}$, then $W_k(\chi_A) = W_k(\alpha I_{2n}) = \{\alpha\}$. Since $\alpha \in \mathbb{R}$, for every $X \in \mathcal{X}_{n\times k}, \ \alpha X = X\alpha$, and hence, $W^k(A) = \{\alpha\} = W^k_{\mathbb{C}}(A)$. So, $W_k(\chi_A) = W^k_{\mathbb{C}}(A) = \{\alpha\}$, and hence, the set equality holds.
Finally, the first assertion in (c) follows from Corollary 2.6. The inclusion $\mathbb{R} \cap W^k_{\mathbb{C}}(A) \subseteq W^k_{\mathbb{R}}(A)$ follows from (2.2) and (2.3). Now, let $a_0 \in W^k_{\mathbb{R}}(A)$. Then there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_0 + a_1 i + a_2 j + a_3 k \in W^k(A)$. So, $a_0 + a_1 i \in W^k_{\mathbb{C}}(A)$. Then by the first assertion, $a_0 \in W^k_{\mathbb{C}}(A)$. Therefore, $a_0 \in W^k_{\mathbb{C}}(A) \bigcap \mathbb{R}$. So, the proof is complete. $\square$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

By [2, p. 280], $W^1_{\mathbb{C}}(A) = W_1(\chi_{_A})$. The following example shows that if $k > 1$, then the set equality $W^k_{\mathbb{C}}(A) = W_k(\chi_{_A})$ dose not hold in general.

EXAMPLE 2.17. Let $A = \begin{pmatrix} -2 & 4j \\ -4j & 6 \end{pmatrix} \in M_2(\mathbb{H})$. Then

$$\chi_{_A} = \begin{pmatrix} -2 & 0 & 0 & 4 \\ 0 & 6 & -4 & 0 \\ 0 & -4 & -2 & 0 \\ 4 & 0 & 0 & 6 \end{pmatrix}$$

is a Hermitian matrix with eigenvalues $h_1 = h_2 = 2 - 4\sqrt{2} \le h_3 = h_4 = 2 + 4\sqrt{2}$. Therefore, by Theorem 2.13, $W^2_{\mathbb{C}}(A) = W^2(A) = \{2\} \ne [2-4\sqrt{2}, 2+4\sqrt{2}] = W_2(\chi_{_A})$.

At the end of this section, we are going to give a necessary and sufficient condition for the convexity of $k-$numerical range of a quaternion matrix. For this, we need the following lemma.

LEMMA 2.18. *Let $A \in M_n(\mathbb{H})$ be such that $W^k(A) \cap \mathbb{C}$ is convex. If $a + p \in W^k(A)$, where $a \in \mathbb{R}$ and $p \in \mathbb{P}$, then for any $x \in \mathbb{P}$ with $|x| \le |p|$, $a + x \in W^k(A)$.*

Proof. The result trivially holds if $p = 0$. Now, we assume that $p \ne 0$. By Theorem 2.5 (c), we have $a \pm |p|i \in W^k(A) \cap \mathbb{C}$. Therefore, the convexity of $W^k(A) \cap \mathbb{C}$ implies that $a + t|p|i \in W^k(A) \cap \mathbb{C}$ for any $t \in [-1, 1]$. Now, let $x \in \mathbb{P}$ with $|x| \le |p|$ be given. Then by setting $t = \frac{|x|}{|p|} \le 1$, we have $a + |x|i = a + t|p|i \in W^k(A) \cap \mathbb{C}$. Therefore, by Theorem 2.5 (c), $a + x \in W^k(A)$. □

THEOREM 2.19. *Let $A \in M_n(\mathbb{H})$. Then $W^k(A)$ is convex if and only if $W^k(A) \cap \mathbb{C}$ is convex.*

Proof. Let $W^k(A) \cap \mathbb{C}$ is convex. We will show that $W^k(A)$ is also convex. For this, let $x_1 := a + p \in W^k(A)$, $x_2 := b + q \in W^k(A)$ and $\theta \in [0, 1]$, where $a, b \in \mathbb{R}$ and $p, q \in \mathbb{P}$, be given. Then, by Theorem 2.5 (c), we have $a + |p|i$, $b + |q|i \in W^k(A) \cap \mathbb{C}$. Therefore, the convexity of $W^k(A) \cap \mathbb{C}$ implies that

$$(\theta a + (1 - \theta)b) + (\theta|p| + (1 - \theta)|q|)i \in W^k(A) \cap \mathbb{C}.$$

Since $|Im(\theta x_1 + (1 - \theta)x_2)| = |\theta p + (1 - \theta)q| \le \theta|p| + (1 - \theta)|q|$, Lemma 2.18 implies that

$$\theta x_1 + (1 - \theta)x_2 = (\theta a + (1 - \theta)b) + (\theta p + (1 - \theta)q) \in W^k(A).$$

Therefore, $W^k(A)$ is convex.

The converse is trivial, and so, the proof is complete. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

3. On numerical range of normal quaternion matrices. In this section, we give a new description of the 1-numerical range of normal quaternion matrices.

THEOREM 3.1. *Let $A \in M_n(\mathbb{H})$ be a normal matrix with the standard right eigenvalues $\lambda_1, \ldots, \lambda_n$, counting multiplicities. Then*

$$W(A) = \bigcup_{a_i \in [\lambda_i]} conv(\{a_1, \ldots, a_n\}).$$

Proof. Since A is normal, by [21, Corollary 6.2], there exists a unitary matrix $U \in \mathcal{U}_n$ such that $D = U^*AU = diag(\lambda_1, \ldots, \lambda_n)$. Now, let $\mu \in W(A) = W(D)$ be given. Then there exists a $x = (x_1, \ldots, x_n)^T \in \mathbb{H}^n$ such that $\sum_{i=1}^n |x_i|^2 = 1$ and $\mu = x^*Dx = \sum_{i=1, |x_i| \neq 0}^n |x_i|^2 (\frac{\bar{x_i}}{|x_i|} \lambda_i \frac{x_i}{|x_i|})$. So, $\mu \in \bigcup_{a_i \in [\lambda_i]} conv(\{a_1, \ldots, a_n\})$, because for any $1 \leq i \leq n$, $|x_i|^2 \geq 0$, $\sum_{i=1}^n |x_i|^2 = 1$, *and* $\frac{\bar{x_i}}{|x_i|} \lambda_i \frac{x_i}{|x_i|} \in [\lambda_i]$. Therefore, $W(A) \subseteq \bigcup_{a_i \in [\lambda_i]} conv(\{a_1, \ldots, a_n\})$.
Conversely, let $\mu \in \bigcup_{a_i \in [\lambda_i]} conv(\{a_1, \ldots, a_n\})$ be given. Then there are nonnegative real numbers $t_1, \ldots, t_n \in \mathbb{R}$ summing to 1, and $x_1, \ldots, x_n \in \mathbb{H}$ such that for any $1 \leq i \leq n$, $|x_i|^2 = 1$, *and* $\mu = \sum_{i=1}^n t_i (\bar{x_i} \lambda_i x_i)$. Now, by setting $x = (\sqrt{t_1} x_1, \ldots, \sqrt{t_n} x_n)^T \in \mathbb{H}^n$, we have $x^*x = 1$ and $\mu = x^*Dx \in W(D) = W(A)$. So, $\bigcup_{a_i \in [\lambda_i]} conv(\{a_1, \ldots, a_n\}) \subseteq W(A)$.
Hence, the proof is complete. $\square$

COROLLARY 3.2. *Let $A \in M_n(\mathbb{H})$ be a normal matrix with the right spectrum $\sigma_r(A)$ as in Definition 2.3. Then*

$$conv(\sigma_r(A)) = conv(W(A)).$$

Proof. By Theorem 2.5 (e), $\sigma_r(A) \subseteq W(A)$, and hence,

$$conv(\sigma_r(A)) \subseteq conv(W(A)).$$

By Theorems 3.1 and 2.5 (e), we have $W(A) \subseteq conv(\sigma_r(A))$. So, the converse inclusion also holds, and hence, the proof is complete. $\square$

At the end of this section and in the following example, using Theorem 3.1, we find the $1-$numerical range of two normal quaternion matrices.

EXAMPLE 3.3. (a) Let $A = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{H})$. Then A is a normal matrix with the standard eigenvalues $0,\ i$. Since $[0] = \{0\}$, by Theorem 3.1, we have

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

$$
\begin{aligned}
W(A) &= \bigcup_{a_1\in[i],a_2=0} conv(\{a_1,a_2\}) \\
&= \{t(\alpha_1 i+\alpha_2 j+\alpha_3 k)+(1-t).0 : 0\le t\le 1, \alpha_1^2+\alpha_2^2+\alpha_3^2=1\} \\
&= \{x_1 i+x_2 j+x_3 k : x_1^2+x_2^2+x_3^2\le 1\},
\end{aligned}
$$

which is convex.
(b) Let $B=\begin{pmatrix} i & 0\\ 0 & 1\end{pmatrix}\in M_2(\mathbb{H})$. Then B is a normal matrix with standard eigenvalues $i,1$. Since $[1]=\{1\}$, by Theorem 3.1, we have

$$
\begin{aligned}
W(B) &= \bigcup_{a_1=1,a_2\in[i]} conv(\{a_1,a_2\}) \\
&= \{t(\alpha_1 i+\alpha_2 j+\alpha_3 k)+(1-t).1 : 0\le t\le 1, \alpha_1^2+\alpha_2^2+\alpha_3^2=1\} \\
&= \{x_0+x_1 i+x_2 j+x_3 k : 0\le x_0\le 1, x_1^2+x_2^2+x_3^2=(1-x_0)^2\}.
\end{aligned}
$$

Obviously, $W(B)$ is not convex.
By setting $X=\begin{pmatrix} \frac{1}{2}+\frac{1}{2}i & \frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}i+\frac{1}{\sqrt{10}}j \\ \frac{1}{\sqrt{2}}j & \frac{1}{2\sqrt{5}}+\frac{1}{2\sqrt{5}}i-\frac{\sqrt{2}}{\sqrt{5}}j\end{pmatrix}\in\mathcal{X}_{2\times 2}$, we see that

$$
\lambda=\frac{1}{2}+\frac{2}{5}i+\frac{1}{5\sqrt{2}}j+\frac{1}{5\sqrt{2}}k=\frac{1}{2}tr(X^*BX)\in W^2(B),
$$

and $\lambda\notin W(B)$. So, $W^2(B)$ is not a subset of $W(B)$; see Proposition 2.7.
It is clear that $\frac{1}{2}+\frac{1}{3}i+\frac{\sqrt{5}}{6}j\in W^1(B)$, and $1+i-(\frac{1}{2}+\frac{1}{3}i+\frac{\sqrt{5}}{6}j)=\frac{1}{2}+\frac{2}{3}i-\frac{\sqrt{5}}{6}j\notin W^1(B)$. Therefore, $(n-k)W^{n-k}(B)\neq tr(B)-kW^k(B)$, where $k=1$ and $n=2$; see Proposition 2.10.

4. Conclusions and future work. In the complex case, the notion of numerical range is useful in studing and understanding of complex matrices, and has many applications in numerical analysis, differential equations, systems theory, etc; e.g., see [7] and its references. Unlike the complex case, the quaternion numerical range may not be convex even for a normal quaternion matrix. In this paper, we have given some fundamental properties and a necessary and sufficient condition for the convexity of the higher numerical ranges of quaternion matrices. We have also given a description for the $k-$numerical range of Hermitian matrices and for the $1-$numerical range of normal quaternion matrices. There are many open problems in the study of the higher numerical ranges of quaternion matrices. For example, it is very nice if one can give a description for the structure of the $k-$numerical range of quaternion matrices. Also, it is important if one can characterize the shape of the $k-$numerical range of skew-Hermitian quaternion matrices.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 889-904, December 2015

ELA

Acknowledgment. The authors wish to express their gratitude to anonymous referees for helpful comments and useful suggestions.

REFERENCES

[1] S.L. Adler. *Quaternionic Quantum Mechanics and Quantum Fields.* Oxford University Press, New York, 1995.

[2] Y.H. Au-Yeung. A short proof of a theorem on the numerical range of a normal quaternionic matrix. *Linear Multilinear Algebra*, 39:279–284, 1995.

[3] Y.H. Au-Yeung and L.S. Siu. Quaternionic numerical range and real subspaces. *Linear Multilinear Algebra*, 45:317–327, 1999.

[4] P.M. Cohn. *Skew-Fields: Theory of General Division Rings.* Cambridge University Press, Cambridge, 1995.

[5] J.H. Conway and D.A. Smith. *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry.* A.K. Peters, Natick, 2002.

[6] K. Gurlebeck and W. Sprossig. *Quaternionic Analysis and Elliptic Boundary Value Problems.* Birkhauser Verlag, Boston, 1990.

[7] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[8] J.E. Jamison. *Numerical Range and Numerical Radius in Quaternionic Hilbert Spaces.* PhD Thesis, University of Missouri, 1972.

[9] G. Kamberov, P. Norman, F. Pedit, and U. Pinkall. *Quaternions, spinors, and surfaces.* Contemporary Mathematics, Vol. 299, Amer. Math. Soc., Province, 2002.

[10] R. Kippenhahn. Uber die Wertvorrat einer Matrix. *Math. Nachr.*, 6: 193–228, 1951.

[11] J.B. Kuipers. *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace, and Virtual Reality.* Princeton University Press, Princeton, 2002.

[12] C.K. Li, I. Spitkovsky, and S. Shukla. Equality of higher numerical ranges of matrices and a conjecture of Kippenhahn on Hermitian pencils. *Linear Algebra Appl.*, 270:323–349, 1998.

[13] N. Mackey. Hamilton and Jacobi meet again; quaternions and the eigenvalues problem. *SIAM J. Matrix Anal. Appl.*, 16:421–435, 1995.

[14] G. Nebe. Finite quaternionic matrix groups. *Represent. Theory*, 2:106–223, 1998.

[15] S. Qaisar and L. Zou. Distribuition for the standard eigenvalues of quaternion matrices. *Int. Math. Forum*, 7:831–838, 2012.

[16] L. Rodman. Pairs of quaternionic selfadjoint operators with numerical range in a halfplane. *Linear Multilinear Algebra*, 56:179–184, 2008.

[17] L.S. Siu. *A Study of Polynomials, Determinants, Eigenvalues and Numerical Ranges over Quaternions.* PhD Thesis, University of Hong Kong, 1997.

[18] W. So, R.C. Thompson, and F. Zhang. The numerical range of normal matrices with quaternion entries. *Linear Multilinear Algebra*, 37:175–195, 1994.

[19] B.L. van der Waerden. Hamilton's discovery of quaternions. *Math. Magazine*, 49:227–234, 1976.

[20] J.P. Ward. *Quaternions and Cayley Numbers.* Mathematics Appl., Vol. 403, Kluwer Academic Publishers, Dordrecht, 1997.

[21] F. Zhang. Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251:21–57, 1997.

[22] F. Zhang. On numerical range of normal matrices of quaternions. *J. Math. Phys. Sci.*, 29:235–251, 1995.