

MATRICES OF MINIMUM NORM SATISFYING CERTAIN PRESCRIBED BAND AND SPECTRAL RESTRICTIONS – AN EXTREMAL CHARACTERIZATION OF THE DISCRETE PERIODIC LAPLACIAN*

MARCOS V. TRAVAGLIA[†]

Abstract. This paper has been motivated by the curiosity that the circulant matrix $\text{Circ}(1/2, -1/4, 0, \dots, 0, -1/4)$ is the $n \times n$ positive semidefinite, tridiagonal matrix A of smallest Euclidean norm having the property that $Ae = 0$ and $Af = f$, where e and f are, respectively, the vector of all 1s and the vector of alternating 1 and -1 s. It then raises the following question (minimization problem): What should be the matrix A if the tridiagonal restriction is replaced by a general bandwidth $2r + 1$ ($1 \leq r \leq \frac{n}{2} - 1$)? It is first easily shown that the solution of this problem must still be a circulant matrix. Then the determination of the first row of this circulant matrix consists in solving a least-squares problem having $\frac{n}{2} - 1$ nonnegative variables (Nonnegative Orthant) subject to $\frac{n}{2} - r$ linear equations. Alternatively, this problem can be viewed as the minimization of the norm of an even function vanishing at the points $|i| > r$ of the set $\{-\frac{n}{2} + 1, \dots, -1, 0, 1, \dots, \frac{n}{2}\}$, and whose Fourier-transform is nonnegative, vanishes at zero, and assumes the value one at $\frac{n}{2}$. Explicit solutions are given for the special cases of $r = \frac{n}{2}$, $r = \frac{n}{2} - 1$, and $r = 2$. The solution for the particular case of $r = 2$ can be physically interpreted as the vibrational mode of a ring-like chain of masses and springs in which the springs link both the nearest neighbors (with positive stiffness) and the next-nearest neighbors (with negative stiffness). The paper ends with a numerical illustration of the six cases ($1 \leq r \leq 6$) corresponding to $n = 12$.

Key words. Minimization problem involving matrices, Circulant matrices, Banded matrices, Laplacian matrices, Least-norm problems, Restriction on the Fourier-transform, Negative stiffness.

AMS subject classifications. 15A18.

1. Introduction. A matrix of order n is said to be *circulant* if it is of the form

$$(1.1) \quad \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ c_1 & c_2 & \cdots & & c_0 \end{bmatrix},$$

where c_0, c_1, \dots , and c_{n-1} are given real numbers; it is simply denoted by $\text{Circ}(c_0, c_1, \dots, c_{n-1})$.

*Received by the editors on May 8, 2015. Accepted for publication on August 22, 2015. Handling Editor: Bryan L. Shader.

[†]Departamento de Matemática, Centro de Ciências da Natureza, Universidade Federal do Piauí-UFPI. CEP 64049-550 Teresina - PI, Brazil (mvtravaglia@ufpi.edu.br).

The one-dimensional *discrete periodic Laplacian*, denoted here by \mathcal{L} , is the matrix $\text{Circ}(2, -1, 0, \dots, 0, -1)$. Observe that \mathcal{L} is a periodic tridiagonal banded matrix. Moreover, the matrix \mathcal{L} has the interesting property that its spectrum is nonnegative and two of its eigenvectors, $e := (1, 1, \dots, 1)^T$ and $f := (1, -1, \dots, 1, -1)^T$, are associated with the smallest $\lambda_e = 0$ and largest $\lambda_f = 4$ eigenvalues, respectively. Throughout this paper, we assume that n is even and greater than or equal to 4; otherwise we cannot ensure that 4 is the corresponding eigenvalue of f .

Physically, we can interpret these properties by considering a ring-like chain of n point masses (with identical masses $m = 1$), and assume that only the nearest neighbor pairs of them are linked by a spring with (positive) stiffness ($k_1 = -c_1 = 1$). In this physical context, the potential energy stored in the springs of the chain for a given vector $d \in \mathbb{R}^n$ of the displacement (from equilibrium position) corresponds to the value $\frac{1}{2} d^T \mathcal{L} d$. Due to the stability of the system, it expects \mathcal{L} to be a positive semidefinite matrix. Indeed, for the Laplacian \mathcal{L} we have $\frac{1}{2} d^T \mathcal{L} d = \frac{1}{2} (d_1 - d_n)^2 + \frac{1}{2} \sum_{j=1}^n (d_j - d_{j-1})^2$, which is a sum of squares; therefore nonnegative. Furthermore, the eigenvector $e := (1, 1, \dots, 1)^T$ corresponds to the following motion of the entire chain: Every point mass is displaced by the same quantity in the clockwise direction. Note that in this mode of motion, there is no contraction or extension of any spring (just rigid motion), and so the potential energy is zero. By contrast, for the eigenvector f we have the following interpretation: The point masses located at odd positions are clockwise displaced, and the ones located at even positions are counter clockwise displaced. In this case, the chain has its maximum potential energy which is equal to the eigenvalue $\lambda_f = 4 > 0$.

Motivated by this peculiarity of the Laplacian matrix, we propose the following *matrix-minimum-norm problem*: For a given bandwidth $2r+1$ with $r \in \{1, 2, \dots, n/2\}$ and $\beta > 0$ find the following minimizer or the minimum-norm solution:

$$(1.2) \quad \overline{X}(n, b, \beta) := \operatorname{argmin} \left\{ \|X\|^2 : X \in \mathbb{B}^{(r)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(\beta)} \right\}.$$

In the above expression, $\mathbb{B}^{(r)}$ is the set of all banded matrices with bandwidth $2r+1$ where each upper diagonal is extended to the length n . More precisely, $\mathbb{B}^{(r)} := \{X \in \mathbb{R}^{n \times n} : X_{ij} = 0 \text{ if } \operatorname{dist}_n(j-i) > r\}$, where $\operatorname{dist}_n(j-i) := \min\{j-i \bmod n, i-j \bmod n\}$; Note that “ dist_n ” is not exactly the standard distance from the entry ij to the main diagonal (entry ii) because we are considering the i -th row as a circular ring. By \mathbb{S}_+ we mean the set of all symmetric positive semidefinite matrices, and the set $\mathbb{E}^{(\beta)}$ is defined as $\mathbb{E}^{(\beta)} := \{X \in \mathbb{R}^{n \times n} : Xe = 0 \text{ and } Xf = \beta f\}$. The motivation to choose the letter \mathbb{E} for this last set comes from the fact that the pairs $(e, 0)$ and (f, β) must be in the set of the eigenpairs of a given matrix in $\mathbb{E}^{(\beta)}$. The norm in (1.2) is defined as $\|X\| = \sqrt{\sum_{i,j=0}^{n-1} X_{ij}^2}$ and is called the Frobenius or Euclidean norm.

As we are about to see (in Section 2), the matrix $\frac{\beta}{4}\mathcal{L}$ is the solution of (1.2) for $r = 1$ (tridiagonal). That is, the problem (1.2) gives an extremal characterization of \mathcal{L} . Concerning the case $r = 2$, its solution in the mass-spring interpretation presents negative stiffness. Moreover, we can find applications of the case $r = 2$, for instance, in the study of surface effects in Material Sciences, see [4].

From the point of view of graph theory, problem (1.2) consists in finding the positive semidefinite matrix A of smallest Euclidian norm whose graph is a subgraph of the Cayley graph $\Gamma(\mathbb{Z}_n, \{1, 2, \dots, r, n-1, \dots, n-r\})$ having the property that $Ae = 0$ and $Af = \beta f$.

The paper is organized as follows. In Section 2, we make a preliminary analysis on the foregoing problem. Namely, after ensuring the existence and uniqueness to the problem, we show that its solution must be a circulant matrix. This allows us to reduce the matrix-norm-minimization problem to a vector-norm-minimization one. In Section 3, we recall the discrete Fourier-transform and its relation to the eigenvalues of circulant matrices. This will permit us to rewrite the vector-norm-minimization problem as a least-squares problem. In Sections 4 and 5, we explicitly solve the least-squares problem for the cases $r = n/2 - 1$ and $r = 2$, respectively. A numerical illustration for the case $n = 12$ and r varying from 1 to 6 is given in Section 6. Conclusions and outlook are presented in Section 7.

2. Immediate properties of the matrix minimization problem. First, note that $\mathbb{B}^{(r)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(\beta)}$, i.e., the feasible set of (1.2), is nonempty because $\frac{\beta}{4}\mathcal{L} \in \mathbb{B}^{(1)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(\beta)} \subset \mathbb{B}^{(r)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(\beta)}$ for any $r \in \{1, 2, \dots, n/2\}$. Second, it is easy to see that (1.2) admits a unique solution since the objective function $\|\cdot\|$ is strict convex and coersive, and the set $\mathbb{B}^{(r)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(\beta)}$ is closed and convex. Third, an alternating projective method as described in [1], [2], [5], [6], [7] and [8] converges to the solution. Fourth, the solution $\overline{X}(n, b, \beta)$ is always a circulant matrix. The proof of this last fact follows easily by the triangular inequality and the unitary invariance of the Frobenius norm; namely, for any $X \in \mathbb{R}^{n \times n}$ we have

$$(2.1) \quad \left\| \frac{1}{n}X + \frac{1}{n}TXT^{-1} + \dots + \frac{1}{n}T^{(n-1)}XT^{-(n-1)} \right\| \leq \|X\|,$$

where $T := \text{Circ}(0, 1, 0, \dots, 0)$ is the standard periodic shift operator. Note that the expression inside the norm on the left-hand side of (2.1) is what we obtain by averaging X over the diagonals. Recall that we are considering that all diagonals have the same length n ; except for the main diagonal, we extend the other diagonals periodically. Since this averaging process leaves the feasible set of (1.2) invariant and does not increase the norm (see the inequality (2.1) above), we conclude that the minimizer must be a circulant matrix.

Now, we can rewrite the minimization problem (1.2) as follows. Find $\bar{x} \in \mathbb{R}^{r+1}$, that is, $\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_r$ such that

$$(2.2) \quad \bar{x}(n, r, \beta) := \operatorname{argmin} \left\{ x_0^2 + 2x_1^2 + \dots + 2x_r^2 : \begin{array}{l} \operatorname{Circ}(x_0, \dots, x_r, 0, \dots, 0, x_r, \dots, x_1) \in \mathbb{S}_+, \\ x_0 + 2x_1 + \dots + 2x_r = 0, \text{ and} \\ x_0 - 2x_1 + 2x_2 \dots + 2(-1)^r x_r = \beta. \end{array} \right\}.$$

Note that the minimizers of problems (1.2) and (2.2) are related to each other as $\bar{X}(n, b, \beta) = \operatorname{Circ}(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_r, 0, \dots, 0, \bar{x}_r, \dots, \bar{x}_1)$. Moreover, for the special case of $r = 1$ (tridiagonal), it is easy to see that the solution of (2.2) is given by $\bar{x}_0 = \frac{\beta}{2}$ and $\bar{x}_1 = -\frac{\beta}{4}$. This confirms that $\frac{\beta}{4}\mathcal{L}$ is the solution of (1.2) for $r = 1$.

Note that the first two conditions defining the feasible set of (2.2) remain invariant if each x_i is scaled by the factor $1/\beta > 0$. Taking this fact and the homogeneity of the objective function into account, we can fix $\beta = 1$ because, for general $\beta > 0$, the solutions $\bar{x}(n, r, \beta)$ and $\bar{x}(n, r, 1)$ are related to each other as $\bar{x}(n, r, \beta) = \beta \bar{x}(n, r, 1)$. That is, the solution \bar{x} of (2.2) depends on β linearly. From now on we simply write $\bar{x}(n, r)$ and $\bar{X}(n, r)$ instead of $\bar{x}(n, r, 1)$ and $\bar{X}(n, r, 1)$, respectively.

Keeping in mind the above observations, we shall be concerned with the following questions.

QUESTION 2.1. For given $n \geq 4$ even and $r = 1, 2, \dots, n/2$ determine the $r + 1$ entries of the minimizer $\bar{x}(n, r)$ of (2.2), that is, find $\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_r$?

QUESTION 2.2. What can we say about the eigenvalues (and their algebraic multiplicity) of the matrix $\bar{X}(n, r) = \operatorname{Circ}(\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_r, 0, \dots, 0, \bar{x}_r, \dots, \bar{x}_1)$?

3. Discrete Fourier-transform for symmetric circulant matrices. In this section, we recall the discrete Fourier-transform and its relation to the particular case of symmetric circulant matrices. These relations will enable us to rewrite the problem (2.2) as a least-squares problem (see Proposition 3.3 below). The solution of this least-squares problem are exactly the eigenvalues of the minimal circulant matrix which solves the original problem (1.2).

It is well-known that the eigenvalues $\lambda_i(x)$, $i = 0, 1, \dots, n-1$, of an $n \times n$ circulant matrix $X = \operatorname{Circ}(x)$, $x = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n$, are given by $\lambda_i(x) = \sum_{j=0}^{n-1} u_{ij} x_j$ with $u_{ij} = e^{\sqrt{-1}\theta ij}$, where $\theta = \theta_n = \frac{2\pi}{n}$. That is, the vector of eigenvalues $\lambda \in \mathbb{R}^n$ and the vector $x \in \mathbb{R}^n$ are related to each other as $\lambda = Ux$. The complex matrix U is known as the discrete Fourier transform (DFT) and its inverse satisfies $U^{-1} = \frac{1}{n}U^H$. For a given $x \in \mathbb{R}^n$, it is important to emphasize that the eigenvalues $\lambda_i(x)$ are, in general, neither in increasing nor in decreasing order with relation to the index i .

In the case of *symmetric circulant* matrices, we introduce the notation *Scirc* as follows. For given $x = (x_0, x_1, \dots, x_{n/2}) \in \mathbb{R}^{\frac{n}{2}+1}$ (n even) we define the following matrix:

$$(3.1) \quad \text{Scirc}(x) := \text{Circ}(x_0, x_1, \dots, x_{n/2-1}, x_{n/2}, x_{n/2-1}, \dots, x_2, x_1).$$

It is easy to show that the spectrum of $\text{Scirc}(x)$ has the symmetry $\lambda_i(x) = \lambda_{n-i}(x)$, $i = 1, 2, \dots, \frac{n}{2} - 1$. This means that for symmetric circulant matrices $\text{Scirc}(x)$, $x \in \mathbb{R}^{n/2+1}$, it suffices to know only the eigenvalues $\lambda_0(x), \lambda_1(x), \dots, \lambda_{n/2}(x)$. It is easy to see that the vector of these $n/2 + 1$ eigenvalues $-\lambda \in \mathbb{R}^{n/2+1}$ are related to $x \in \mathbb{R}^{n/2+1}$ by the equation

$$(3.2) \quad (\lambda_0, \lambda_1, \dots, \lambda_{n/2})^T = V(x_0, x_1, \dots, x_{n/2})^T,$$

where V is an $\frac{n}{2} + 1 \times \frac{n}{2} + 1$ real matrix given by

$$(3.3) \quad V = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \cos\left(\left(\frac{n}{2}-1\right)\theta\right) & 1 \\ 1 & 2 \cos(\theta) & 2 \cos(2\theta) & \cdots & 2 \cos\left(2\left(\frac{n}{2}-1\right)\theta\right) & -1 \\ 1 & 2 \cos(2\theta) & 2 \cos(4\theta) & \cdots & 2 \cos\left(2\left(\frac{n}{2}-1\right)\theta\right) & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 \cos\left(\left(\frac{n}{2}-1\right)\theta\right) & 2 \cos\left(2\left(\frac{n}{2}-1\right)\theta\right) & \cdots & 2 \cos\left(\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-1\right)\theta\right) & (-1)^{\frac{n}{2}-1} \\ 1 & -2 & 2 & \cdots & 2(-1)^{\frac{n}{2}-1} & (-1)^{\frac{n}{2}} \end{bmatrix}.$$

That is, according to the j -th column, the entries of the matrix V are given as:

$$(3.4) \quad \{V_{ij}\}_{i,j=0}^{n/2} = \begin{cases} 1, & \text{if } j = 0, \\ 2 \cos(i j \theta), & \text{if } j = 1, 2, \dots, \frac{n}{2} - 1, \\ (-1)^i, & \text{if } j = n/2. \end{cases}$$

Moreover, it is also easy to see that the inverse of the matrix V satisfies

$$(3.5) \quad V^{-1} = \frac{1}{n} V.$$

Since the Frobenius norm has the property $\|A\|^2 = \sum \lambda_j^2(A)$ for any symmetric matrix A , we establish the following remark which is nothing other than the Parseval identity for symmetric circulant matrices.

REMARK 3.1 (Parseval). Let x be in $\mathbb{R}^{\frac{n}{2}+1}$ and let $\text{Scirc}(x)$ be its corresponding $n \times n$ symmetric circulant matrix. For the sake of brevity, write simply $\lambda_i(x)$ for the eigenvalue $\lambda_i(\text{Scirc}(x))$. Then the following identity holds for the eigenvalues of $\text{Scirc}(x)$:

$$\begin{aligned} \lambda_0^2(x) + 2\lambda_1^2(x) + \cdots + 2\lambda_{\frac{n}{2}-1}^2(x) + \lambda_{\frac{n}{2}}^2(x) &= \|\text{Scirc}(x)\|^2 \\ &= n \left(x_0^2 + 2x_1^2 + \cdots + 2x_{\frac{n}{2}-1}^2 + x_{\frac{n}{2}}^2 \right). \end{aligned}$$

REMARK 3.2. It follows immediately from (3.2) that an $n \times n$ symmetric circulant matrix $X = \text{Scirc}(x)$, $x \in \mathbb{R}^{n/2+1}$, is positive semidefinite if and only if all $\frac{n}{2} + 1$ entries of the vector Vx are nonnegative. Moreover, if $\text{Scirc}(x) \in \mathbb{E}^{(1)}$, then its eigenvalues $\lambda_0(x)$ and $\lambda_{\frac{n}{2}}(x)$ are, respectively, 0 and 1.

Using (2.2), (3.2)–(3.5), and Remarks 3.1 and 3.2, we can easily prove that the minimization problem (1.2) can be rewritten in terms of λ as a least-squares problem with $\frac{n}{2} - 1$ nonnegative variables and $\frac{n}{2} - r$ linear restrictions. More precisely, we have the following proposition.

PROPOSITION 3.3 (Vector-minimum-norm problem). *Let $n \geq 4$ be an even number and $r = 1, 2, \dots, \frac{n}{2}$. Then $\text{argmin}\{\|X\| : X \in \mathbb{B}^{(r)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(1)}\}$, the solution of (1.2), is given by*

$$(3.6) \quad \bar{X}(n, r) = \frac{1}{n} \text{Circ}((V\bar{\lambda})_0, (V\bar{\lambda})_1, \dots, (V\bar{\lambda})_r, 0, \dots, 0, (V\bar{\lambda})_r, \dots, (V\bar{\lambda})_1),$$

where $\bar{\lambda} = (0, \bar{\lambda}_1, \dots, \bar{\lambda}_{\frac{n}{2}-1}, 1)^T$ and its $\frac{n}{2} - 1$ entries $\bar{\lambda}_1, \dots, \bar{\lambda}_{\frac{n}{2}-1}$ are the solution of the following least-squares problem:

Find the vector

$$(3.7) \quad (\bar{\lambda}_1, \dots, \bar{\lambda}_{\frac{n}{2}-1}) = \text{argmin} \left\{ \sum_{j=1}^{\frac{n}{2}-1} \lambda_j^2 : \lambda_j \geq 0 \text{ and } A\lambda = b \right\},$$

where the matrix $A \in \mathbb{R}^{(\frac{n}{2}-r) \times (\frac{n}{2}-1)}$ and the vector $b \in \mathbb{R}^{\frac{n}{2}-r}$ are given by

$$(3.8) \quad A_{ij} := 2 \cos\left(\frac{2\pi}{n}(i+r)j\right), \quad i = 1, \dots, \frac{n}{2} - r \text{ and } j = 1, \dots, \frac{n}{2} - 1,$$

and

$$(3.9) \quad b_i := (-1)^{i+r+1}, \quad i = 1, \dots, \frac{n}{2} - r,$$

respectively.

Proof. Note that the expression (3.6) comes from the reduction of the minimization problem (1.2) to the case in which the variable is a symmetric circulant matrix; see Section 2. Replacing x by $\frac{1}{n}V\lambda = V^{-1}\lambda$ in the reduced problem (2.2), we can rewrite it in terms of the variable λ with the aid of previous remarks. As a result we shall obtain the problem (3.7). To see this in more detail, note firstly that $\bar{\lambda}_0$ and $\bar{\lambda}_{\frac{n}{2}}$ do not appear in the problem (3.7) since they are, respectively, 0 and 1; see Remark 3.2. This explains why the matrix A in (3.7) has $n/2 - 1$ columns. To see that A has $n/2 - r$ rows, note that equation $A\lambda = b$ in (3.7) is the same as the $n/2 - r$ conditions $\bar{x}_{r+1} = \dots = \bar{x}_{n/2} = 0$ which correspond to $(V^{-1}\bar{\lambda})_i = 0$ (equivalently,

$(V\bar{\lambda})_i = 0$), for $i = r + 1, \dots, n/2$. These last $n/2 - r$ equations come from the fact that $X = \text{Scirc}(x) \in \mathbb{B}^{(r)}$. To see why the entries of the matrix A and the entries of the vector b are given, respectively, by (3.8) and (3.9), recall the entries of the matrix V in (3.3) and plug $\bar{\lambda} = (0, \bar{\lambda}_1, \dots, \bar{\lambda}_{\frac{n}{2}-1}, 1)^T$ into $(V\bar{\lambda})_i = 0$, for $i = r + 1, \dots, n/2$. We close the proof by noting that the objective function $\sum_{j=1}^{n/2-1} \lambda_j^2$ and the condition $\lambda_j \geq 0$ come from the Remarks 3.1 and 3.2, respectively. \square

As a direct consequence of the above proposition, we have the following corollary.

COROLLARY 3.4 (The case $r = \frac{n}{2}$). *Let $n \geq 4$ be an even number and $r = \frac{n}{2}$. Then $\text{argmin}\{\|X\| : X \in \mathbb{B}^{(r)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(1)}\}$, the solution of (1.2), is given by*

$$\bar{X}(n, r = n/2) = \text{Circ}\left(\frac{1}{n}, -\frac{1}{n}, \dots, \frac{1}{n}, -\frac{1}{n}\right).$$

Proof. Since $r = n/2$, there is no restriction equation in the minimization problem (3.7), and so its solution is given by $0 = \bar{\lambda}_1 = \dots = \bar{\lambda}_{n/2-1}$. Therefore, $\bar{\lambda} = (0, 0, \dots, 0, 1) \in \mathbb{R}^{\frac{n}{2}+1}$. Replace this $\bar{\lambda}$ into (3.6) and recalling the entries of the matrix V above, we obtain the claim. \square

The problem (3.7) consists of the minimization of Euclidean distance from the zero vector to the nonnegative orthant subject to linear constraints. It is a particular case of least-squares programming. As far as we know, for general matrix A and vector b , there is no closed-formula expression for its solution¹. In the case of algorithms, we can refer to the MATLAB function `lsqlin`. The MATLAB function `lsqlin(C, d, A_ineq, b_ineq, A, b, lb, ub)` solves numerically the minimization problem $\text{argmin}\{\|C\lambda - d\| : A_{\text{ineq}}\lambda \leq b_{\text{ineq}}, A\lambda = b \text{ and } lb \leq \lambda \leq ub\}$. Note that the problem (3.7)–(3.9) corresponds to the particular choice of $C = I$, $d = 0$, $A_{\text{ineq}} = 0$, $b_{\text{ineq}} = 0$, $lb = 0$, $ub = +\infty$, and A and b given respectively by (3.8) and (3.9).

Nevertheless, for the special case of $r = \frac{n}{2} - 1$, that is, when A is a row-matrix a^T and b is a scalar, the solution of the problem (3.7)–(3.9) has a closed-formula expression given by Lemma 4.1 below.

We finish the present section by mentioning the following interesting remark.

REMARK 3.5 (Minimization of even function with restriction on its Fourier-transform). If we consider the set with n points $\Omega := \{-\frac{n}{2} + 1, \dots, -1, 0, 1, \dots, \frac{n}{2}\}$, then to obtain the solution of minimization problem (1.2)

$$\bar{X}(n, r) = \text{Circ}(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_r, 0, \dots, 0, \bar{x}_r, \dots, \bar{x}_1)$$

¹A natural candidate to the solution would be $[A^\dagger b]_+$, where $[\cdot]_+$ and \dagger denote, respectively, the positive part and the pseudo-inverse. This try comes from the fact that $A^\dagger b$ solves the problem $\text{argmin}\{\|\lambda\|^2 : A\lambda = b\}$, and the positive part ensures the nonnegativity of the solution. However, it is easy to furnish a counterexample showing that $[A^\dagger b]_+$ does not solve the problem (3.7)–(3.9).

corresponds to find \bar{x}_i , $i \in \Omega$, satisfying

(3.10)

$$\operatorname{argmin} \left\{ \|x\|^2 : x : \Omega \rightarrow \mathbb{R} \text{ with } x_i = x_{-i}, x_i = 0 \text{ for } |i| > r, \hat{x}_0 = 0, \hat{x}_{n/2} = 1 \text{ and } \hat{x}_i \geq 0 \right\},$$

or, equivalently, to find $\bar{\lambda}_i$, $i \in \Omega$, satisfying

(3.11)

$$\operatorname{argmin} \left\{ \|\lambda\|^2 : \lambda : \Omega \rightarrow \mathbb{R} \text{ with } \lambda_i = \lambda_{-i}, \bar{\lambda}_i = 0 \text{ for } |i| > r, \lambda_0 = 0, \lambda_{n/2} = 1 \text{ and } \lambda_i \geq 0 \right\}.$$

In (3.10) and (3.11), the symbols “ $\hat{\cdot}$ ” and “ $\bar{\cdot}$ ” mean $\hat{x} = Ux$ (the direct Fourier-transform) and $\bar{\lambda} = U^{-1}\lambda$ (the inverse Fourier-transform), respectively. *Problems of the type (3.10) and (3.11) - find the closest sequence \bar{x}_i , $i \in \Omega$, whose Fourier-transform is nonnegative - appear in Filter Synthesis; see for instance [3].*

4. The case of upper bandwidth equal to $\frac{n}{2} - 1$. In this section, we explicitly solve the minimization problem (1.2) by applying Proposition 3.3 to the particular case of $r = \frac{n}{2} - 1$. Note that in this case the matrices of $\mathbb{B}^{(r=n/2-1)}$ have at least one upper diagonal of zeros. According to Proposition 3.3 (see (3.7), (3.8), and (3.9)), we should solve the following least-squares problem with $\frac{n}{2} - 1$ nonnegative variables and exactly one equation:

(4.1)

$$(\bar{\lambda}_1, \dots, \bar{\lambda}_{\frac{n}{2}-1}) = \operatorname{argmin} \left\{ \sum_{j=1}^{n/2-1} \lambda_j^2 : \lambda_j \geq 0 \text{ and } \sum_{j=1}^{n/2-1} 2(-1)^j \lambda_j = (-1)^{n/2+1} \right\}.$$

We next solve (4.1) with the help of the following three lemmas.

Let $[a]_+$ denote the positive part of a given vector $a \in \mathbb{R}^r$; that is, $[a]_+$ is the vector in \mathbb{R}^r whose entries are defined by $([a]_+)_j := \max\{0, a_j\} = [a_j]_+$, $j = 1, 2, \dots, r$.

LEMMA 4.1. *Let a be a vector in \mathbb{R}^r such that $[a]_+ \neq 0$ and consider the minimization problem*

$$(4.2) \quad \operatorname{argmin} \{ \|\lambda\|^2 : \lambda \in \mathbb{R}_+^r \text{ and } \langle a, \lambda \rangle = 1 \}.$$

Then

- i. The vector $\frac{1}{\|[a]_+\|^2} [a]_+$ is the solution to the problem (4.2).
- ii. The more general problem $\operatorname{argmin} \{ \|\lambda\|^2 : \lambda \in \mathbb{R}_+^r \text{ and } \langle a, \lambda \rangle = b \}$ has as solution $\bar{\lambda} = \frac{|b|}{\|[\operatorname{sign}(b)a]_+\|^2} [\operatorname{sign}(b)a]_+$ and $\bar{\lambda} = 0$ in the cases $b \neq 0$ and $b = 0$, respectively.

Proof. Let D_a denote the feasible set of (4.2), that is,

$$D_a = \{ \lambda \in \mathbb{R}_+^r \text{ and } \langle a, \lambda \rangle = 1 \}.$$

Since by hypothesis $[a]_+ \neq 0$, it makes sense to pick the point $\frac{1}{\|[a]_+\|^2} [a]_+$. Moreover, it is easy to see that $\frac{1}{\|[a]_+\|^2} [a]_+ \in D_a$, and so $D_a \neq \emptyset$. We claim that $\frac{1}{\|[a]_+\|^2} [a]_+$ is the point from D_a which is closest to the point 0. In other words, the point $\frac{1}{\|[a]_+\|^2} [a]_+$ is the projection of the point 0 onto the subset D_a . Since D_a is closed and convex, the previous claim is according to Kolmogorov's criterium (see [6, Theorem 2.8]) equivalent to show that the point $\frac{1}{\|[a]_+\|^2} [a]_+$ satisfies the following two conditions:

$$(4.3) \quad \frac{1}{\|[a]_+\|^2} [a]_+ \in D_a$$

and

$$(4.4) \quad \langle \mu - \frac{1}{\|[a]_+\|^2} [a]_+, 0 - \frac{1}{\|[a]_+\|^2} [a]_+ \rangle \leq 0 \quad \text{for all } \mu \in D_a.$$

We have already verified the condition (4.3). We now check the condition (4.4). Let μ be an arbitrary element of the set D_a , that is, μ satisfies $1 = \sum_{j=1}^r a_j \mu_j$ with $\mu_j \geq 0$ for all $j = 1, \dots, r$. Since $a_j \leq [a_j]_+$, we have $\sum_{j=1}^r [a_j]_+ \mu_j \geq \sum_{j=1}^r a_j \mu_j$. Hence,

$$(4.5) \quad 1 - \langle [a]_+, \mu \rangle = 1 - \sum_{j=1}^r [a_j]_+ \mu_j \leq 1 - \sum_{j=1}^r a_j \mu_j = 0 \quad \text{for all } \mu \in D_a.$$

We now rewrite the left-hand side of (4.4) as $\|[a]_+\|^{-2} (1 - \langle [a]_+, \mu \rangle)$. From this last fact and (4.5) it follows that the left-hand side of (4.4) is less than or equal to zero. This proves item *i* of the lemma. The proof of item *ii* for $b \neq 0$ follows from item *i* in which the variable λ in (4.2) is scaled by the factor $1/|b|$. For $b = 0$ it follows by direct verification that $\bar{\lambda} = 0$ is the solution in that case. \square

LEMMA 4.2. *The $n/2 - 1$ entries of the solution of problem (4.1), $\bar{\lambda}_1, \dots, \bar{\lambda}_{n/2-1}$, are given by*

$$(4.6) \quad \bar{\lambda}_j = \begin{cases} \frac{1}{2^r} (1 + (-1)^j), & \text{if } r \text{ is even;} \\ \frac{1}{2^{(r+1)}} (1 + (-1)^{j+1}), & \text{if } r \text{ is odd.} \end{cases}$$

Proof. The case r even: According to Lemma 4.1 we have that the minimizer of (4.1) is given by

$$(4.7) \quad \bar{\lambda}_j = \frac{1}{\|[a]_+\|^2} [a_j]_+ \quad \text{with } a_j = A_{1j} = 2(-1)^j.$$

Inserting the identity $[(-1)^j]_+ = \frac{1}{2} + \frac{1}{2}(-1)^j$ into $[a_j]_+$, we get

$$(4.8) \quad [a_j]_+ = 2 \left(\frac{1}{2} + \frac{1}{2}(-1)^j \right) = 1 + (-1)^j.$$

Recalling that $a_j = 2(-1)^j$, we have $\|[a]_+\|^2 = 4 \sum_{j=1}^r [(-1)^j]_+^2$. Inserting the identity $[(-1)^j]_+^2 = \frac{1}{2} + \frac{1}{2}(-1)^j$ into the last expression, and considering that r is an even number, it follows that

$$(4.9) \quad \|[a]_+\|^2 = 4 \left(\sum_{j=1}^r \frac{1}{2} + \sum_{j=1}^r \frac{1}{2}(-1)^j \right) = 4 \left(\frac{r}{2} + 0 \right) = 2r.$$

Plugging (4.9) and (4.8) into (4.7), we obtain the claim, that is,

$$\bar{\lambda}_j = \frac{1}{2r} (1 + (-1)^j).$$

The case r odd: In this case we have “ -1 ” at the right-hand side of the restriction equation in (4.1). By multiplying both sides of this equality by “ -1 ”, we can apply Lemma 4.1 for $a_j = A_{1j} = -2(-1)^j = 2(-1)^{j+1}$. For these values of a_j we should compute $[a_j]_+$ and $\|[a]_+\|^2$, and then make the ratio between them. By the help of the identity $[(-1)^{j+1}]_+ = \frac{1}{2} + \frac{1}{2}(-1)^{j+1}$, we can easily find that $[a_j]_+ = 1 + (-1)^{j+1}$. Further, using the identity $[(-1)^{j+1}]_+^2 = \frac{1}{2} + \frac{1}{2}(-1)^{j+1}$, and considering that r is an odd number, we get $\|[a]_+\|^2 = 4 \left(\sum_{j=1}^r \frac{1}{2} + \sum_{j=1}^r \frac{1}{2}(-1)^{j+1} \right) = 4 \left(\frac{r}{2} + \frac{1}{2} \right) = 2(r+1)$. Hence, $\bar{\lambda}_j = \frac{[a_j]_+}{\|[a]_+\|^2} = \frac{1 + (-1)^{j+1}}{2(r+1)}$. \square

To obtain the explicit solution to the problem (1.2) for the special case of $r = \frac{n}{2} - 1$, we need yet a third lemma which establishes two trigonometric identities for certain sums of cosines.

LEMMA 4.3 (Trigonometric Identities).

A. If $n/2$ is even, then for any $p = 1, 2, \dots, \frac{n}{2} - 1$, we have

$$(4.10) \quad \sum_{q=1}^{n/4} \cos \left(\frac{2\pi}{n} p (2q - 1) \right) = 0.$$

B. If $n/2$ is odd, then for any $p = 1, 2, \dots, \frac{n}{2} - 1$, we have

$$(4.11) \quad \sum_{q=1}^{n/4 - 1/2} \cos \left(\frac{4\pi}{n} p q \right) = -\frac{1}{2}.$$

Proof. See appendix A. \square

Inserting the conclusions of the Lemmas 4.2 and 4.3 into the Proposition 3.3, we obtain the following theorem.

THEOREM 4.4 (The case $r = \frac{n}{2} - 1$). *The minimization problem (1.2) has the following explicit solution in the case of $r = \frac{n}{2} - 1$, according to whether $n/2$ is even or odd.*

A. *If $\frac{n}{2}$ is even, then the minimizer of (1.2) is given by*

$$(4.12) \quad \overline{X}(n, r = n/2 - 1) = \text{Circ} \left(\frac{2}{n}, -\frac{1}{n}, \frac{1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n}, 0, -\frac{1}{n}, \dots, -\frac{1}{n}, \frac{1}{n}, -\frac{1}{n} \right).$$

B. *If $\frac{n}{2}$ is odd, then the minimizer of (1.2) is given by*

$$(4.13) \quad \overline{X}(n, r = n/2 - 1) = \text{Circ} \left(\frac{2}{n}, \overline{x}_1, \dots, \overline{x}_{\frac{n}{2}-1}, 0, \overline{x}_{\frac{n}{2}-1}, \dots, \overline{x}_1 \right),$$

where

$$(4.14) \quad \overline{x}_i = \frac{1}{n} \left[(-1)^i - \frac{2}{n-2} \right], \text{ for } i = 1, 2, \dots, \frac{n}{2} - 1.$$

Proof. Proof of (4.12): Since we are assuming in this case that $n/2$ is an even number, and $r = n/2 - 1$, we have that r is an odd number. According to (4.6) the minimizer of the problem (4.1) is given by $\overline{\lambda}_j = \frac{1}{2(r+1)} (1 + (-1)^{j+1})$, $j = 1, 2, \dots, r = n/2 - 1$. Inserting this expression into (3.6) and recalling (3.2) – (3.5), we obtain that the minimizer $\overline{X}(n, r = n/2 - 1) = \text{Circ}(\overline{x}_0, \overline{x}_1, \dots, \overline{x}_r, 0, \overline{x}_r, \dots, \overline{x}_1)$ has its $r + 1$ nonzero entries given by

$$(4.15) \quad \overline{x}_i = \frac{1}{n} \left\{ (-1)^i + \sum_{j=1}^{n/2-1} 2 \cos\left(\frac{2\pi}{n} i j\right) \frac{1}{2(r+1)} (1 + (-1)^{j+1}) \right\},$$

for $i = 0, 1, 2, \dots, r = n/2 - 1$. Note first that the term $1 + (-1)^{j+1}$ in the above expression is zero if the index j is an even number and 2 if the index j is an odd number. Taking this fact into account, and introducing the new index q , which is related with the odd j 's as $j = 2q - 1$, we can rewrite (4.15) as:

$$(4.16) \quad \overline{x}_i = \frac{1}{n} \left\{ (-1)^i + \sum_{q=1}^{n/4} 2 \cos\left(\frac{2\pi}{n} i (2q - 1)\right) \frac{2}{2(r+1)} \right\},$$

for $i = 0, 1, 2, \dots, r = n/2 - 1$. For $i = 0$, it is easy to check that the result of the above summation, “ $\sum_{q=1}^{n/4} \dots$ ”, is equal to 1 since $r = n/2 - 1$. This accounts for the number 2 in the numerator of the first entry in the expression (4.12) of the minimizer $\overline{X}(n, r = n/2 - 1)$. On the other hand, according to Lemma 4.3, i.e., (4.10), we have that the above summation, “ $\sum_{q=1}^{n/4} \dots$ ”, is zero for $i = 1, 2, 3, \dots, n/2 - 1$. This proves that $\overline{x}_i = \frac{1}{n} (-1)^i$ for $i = 1, 2, \dots, n/2 - 1$, which completes the proof of (4.12).

Proof of (4.13)–(4.14): Since we are assuming in this case that $n/2$ is an odd number, and $r = n/2 - 1$, we have that r is an even number. According to (4.6), the

minimizer of the problem (4.1) is given by $\bar{\lambda}_j = \frac{1}{2^r} (1 + (-1)^j)$, $j = 1, 2, \dots, r = n/2 - 1$. Recall that according to the Proposition 3.3 the minimizer $\bar{X}(n, r = n/2 - 1)$ of the problem (1.2) is given by (3.6). Inserting the above expression for $\bar{\lambda}_j$ into (3.6) and recalling (3.2) – (3.5), we obtain that $\bar{X}(n, r = n/2 - 1) = \text{Circ}(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_r, 0, \bar{x}_r, \dots, \bar{x}_1)$ with

$$(4.17) \quad \bar{x}_i = \frac{1}{n} \left\{ (-1)^{i-1} + \sum_{j=1}^{n/2-1} 2 \cos\left(\frac{2\pi}{n} i j\right) \frac{1}{2^r} (1 + (-1)^j) \right\},$$

for $i = 0, 1, 2, \dots, r = n/2 - 1$. Note first that the term $1 + (-1)^j$ in the above expression is zero if the index j is an odd number, and 2 if the index j is an even number. Considering this fact, and introducing the new index q , which is related with the even j 's as $j = 2q$, we can rewrite (4.17) as:

$$(4.18) \quad \bar{x}_i = \frac{1}{n} \left\{ (-1)^{i-1} + \sum_{q=1}^{n/4-1/2} 2 \cos\left(\frac{2\pi}{n} i 2q\right) \frac{2}{2^r} \right\}, \quad i = 0, 1, 2, \dots, r = n/2 - 1.$$

Again, for $i = 0$, we check that the above summation, “ $\sum_{q=1}^{n/4-1/2} \dots$ ”, is equal to 1 since $r = n/2 - 1$. This accounts for the number 2 in the numerator of the first entry in the expression (4.13). On the other hand, according to Lemma 4.3, namely (4.11), we have that this above summation is now equal to $2 \times (-1/2) \times \frac{2}{2^r} = -\frac{1}{r}$. Since $r = n/2 - 1$ it follows that $\bar{x}_i = \frac{1}{n} \left[(-1)^{i-1} - \frac{2}{n-2} \right]$ for $i = 1, 2, \dots, n/2 - 1$. This proves (4.14). \square

5. The case $r = 2$ (pentadiagonal matrix variables), next-nearest neighbors, and negative stiffness. In this section, we find an explicit solution to the matrix-minimum-norm problem (1.2) for the particular case of $r = 2$. In the mass-spring interpretation this corresponds to a ring-like chain of point masses with two different stiffness. Namely, springs with stiffness $k_1 = -x_1$ linking nearest-neighbor point masses, and springs with stiffness $k_2 = -x_2$ linking next-nearest neighbor ones. It is interesting to mention that the system can be stable – this means that its potential energy $U(d) = \frac{1}{2} d^T X d$ is always positive; equivalently, X is positive semidefinite – even when $k_2 < 0$ (negative stiffness). In fact, the next result shows that the solution of the minimization problem (1.2) has a mass-spring interpretation with negative k_2 in the case of $r = 2$.

THEOREM 5.1. *Let $n \geq 6$ be an even number and $r = 2$. Then $\text{argmin}\{\|X\| : X \in \mathbb{B}^{(r=2)} \cap \mathbb{S}_+ \cap \mathbb{E}^{(1)}\}$ is given by*

$$(5.1) \quad \bar{X}(n, r = 2) = \text{Circ}(\bar{x}_0, \bar{x}_1, \bar{x}_2, 0, \dots, 0, \bar{x}_2, \bar{x}_1),$$

where

- i. $\bar{x}_0 = \frac{1}{2} - \frac{1}{4[1+\cos(\frac{2\pi}{n})]} > 0$;
- ii. $\bar{x}_1 = -\frac{1}{4} < 0$ which corresponds to $k_1 = 1/4 > 0$;
- iii. $\bar{x}_2 = \frac{1}{8[1+\cos(\frac{2\pi}{n})]} > 0$ which corresponds to $k_2 < 0$ (negative stiffness).

Furthermore, the eigenvalues $\bar{\lambda}_i$, $i = 0, 1, \dots, n/2$ of the minimizer \bar{X} are monotonically distributed as

$$(5.2) \quad 0 = \bar{\lambda}_0 = \bar{\lambda}_1 < \bar{\lambda}_2 < \dots < \bar{\lambda}_{\frac{n}{2}-1} < \bar{\lambda}_{\frac{n}{2}} = 1.$$

More precisely, the difference between two consecutive $\bar{\lambda}_i$'s is given by the formula:

$$(5.3) \quad \bar{\lambda}_i - \bar{\lambda}_{i-1} = \frac{2 \sin(2\pi/n)}{1+\cos(2\pi/n)} \sin((2i-1)\pi/n) \sin(i\pi/n) \sin((i-1)\pi/n),$$

for $i = 1, 2, \dots, \frac{n}{2}$.

REMARK 5.2. From the items *i*, *ii*, and *iii* of Theorem 5.1, we can directly obtain the asymptotic behavior as n goes to infinity for the solution with $r = 2$. Namely, we have $\bar{x}_0(n \rightarrow \infty, r = 2) = 3/8$, $\bar{x}_1(n \rightarrow \infty, r = 2) = -1/4$, and $\bar{x}_2(n \rightarrow \infty, r = 2) = 1/16$. Interestingly, this solution corresponds to the non-generic case $\mu = -1$ which appears in the study of infinite chains presented in [4]. Those authors defined the parameter μ as $\mu := \alpha/(4\gamma)$ (see [4, equation (3.5)] and also the matrix presented in the appendix of that paper), where, in their notation, the moduli α and γ correspond to our $-\bar{x}_1(n \rightarrow \infty, r = 2) = 1/4$ and $-\bar{x}_2(n \rightarrow \infty, r = 2) = -1/16$, respectively.

REMARK 5.3. By contrast, for $r = 3$ the distribution of eigenvalues $(\bar{\lambda}_i, i = 0, 1, \dots, n/2)$ does not need to be monotonic anymore. See for instance the dashed line of the third graph of Figure 6.1 (case $n = 12$ and $r = 3$).

Proof of Theorem 5.1. Recall first that according to Section 2, the minimizer \bar{X} must be a circulant matrix. Note also that the zero-entries in (5.1) come from the fact that $\bar{X} \in \mathbb{B}^{(r)}$ with $r = 2$. According to (2.2) and Remark 3.2, the nonzero entries of \bar{X} , that is, \bar{x}_0 , \bar{x}_1 , and \bar{x}_2 are the solution of the following minimization problem:

$$(5.4) \quad (\bar{x}_0, \bar{x}_1, \bar{x}_2) = \operatorname{argmin} \left\{ \begin{array}{ll} a) & x_0 + \sum_{j=1}^{r=2} 2 \cos(\theta_{ij}) x_j \geq 0 \\ & \text{for } i = 1, \dots, n/2 - 1; \\ b) & x_0 + 2x_1 + 2x_2 = 0 \\ & \text{(which corresponds to } i = 0); \\ c) & x_0 - 2x_1 + 2x_2 = 1 \\ & \text{(which corresponds to } i = n/2). \end{array} : x_0^2 + 2x_1^2 + 2x_2^2 \right\},$$

where we recall that $\theta = \frac{2\pi}{n}$. By subtracting equation c from b in (5.4) we obtain *ii*, that is, $\bar{x}_1 = -1/4$. Replace this last result into b or c, we establish that \bar{x}_0 and \bar{x}_2

are related to each other as

$$(5.5) \quad \bar{x}_0 = \frac{1}{2} - 2\bar{x}_2.$$

In the next, we will find \bar{x}_2 . Replace $\bar{x}_1 = -1/4$ and (5.5) into (5.4), we obtain that \bar{x}_2 must be the solution of the problem

$$\bar{x}_2 = \operatorname{argmin} \left\{ \left(\frac{1}{2} - 2x_2 \right)^2 + \frac{1}{8} + 2x_2^2 : \left(\frac{1}{2} - 2x_2 \right) - \frac{1}{2} \cos(\theta i) + 2x_2 \cos(2\theta i) \geq 0, \right. \\ \left. i = 1, \dots, \frac{n}{2} - 1 \right\}.$$

This, after some simplifications, gives

$$\bar{x}_2 = \operatorname{argmin} \left\{ \left(x_2 - \frac{1}{6} \right)^2 : \frac{1}{2} (1 - \cos(\theta i)) - 2(1 - \cos(2\theta i)) x_2 \geq 0, \right. \\ \left. i = 1, \dots, \frac{n}{2} - 1 \right\}.$$

Noting that $1 - \cos(2\theta i) > 0$ for $i = 1, \dots, n/2 - 1$ since $\theta = \frac{2\pi}{n}$ and $n \geq 6$, and using the trigonometric identity

$$(5.6) \quad 1 - \cos(2\theta) = 2(1 + \cos(\theta))(1 - \cos(\theta)),$$

we obtain

$$(5.7) \quad \bar{x}_2 = \operatorname{argmin} \left\{ \left(x_2 - \frac{1}{6} \right)^2 : x_2 \leq \frac{1}{8[1 + \cos(\theta i)]}, i = 1, \dots, \frac{n}{2} - 1 \right\}.$$

Recalling again that $\theta = \frac{2\pi}{n}$, it is easy to see that $\frac{1}{8[1 + \cos(\theta i)]} < \frac{1}{8[1 + \cos(\theta(i+1))]}$ for $i = 1, \dots, \frac{n}{2} - 2$. From this inequality follows that

$$(5.8) \quad \bar{x}_2 = \operatorname{argmin} \left\{ \left(x_2 - \frac{1}{6} \right)^2 : x_2 \leq \frac{1}{8[1 + \cos(\theta)]} \right\}.$$

Noting that $\frac{1}{8[1 + \cos(\theta)]} < \frac{1}{6}$ since $\theta = \frac{2\pi}{n}$ and $n \geq 6$, we have from above expression that $\bar{x}_2 = \frac{1}{8[1 + \cos(\theta)]}$. This closes the proof of *iii*. The expression *i* follows from *iii* and (5.5).

Proof of the distribution (5.2): Recall that $\bar{\lambda}_i$ are related to \bar{x}_i , $i = 0, 1, \dots, n/2$, as $\bar{\lambda} = V \bar{x}$ (see (3.2), (3.3) and (3.4)). From this relation it follows immediately that $\bar{\lambda}_0 = 0$ since $\bar{\lambda}_0 = \sum_{j=0}^{n/2} V_{0j} \bar{x}_j = \bar{x}_0 + 2\bar{x}_1 + 2\bar{x}_2 = 0$ (see restriction *b* of (5.4)), and $\bar{\lambda}_{n/2} = 1$ since $\bar{\lambda}_{n/2} = \sum_{j=0}^{n/2} V_{n/2j} \bar{x}_j = \bar{x}_0 - 2\bar{x}_1 + 2\bar{x}_2 = 1$ (see restriction *c* of (5.4)).

We now prove the formula (5.3), which completes the proof of (5.2), since the right-hand side of (5.3) is equal to zero for $i = 1$ and strict positive for $i = 2, 3, \dots, n/2$. To prove the formula (5.3), we recall that $n \geq 6$ (so $n/2 \geq 3$) and $r = 2$ (thus $0 = \bar{x}_3 = \dots = \bar{x}_{n/2}$), and use (3.4) (so $V_{i0} - V_{i-1,0} = 1 - 1 = 0$). Namely, we have

$$\begin{aligned}\bar{\lambda}_i - \bar{\lambda}_{i-1} &= \sum_{j=0}^{n/2} (V_{i,j} - V_{i-1,j}) \bar{x}_j \\ &= [2 \cos(i\theta) - 2 \cos((i-1)\theta)] \bar{x}_1 + [2 \cos(2i\theta) - 2 \cos(2(i-1)\theta)] \bar{x}_2.\end{aligned}$$

Plugging $\bar{x}_1 = -\frac{1}{4}$ (see *ii* above) and $\bar{x}_2 = \frac{1}{8[1+\cos(\theta)]}$ (see *iii* above) into the last expression, yields

$$\bar{\lambda}_i - \bar{\lambda}_{i-1} = \frac{1}{2} [\cos((i-1)\theta) - \cos(i\theta)] + \frac{1}{4[1+\cos(\theta)]} [\cos(2i\theta) - \cos(2(i-1)\theta)].$$

After using the trigonometric identity $\cos(2\alpha) = 2\cos^2(\alpha) - 1$ for each term of the difference

$[\cos(2i\theta) - \cos(2(i-1)\theta)]$, we obtain

$$\begin{aligned}\bar{\lambda}_i - \bar{\lambda}_{i-1} &= \frac{1}{2} \left\{ [\cos((i-1)\theta) - \cos(i\theta)] + \frac{1}{1+\cos(\theta)} [\cos^2(i\theta) - \cos^2((i-1)\theta)] \right\} \\ &= \frac{1}{2} [\cos((i-1)\theta) - \cos(i\theta)] \left\{ 1 - \frac{\cos((i-1)\theta) + \cos(i\theta)}{1+\cos(\theta)} \right\} \\ &= \frac{1}{2} \left[\frac{\cos((i-1)\theta) - \cos(i\theta)}{1+\cos(\theta)} \right] \{ \cos(0) - \cos((i-1)\theta) + \cos(\theta) - \cos(i\theta) \}.\end{aligned}$$

First, we use the identity for the difference of cosines in terms of products of sines, three times; second, we apply the identity for the sum of two sines in terms of product of sine and cosine; and third, we use the elementary identity $\sin(1/2\theta) \cos(1/2\theta) = 1/2 \sin(\theta)$. Namely, we have

$$\begin{aligned}\bar{\lambda}_i - \bar{\lambda}_{i-1} &= \frac{\sin((i-\frac{1}{2})\theta) \sin(\frac{1}{2}\theta)}{1+\cos(\theta)} 2 \left\{ \sin^2(\frac{i-1}{2}\theta) + \sin(\frac{i+1}{2}\theta) \sin(\frac{i-1}{2}\theta) \right\} \\ &= 2 \frac{\sin((i-\frac{1}{2})\theta) \sin(\frac{1}{2}\theta) \sin(\frac{i-1}{2}\theta)}{1+\cos(\theta)} \{ \sin(\frac{i-1}{2}\theta) + \sin(\frac{i+1}{2}\theta) \} \\ &= 4 \frac{\sin((i-\frac{1}{2})\theta) \sin(\frac{1}{2}\theta) \sin(\frac{i-1}{2}\theta)}{1+\cos(\theta)} \sin(\frac{i}{2}\theta) \cos(\frac{1}{2}\theta) \\ &= \frac{2}{1+\cos(\theta)} \sin(\theta) \sin((i-\frac{1}{2})\theta) \sin(\frac{i-1}{2}\theta) \sin(\frac{i}{2}\theta).\end{aligned}$$

This closes the proof of the formula (5.3). \square

6. Numerical illustration for the case $n = 12$ and $1 \leq r \leq 6$. The purpose of this section is to illustrate the results of Theorem 5.1, Theorem 4.4, and Corollary 3.4. More precisely, we present below the six graphics corresponding, respectively, to $r = 1, 2, 3, 4, 5$, and 6 for the case $n = 12$.

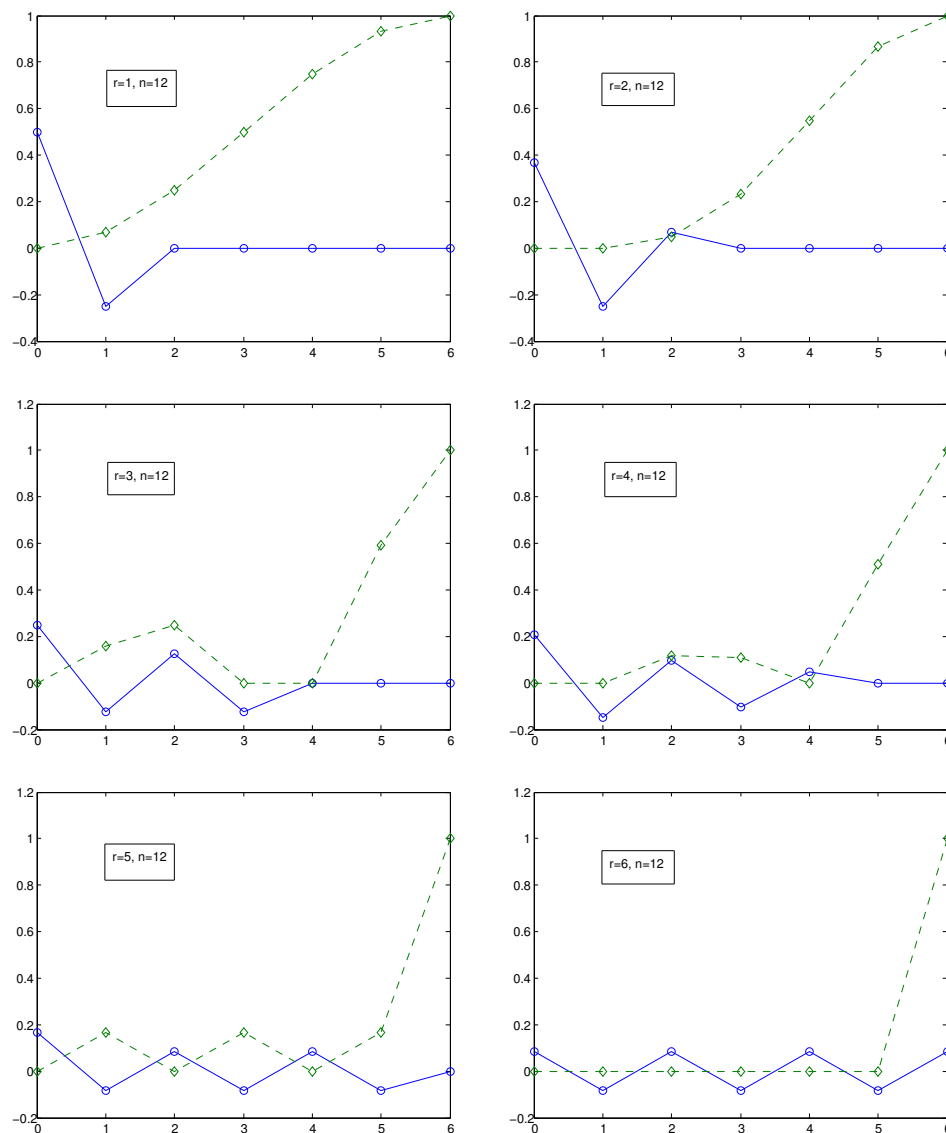


FIG. 6.1. The six pictures above were numerically obtained by the MATLAB toolbox `lsqlin`; see Section 3 for a discussion on algorithms. They show the first row (solid line) and its Fourier-transform (dashed line) of the circulant matrix which solves the minimization problem (1.2) in the case of $n = 12$ and $\beta = 1$ for $r = 1, 2, 3, 4, 5$, and 6 , respectively. Note that the case $r = 1$ corresponds to the standard Laplacian $\frac{1}{4}\mathcal{L}$; observe also that the case $r = 2$, which is illustrated by the second picture, is in agreement with Theorem 5.1. The illustration for the cases $r = 5$ and $r = 6$ are in agreement with Theorem 4.4 and Corollary 3.4, respectively.

7. Conclusions and outlook. In this paper, we proposed a matrix-minimum-norm problem which for the particular case of $r = 1$ characterizes the discrete periodic Laplacian operator. We obtained explicit solutions for other special cases of $r = n/2$, $r = n/2 - 1$, and $r = 2$. Its solutions can be interpreted as the stiffness values which minimize the potential energy of a mass-spring system with r -nearest neighbors. It seems plausible that the eigenvalue $\beta = 1$ of the solutions is simple and the largest one, but we have been able to establish this only in the special cases above mentioned. Questions on explicit solutions for other cases of r , and for the asymptotic case in which $r = n/4$ and n goes to infinity, should be of interest in a future research.

Appendix A. Proof of Lemma 4.3..

The proof of this lemma goes as follows. Define the value $\varphi_{n,p} := \frac{4\pi}{n} p$. Note that $\varphi_{n,p} \in (0, 2\pi)$. To see this, recall first that p takes only the values $1, 2, \dots, n/2 - 1$, so $0 < \frac{4\pi}{n} \leq \frac{4\pi}{n} p = \varphi_{n,p} < 2\pi - \frac{4\pi}{n} < 2\pi$. Hence, $\exp(\sqrt{-1} \varphi_{n,p}) \neq 1$. This last fact allows us to apply the geometric series sum formula for Q summands. Namely,

$$\begin{aligned} & \sum_{q=1}^Q \exp(\sqrt{-1} q \varphi_{n,p}) \\ &= \frac{\exp(\sqrt{-1} (Q+1) \varphi_{n,p}) - \exp(\sqrt{-1} \varphi_{n,p})}{\exp(\sqrt{-1} \varphi_{n,p}) - 1} \\ &= \frac{\exp(-\sqrt{-1} \frac{1}{2} \varphi_{n,p})}{\exp(-\sqrt{-1} \frac{1}{2} \varphi_{n,p})} \frac{\exp(\sqrt{-1} (Q+1) \varphi_{n,p}) - \exp(\sqrt{-1} \varphi_{n,p})}{\exp(\sqrt{-1} \varphi_{n,p}) - 1} \\ &= \frac{\exp(\sqrt{-1} (Q+1/2) \varphi_{n,p}) - \exp(\sqrt{-1} \frac{1}{2} \varphi_{n,p})}{2 \sqrt{-1} \sin(1/2 \varphi_{n,p})} \\ &= \frac{\sin((Q+1/2) \varphi_{n,p}) - \sin(1/2 \varphi_{n,p})}{2 \sin(1/2 \varphi_{n,p})} + \frac{\cos(1/2 \varphi_{n,p}) - \cos((Q+1/2) \varphi_{n,p})}{2 \sin(1/2 \varphi_{n,p})} \sqrt{-1}. \end{aligned}$$

Recalling that $\varphi_{n,p} := \frac{4\pi}{n} p$, and using the uniqueness of the decomposition of a complex number into real and imaginary parts, we have

$$(A.1) \quad \sum_{q=1}^Q \cos\left(\frac{2\pi}{n} p 2q\right) = \frac{\sin\left((Q+1/2) \frac{4\pi}{n} p\right) - \sin\left(\frac{2\pi}{n} p\right)}{2 \sin\left(\frac{2\pi}{n} p\right)}$$

and

$$(A.2) \quad \sum_{q=1}^Q \sin\left(\frac{2\pi}{n} p 2q\right) = \frac{\cos\left(\frac{2\pi}{n} p\right) - \cos\left((Q+1/2) \frac{4\pi}{n} p\right)}{2 \sin\left(\frac{2\pi}{n} p\right)}.$$

Proof of item B: We need just to plug $Q = n/4 - 1/2$ into the identity (A.1) and note that

$$\sin\left((Q+1/2) \frac{4\pi}{n} p\right) = \sin\left(n/4 \frac{4\pi}{n} p\right) = \sin(\pi p) = 0. \quad \square$$

Proof of item A: We first rewrite $\cos\left(\frac{2\pi}{n}p(2q-1)\right)$ as

$$\cos\left(\frac{2\pi}{n}p\right)\cos\left(\frac{2\pi}{n}p2q\right) + \sin\left(\frac{2\pi}{n}p\right)\sin\left(\frac{2\pi}{n}p2q\right).$$

Then we plug $Q = n/4$ into the identities (A.1) and (A.2), and note that

$$\sin\left((n/4 + 1/2)\frac{4\pi}{n}p\right) \pm \sin\left(\frac{2\pi}{n}p\right) = 0$$

and

$$\cos\left(\frac{2\pi}{n}p\right) \pm \cos\left((n/4 + 1/2)\frac{4\pi}{n}p\right) = 0$$

according to p is odd or even. \square

Acknowledgement. It is a pleasure to thank Bryan L. Shader for his encouragement and crucial suggestions in the first version of the manuscript. Special thanks go to the anonymous referee for pointing out some aspects of graph theory and helping me to improve the final version of the paper. The author would also like to thank Lívio Leandro Avelino de Oliveira for pointing out some typos.

REFERENCES

- [1] J.P. Boyle and R.L. Dykstra. A method for finding projections onto the intersection of convex sets in Hilbert spaces. *Lecture Notes in Statist.*, 37:28–47, 1986.
- [2] F.E. Browder. Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces. *Arch. Rational Mech. Anal.*, 24:82–90, 1967.
- [3] J.A. Cadzow and Y. Sun. Sequences with positive semidefinite Fourier transforms. *IEEE Trans. Acoust. Speech Signal Process.*, ASSP-34:1502–1510, 1986.
- [4] M. Charlotte and L. Truskinovsky. Linear elastic chain with a hyper-pre-stress *J. Mech. Phys. Solids*, 50:217–251, 2002.
- [5] R.L. Dykstra. An algorithm for restricted least squares regression. *J. Amer. Stat. Assoc.*, 78:837–842, 1983.
- [6] R. Escalante and M. Raydan. *Alternating Projection Methods*. Fundamentals of Algorithms. Society for Industrial and Applied Mathematics, 2011.
- [7] B. Halpern. Fixed points of nonexpanding maps. *Bull. Amer. Math. Soc.*, 73:957–961, 1967.
- [8] N.J. Higham. Computing the nearest correlation matrix – A problem from finance. *IMA J. Numer. Anal.*, 22(3):329–343, 2002.